2. Show that

$$\frac{1}{1/z} = z \qquad (z \neq 0).$$

3. Use the associative and commutative laws for multiplication to show that

$$(z_1 z_2)(z_3 z_4) = (z_1 z_3)(z_2 z_4).$$

4. Prove that if $z_1 z_2 z_3 = 0$, then at least one of the three factors is zero.

Suggestion: Write $(z_1 z_2) z_3 = 0$ and use a similar result (Sec. 3) involving two factors.

5. Derive expression (6), Sec. 3, for the quotient z_1/z_2 by the method described just after it.

6. With the aid of relations (10) and (11) in Sec. 3, derive the identity

$$\left(\frac{z_1}{z_3}\right)\left(\frac{z_2}{z_4}\right) = \frac{z_1 z_2}{z_3 z_4} \qquad (z_3 \neq 0, z_4 \neq 0).$$

7. Use the identity obtained in Exercise 6 to derive the cancellation law

$$\frac{z_1 z}{z_2 z} = \frac{z_1}{z_2} \qquad (z_2 \neq 0, z \neq 0).$$

8. Use mathematical induction to verify the binomial formula (13) in Sec. 3. More precisely, note that the formula is true when $n = 1$. Then, assuming that it is valid when $n = m$ where m denotes any positive integer, show that it must hold when $n = m + 1$.

Suggestion: When $n = m + 1$, write

$$\begin{aligned}(z_1 + z_2)^{m+1} &= (z_1 + z_2)(z_1 + z_2)^m = (z_2 + z_1)\sum_{k=0}^{m}\binom{m}{k} z_1^k z_2^{m-k} \\ &= \sum_{k=0}^{m}\binom{m}{k} z_1^k z_2^{m+1-k} + \sum_{k=0}^{m}\binom{m}{k} z_1^{k+1} z_2^{m-k}\end{aligned}$$

and replace k by $k - 1$ in the last sum here to obtain

$$(z_1 + z_2)^{m+1} = z_2^{m+1} + \sum_{k=1}^{m}\left[\binom{m}{k} + \binom{m}{k-1}\right] z_1^k z_2^{m+1-k} + z_1^{m+1}.$$

Finally, show how the right-hand side here becomes

$$z_2^{m+1} + \sum_{k=1}^{m}\binom{m+1}{k} z_1^k z_2^{m+1-k} + z_1^{m+1} = \sum_{k=0}^{m+1}\binom{m+1}{k} z_1^k z_2^{m+1-k}.$$

4. VECTORS AND MODULI

It is natural to associate any nonzero complex number $z = x + iy$ with the directed line segment, or vector, from the origin to the point (x, y) that represents z in the complex plane. In fact, we often refer to z as the point z or the vector z. In Fig. 2 the numbers $z = x + iy$ and $-2 + i$ are displayed graphically as both points and radius vectors.

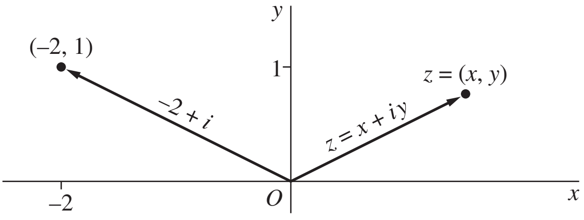

FIGURE 2

When $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$, the sum

$$z_1 + z_2 = (x_1 + x_2) + i(y_1 + y_2)$$

corresponds to the point $(x_1 + x_2, y_1 + y_2)$. It also corresponds to a vector with those coordinates as its components. Hence $z_1 + z_2$ may be obtained vectorially as shown in Fig. 3.

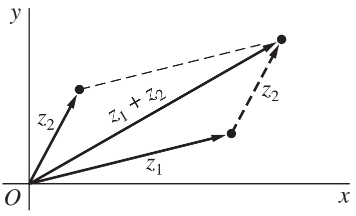

FIGURE 3

Although the product of two complex numbers z_1 and z_2 is itself a complex number represented by a vector, that vector lies in the same plane as the vectors for z_1 and z_2. Evidently, then, this product is neither the scalar nor the vector product used in ordinary vector analysis.

The vector interpretation of complex numbers is especially helpful in extending the concept of absolute values of real numbers to the complex plane. The ***modulus***, or absolute value, of a complex number $z = x + iy$ is defined as the nonnegative real number $\sqrt{x^2 + y^2}$ and is denoted by $|z|$; that is,

$$|z| = \sqrt{x^2 + y^2}. \tag{1}$$

It follows immediately from definition (1) that the real numbers $|z|$, $x = \operatorname{Re} z$, and $y = \operatorname{Im} z$ are related by the equation

$$|z|^2 = (\operatorname{Re} z)^2 + (\operatorname{Im} z)^2. \tag{2}$$

Thus

$$\operatorname{Re} z \le |\operatorname{Re} z| \le |z| \quad \text{and} \quad \operatorname{Im} z \le |\operatorname{Im} z| \le |z|. \tag{3}$$

Geometrically, the number $|z|$ is the distance between the point (x, y) and the origin, or the length of the radius vector representing z. It reduces to the usual absolute value in the real number system when $y = 0$. Note that while *the inequality* $z_1 < z_2$ *is meaningless unless both* z_1 *and* z_2 *are real,* the statement $|z_1| < |z_2|$ means that the point z_1 is closer to the origin than the point z_2 is.

EXAMPLE 1. Since $|-3 + 2i| = \sqrt{13}$ and $|1 + 4i| = \sqrt{17}$, we know that the point $-3 + 2i$ is closer to the origin than $1 + 4i$ is.

The distance between two points (x_1, y_1) and (x_2, y_2) is $|z_1 - z_2|$. This is clear from Fig. 4, since $|z_1 - z_2|$ is the length of the vector representing the number

$$z_1 - z_2 = z_1 + (-z_2);$$

and, by translating the radius vector $z_1 - z_2$, one can interpret $z_1 - z_2$ as the directed line segment from the point (x_2, y_2) to the point (x_1, y_1). Alternatively, it follows from the expression

$$z_1 - z_2 = (x_1 - x_2) + i(y_1 - y_2)$$

and definition (1) that

$$|z_1 - z_2| = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}.$$

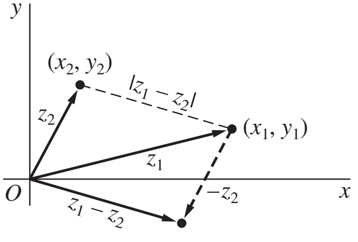

FIGURE 4

The complex numbers z corresponding to the points lying on the circle with center z_0 and radius R thus satisfy the equation $|z - z_0| = R$, and conversely. We refer to this set of points simply as the circle $|z - z_0| = R$.

EXAMPLE 2. The equation $|z - 1 + 3i| = 2$ represents the circle whose center is $z_0 = (1, -3)$ and whose radius is $R = 2$.

Our final example here illustrates the power of geometric reasoning in complex analysis when straightforward computation can be somewhat tedious.

EXAMPLE 3. Consider the set of all points $z = (x, y)$ satisfying the equation

$$|z - 4i| + |z + 4i| = 10.$$

Upon writing this as

$$|z - 4i| + |z - (-4i)| = 10,$$

one can see that it represents the set of all points $P(x, y)$ in the $z = (x, y)$ plane the sum of whose distances from two fixed points $F(0, 4)$ and $F'(0, -4)$ is the constant 10. This is, of course, an ellipse with foci $F(0, 4)$ and $F'(0, -4)$.

5. TRIANGLE INEQUALITY

We turn now to the ***triangle inequality***, which provides an upper bound for the modulus of the sum of two complex numbers z_1 and z_2:

$$|z_1 + z_2| \le |z_1| + |z_2|. \tag{1}$$

This important inequality is geometrically evident in Fig. 3 of Sec. 4 since it is merely a statement that the length of one side of a triangle is less than or equal to the sum of the lengths of the other two sides. We can also see from Fig. 3 that inequality (1) is actually an equality when 0, z_1, and z_2 are collinear. Another, strictly algebraic, derivation is given in Exercise 15, Sec. 6.

An immediate consequence of the triangle inequality is the fact that

$$|z_1 + z_2| \ge ||z_1| - |z_2||. \tag{2}$$

To derive inequality (2), we write

$$|z_1| = |(z_1 + z_2) + (-z_2)| \le |z_1 + z_2| + |- z_2|,$$

which means that

$$|z_1 + z_2| \ge |z_1| - |z_2|. \tag{3}$$

This is inequality (2) when $|z_1| \ge |z_2|$. If $|z_1| < |z_2|$, we need only interchange z_1 and z_2 in inequality (3) to arrive at

$$|z_1 + z_2| \ge -(|z_1| - |z_2|),$$

which is the desired result. Inequality (2) tells us, of course, that the length of one side of a triangle is greater than or equal to the difference of the lengths of the other two sides.

Because $|- z_2| = |z_2|$, one can replace z_2 by $-z_2$ in inequalities (1) and (2) to write

$$|z_1 - z_2| \le |z_1| + |z_2| \quad \text{and} \quad |z_1 - z_2| \ge ||z_1| - |z_2||.$$

In actual practice, however, one need use only inequalities (1) and (2). This is illustrated in the following example.

EXAMPLE 1. If a point z lies on the unit circle $|z| = 1$, inequalities (1) and (2) tell us that

$$|z - 2| = |z + (-2)| \le |z| + |-2| = 1 + 2 = 3$$

and

$$|z - 2| = |z + (-2)| \ge ||z| - |-2|| = |1 - 2| = 1.$$

The triangle inequality (1) can be generalized by means of mathematical induction to sums involving any finite number of terms:

$$|z_1 + z_2 + \cdots + z_n| \le |z_1| + |z_2| + \cdots + |z_n| \qquad (n = 2, 3, \ldots). \tag{4}$$

To give details of the induction proof here, we note that when $n = 2$, inequality (4) is just inequality (1). Furthermore, if inequality (4) is valid when $n = m$, it must also hold when $n = m + 1$ since by inequality (1),

$$\begin{aligned} |(z_1 + z_2 + \cdots + z_m) + z_{m+1}| &\le |z_1 + z_2 + \cdots + z_m| + |z_{m+1}| \\ &\le (|z_1| + |z_2| + \cdots + |z_m|) + |z_{m+1}|. \end{aligned}$$

EXAMPLE 2. Let z denote any complex number lying on the circle $|z| = 2$. Inequality (4) tells us that

$$\left|3 + z + z^2\right| \le 3 + |z| + \left|z^2\right|.$$

Since $|z^2| = |z|^2$, according to Exercise (8),

$$|3 + z + z^2| \le 9.$$

EXAMPLE 3. If n is a positive integer and if $a_0, a_1, a_2, \ldots, a_n$ are complex constants, where $a_n \neq 0$, the quantity

$$P(z) = a_0 + a_1 z + a_2 z^2 + \cdots + a_n z^n \tag{5}$$

is a ***polynomial*** of degree n. We shall show here that for some positive number R, the reciprocal $1/P(z)$ satisfies the inequality

$$\left|\frac{1}{P(z)}\right| < \frac{2}{|a_n|R^n} \quad \text{whenever} \quad |z| > R. \tag{6}$$

Geometrically, this tells us that the modulus of the reciprocal $1/P(z)$ is bounded from above when z is exterior to the circle $|z| = R$. This important property of polynomials will be used later on in Sec. 58 of Chap. 4, and we verify it here since it illustrates the use of inequalities presented in this section, as well as the identities

$$|z_1 z_2| = |z_1||z_2| \quad \text{and} \quad |z^n| = |z|^n \quad (n = 1, 2, \ldots)$$

to be obtained in Exercises 8 and 9.

We first write

$$w = \frac{a_0}{z^n} + \frac{a_1}{z^{n-1}} + \frac{a_2}{z^{n-2}} + \cdots + \frac{a_{n-1}}{z} \qquad (z \neq 0), \tag{7}$$

so that

$$P(z) = (a_n + w)z^n \tag{8}$$

when $z \neq 0$. Next, we multiply through equation (7) by z^n:

$$w z^n = a_0 + a_1 z + a_2 z^2 + \cdots + a_{n-1} z^{n-1}.$$

This tells us that

$$|w||z|^n \leq |a_0| + |a_1||z| + |a_2||z|^2 + \cdots + |a_{n-1}||z|^{n-1},$$

or

$$|w| \leq \frac{|a_0|}{|z|^n} + \frac{|a_1|}{|z|^{n-1}} + \frac{|a_2|}{|z|^{n-2}} + \cdots + \frac{|a_{n-1}|}{|z|}. \tag{9}$$

Now that a sufficiently large positive number R can be found such that each of the quotients on the right in inequality (9) is less than the number $|a_n|/(2n)$ when $|z| > R$, and so

$$|w| < n\frac{|a_n|}{2n} = \frac{|a_n|}{2} \quad \text{whenever} \quad |z| > R.$$

Consequently,

$$|a_n + w| \geq ||a_n| - |w|| > \frac{|a_n|}{2} \quad \text{whenever} \quad |z| > R;$$

and, in view of equation (8),

$$|P_n(z)| = |a_n + w||z|^n > \frac{|a_n|}{2}|z|^n > \frac{|a_n|}{2}R^n \quad \text{whenever} \quad |z| > R. \tag{10}$$

Statement (6) follows immediately from this.

EXERCISES

1. Locate the numbers $z_1 + z_2$ and $z_1 - z_2$ vectorially when

(a) $z_1 = 2i, \quad z_2 = \frac{2}{3} - i$;

(b) $z_1 = (-\sqrt{3}, 1), \quad z_2 = (\sqrt{3}, 0)$;

(c) $z_1 = (-3, 1), \quad z_2 = (1, 4)$;

(d) $z_1 = x_1 + iy_1, \quad z_2 = x_1 - iy_1$.

2. Verify inequalities (3), Sec. 4, involving $\operatorname{Re} z$, $\operatorname{Im} z$, and $|z|$.

3. Use established properties of moduli to show that when $|z_3| \neq |z_4|$,

$$\frac{\operatorname{Re}(z_1 + z_2)}{|z_3 + z_4|} \leq \frac{|z_1| + |z_2|}{||z_3| - |z_4||}.$$

4. Verify that $\sqrt{2}\,|z| \geq |\operatorname{Re} z| + |\operatorname{Im} z|$.

Suggestion: Reduce this inequality to $(|x| - |y|)^2 \geq 0$.

5. In each case, sketch the set of points determined by the given condition:

(a) $|z - 1 + i| = 1$; (b) $|z + i| \leq 3$; (c) $|z - 4i| \geq 4$.

6. Using the fact that $|z_1 - z_2|$ is the distance between two points z_1 and z_2, give a geometric argument that $|z - 1| = |z + i|$ represents the line through the origin whose slope is -1.

7. Show that for R sufficiently large, the polynomial $P(z)$ in Example 3, Sec. 5, satisfies the inequality

$$|P(z)| < 2|a_n||z|^n \quad \text{whenever} \quad |z| > R.$$

Suggestion: Observe that there is a positive number R such that the modulus of each quotient in inequality (9), Sec. 5, is less than $|a_n|/n$ when $|z| > R$.

8. Let z_1 and z_2 denote any complex numbers

$$z_1 = x_1 + iy_1 \quad \text{and} \quad z_2 = x_2 + iy_2.$$

Use simple algebra to show that

$$|(x_1 + iy_1)(x_2 + iy_2)| \quad \text{and} \quad \sqrt{(x_1^2 + y_1^2)(x_2^2 + y_2^2)}$$

are the same and then point out how the identity

$$|z_1 z_2| = |z_1||z_2|$$

follows.

9. Use the final result in Exercise 8 and mathematical induction to show that

$$|z^n| = |z|^n \qquad (n = 1, 2, \ldots),$$

where z is any complex number. That is, after noting that this identity is obviously true when $n = 1$, assume that it is true when $n = m$ where m is any positive integer and then show that it must be true when $n = m + 1$.

6. COMPLEX CONJUGATES

The ***complex conjugate***, or simply the conjugate, of a complex number $z = x + iy$ is defined as the complex number $x - iy$ and is denoted by $\overline{z}$; that is,

$$\overline{z} = x - iy. \tag{1}$$

The number $\overline{z}$ is represented by the point $(x, -y)$, which is the reflection in the real axis of the point (x, y) representing z (Fig. 5). Note that

$$\overline{\overline{z}} = z \quad \text{and} \quad |\overline{z}| = |z|$$

for all z.

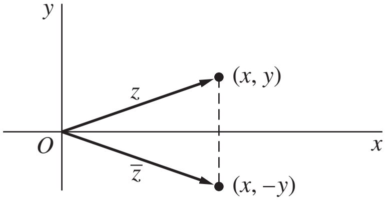

FIGURE 5

If $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$, then

$$\overline{z_1 + z_2} = (x_1 + x_2) - i(y_1 + y_2) = (x_1 - iy_1) + (x_2 - iy_2).$$

So the conjugate of the sum is the sum of the conjugates:

$$\overline{z_1 + z_2} = \overline{z_1} + \overline{z_2}. \tag{2}$$

In like manner, it is easy to show that

$$\overline{z_1 - z_2} = \overline{z_1} - \overline{z_2}, \tag{3}$$

$$\overline{z_1 z_2} = \overline{z_1}\,\overline{z_2}, \tag{4}$$

and

$$\overline{\left(\frac{z_1}{z_2}\right)} = \frac{\overline{z_1}}{\overline{z_2}} \qquad (z_2 \neq 0). \tag{5}$$

The sum $z + \overline{z}$ of a complex number $z = x + iy$ and its conjugate $\overline{z} = x - iy$ is the real number $2x$, and the difference $z - \overline{z}$ is $2iy$. Hence

$$\operatorname{Re} z = \frac{z + \overline{z}}{2} \quad \text{and} \quad \operatorname{Im} z = \frac{z - \overline{z}}{2i}. \tag{6}$$

An important identity relating the conjugate of a complex number $z = x + iy$ to its modulus is

$$z\,\overline{z} = |z|^2, \tag{7}$$

where each side is equal to $x^2 + y^2$. It suggests the method for determining a quotient z_1/z_2 that begins with expression (7), Sec. 3. That method is, of course, based on multiplying both the numerator and the denominator of z_1/z_2 by $\overline{z_2}$, so that the denominator becomes the real number $|z_2|^2$.

EXAMPLE 1. As an illustration,

$$\frac{-1 + 3i}{2 - i} = \frac{(-1 + 3i)(2 + i)}{(2 - i)(2 + i)} = \frac{-5 + 5i}{|2 - i|^2} = \frac{-5 + 5i}{5} = -1 + i.$$

See also the example in Sec. 3.

Identity (7) is especially useful in obtaining properties of moduli from properties of conjugates noted above. We mention that (compare with Exercise 8, Sec. 5)

$$|z_1 z_2| = |z_1||z_2|. \tag{8}$$

Also,

$$\left|\frac{z_1}{z_2}\right| = \frac{|z_1|}{|z_2|} \qquad (z_2 \neq 0). \tag{9}$$

Property (8) can be established by writing

$$|z_1 z_2|^2 = (z_1 z_2)(\overline{z_1 z_2}) = (z_1 z_2)(\overline{z_1}\,\overline{z_2}) = (z_1 \overline{z_1})(z_2 \overline{z_2}) = |z_1|^2 |z_2|^2 = (|z_1||z_2|)^2$$

and recalling that a modulus is never negative. Property (9) can be verified in a similar way.

EXAMPLE 2. Property (8) tells us that $|z^2| = |z|^2$ and $|z^3| = |z|^3$. Hence if z is a point inside the circle centered at the origin with radius 2, so that $|z| < 2$, it follows from the generalized triangle inequality (4) in Sec. 5 that

$$|z^3 + 3z^2 - 2z + 1| \le |z|^3 + 3|z|^2 + 2|z| + 1 < 25.$$

EXERCISES

1. Use properties of conjugates and moduli established in Sec. 6 to show that

(a) $\overline{\overline{z} + 3i} = z - 3i$; ($b$) $\overline{iz} = -i\overline{z}$;

(c) $\overline{(2+i)^2} = 3 - 4i$; ($d$) $|(2\overline{z}+5)(\sqrt{2}-i)| = \sqrt{3}\,|2z+5|$.

2. Sketch the set of points determined by the condition

(a) $\operatorname{Re}(\overline{z} - i) = 2$; ($b$) $|2\overline{z} + i| = 4$.

3. Verify properties (3) and (4) of conjugates in Sec. 6.

4. Use property (4) of conjugates in Sec. 6 to show that

(a) $\overline{z_1 z_2 z_3} = \overline{z_1}\,\overline{z_2}\,\overline{z_3}$; ($b$) $\overline{z^4} = \overline{z}^4$.

5. Verify property (9) of moduli in Sec. 6.

6. Use results in Sec. 6 to show that when z_2 and z_3 are nonzero,

(a) $\overline{\left(\dfrac{z_1}{z_2 z_3}\right)} = \dfrac{\overline{z_1}}{\overline{z_2}\,\overline{z_3}}$; ($b$) $\left|\dfrac{z_1}{z_2 z_3}\right| = \dfrac{|z_1|}{|z_2||z_3|}$.

7. Show that

$$|\operatorname{Re}(2 + \overline{z} + z^3)| \le 4 \qquad \text{when } |z| \le 1.$$

8. It is shown in Sec. 3 that if $z_1 z_2 = 0$, then at least one of the numbers z_1 and z_2 must be zero. Give an alternative proof based on the corresponding result for real numbers and using identity (8), Sec. 6.

9. By factoring $z^4 - 4z^2 + 3$ into two quadratic factors and using inequality (2), Sec. 5, show that if z lies on the circle $|z| = 2$, then

$$\left|\frac{1}{z^4 - 4z^2 + 3}\right| \le \frac{1}{3}.$$

10. Prove that

(a) z is real if and only if $\overline{z} = z$;

(b) z is either real or pure imaginary if and only if $\overline{z}^2 = z^2$.

11. Use mathematical induction to show that when $n = 2, 3, \ldots,$

(a) $\overline{z_1 + z_2 + \cdots + z_n} = \overline{z_1} + \overline{z_2} + \cdots + \overline{z_n}$;

(b) $\overline{z_1 z_2 \cdots z_n} = \overline{z_1}\,\overline{z_2} \cdots \overline{z_n}$.

12. Let $a_0, a_1, a_2, \ldots, a_n$ $(n \geq 1)$ denote *real* numbers, and let z be any complex number. With the aid of the results in Exercise 11, show that

$$\overline{a_0 + a_1 z + a_2 z^2 + \cdots + a_n z^n} = a_0 + a_1 \overline{z} + a_2 \overline{z}^2 + \cdots + a_n \overline{z}^{\,n}.$$

13. Show that the equation $|z - z_0| = R$ of a circle, centered at z_0 with radius R, can be written

$$|z|^2 - 2\,\mathrm{Re}(z\overline{z_0}) + |z_0|^2 = R^2.$$

14. Using expressions (6), Sec. 6, for $\mathrm{Re}\, z$ and $\mathrm{Im}\, z$, show that the hyperbola $x^2 - y^2 = 1$ can be written

$$z^2 + \overline{z}^2 = 2.$$

15. Follow the steps below to give an algebraic derivation of the triangle inequality (Sec. 5)

$$|z_1 + z_2| \leq |z_1| + |z_2|.$$

(a) Show that

$$|z_1 + z_2|^2 = (z_1 + z_2)(\overline{z_1} + \overline{z_2}) = z_1\overline{z_1} + (z_1\overline{z_2} + \overline{z_1\overline{z_2}}) + z_2\overline{z_2}.$$

(b) Point out why

$$z_1\overline{z_2} + \overline{z_1\overline{z_2}} = 2\,\mathrm{Re}(z_1\overline{z_2}) \leq 2|z_1||z_2|.$$

(c) Use the results in parts (a) and (b) to obtain the inequality

$$|z_1 + z_2|^2 \leq (|z_1| + |z_2|)^2,$$

and note how the triangle inequality follows.

7. EXPONENTIAL FORM

Let r and θ be polar coordinates of the point (x, y) that corresponds to a *nonzero* complex number $z = x + iy$. Since $x = r\cos\theta$ and $y = r\sin\theta$, the number z can be written in ***polar form*** as

$$z = r(\cos\theta + i\sin\theta). \tag{1}$$

If $z = 0$, the coordinate θ is undefined; and so it is understood that $z \neq 0$ whenever polar coordinates are used.

In complex analysis, the real number r is not allowed to be negative and is the length of the radius vector for z ; that is, $r = |z|$. The real number θ represents the angle, measured in radians, that z makes with the positive real axis when z is interpreted as a radius vector (Fig. 6). As in calculus, θ has an infinite number of possible values, including negative ones, that differ by integral multiples of 2π. Those values can be determined from the equation $\tan\theta = y/x$, where the quadrant containing the point corresponding to z must be specified. Each value of θ is called an ***argument*** of z, and the set of all such values is denoted by $\arg z$. The ***principal value*** of $\arg z$, denoted by

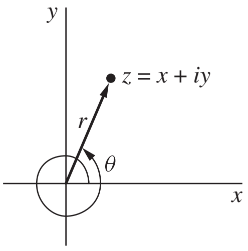

FIGURE 6

Arg z, is the unique value Θ such that $-\pi < \Theta \leq \pi$. Evidently, then,

$$\arg z = \operatorname{Arg} z + 2n\pi \qquad (n = 0, \pm 1, \pm 2, \ldots). \tag{2}$$

Also, when z is a negative real number, Arg z has the value π, not $-\pi$.

EXAMPLE 1. The complex number $-1 - i$, which lies in the third quadrant, has principal argument $-3\pi/4$. That is,

$$\operatorname{Arg}(-1 - i) = -\frac{3\pi}{4}.$$

It must be emphasized that because of the restriction $-\pi < \Theta \leq \pi$ of the principal argument Θ, it is *not* true that $\operatorname{Arg}(-1 - i) = 5\pi/4$.

According to equation (2),

$$\arg(-1 - i) = -\frac{3\pi}{4} + 2n\pi \qquad (n = 0, \pm 1, \pm 2, \ldots).$$

Note that the term Arg z on the right-hand side of equation (2) can be replaced by any particular value of arg z and that one can write, for instance,

$$\arg(-1 - i) = \frac{5\pi}{4} + 2n\pi \qquad (n = 0, \pm 1, \pm 2, \ldots).$$

The symbol $e^{i\theta}$, or $\exp(i\theta)$, is defined by means of ***Euler's formula*** as

$$e^{i\theta} = \cos\theta + i\sin\theta, \tag{3}$$

where θ is to be measured in radians. It enables us to write the polar form (1) more compactly in ***exponential form*** as

$$z = re^{i\theta}. \tag{4}$$

The choice of the symbol $e^{i\theta}$ will be fully motivated later on in Sec. 30. Its use in Sec. 8 will, however, suggest that it is a natural choice.

EXAMPLE 2. The number $-1 - i$ in Example 1 has exponential form

$$-1 - i = \sqrt{2}\exp\left[i\left(-\frac{3\pi}{4}\right)\right]. \tag{5}$$

With the agreement that $e^{-i\theta} = e^{i(-\theta)}$, this can also be written $-1-i = \sqrt{2}\,e^{-i3\pi/4}$. Expression (5) is, of course, only one of an infinite number of possibilities for the exponential form of $-1-i$:

$$(6) \qquad -1-i = \sqrt{2}\exp\left[i\left(-\frac{3\pi}{4} + 2n\pi\right)\right] \qquad (n = 0, \pm 1, \pm 2, \ldots).$$

Note how expression (4) with $r = 1$ tells us that the numbers $e^{i\theta}$ lie on the circle centered at the origin with radius unity, as shown in Fig. 7. Values of $e^{i\theta}$ are, then, immediate from that figure, without reference to Euler's formula. It is, for instance, geometrically obvious that

$$e^{i\pi} = -1, \quad e^{-i\pi/2} = -i, \quad \text{and} \quad e^{-i4\pi} = 1.$$

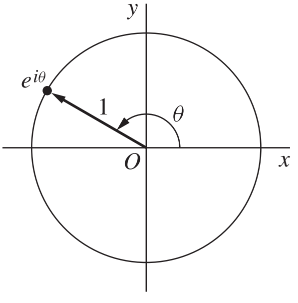

FIGURE 7

Note, too, that the equation

$$(7) \qquad z = Re^{i\theta} \qquad (0 \le \theta \le 2\pi)$$

is a parametric representation of the circle $|z| = R$, centered at the origin with radius R. As the parameter θ increases from $\theta = 0$ to $\theta = 2\pi$, the point z starts from the positive real axis and traverses the circle once in the counterclockwise direction. More generally, the circle $|z - z_0| = R$, whose center is z_0 and whose radius is R, has the parametric representation

$$(8) \qquad z = z_0 + Re^{i\theta} \qquad (0 \le \theta \le 2\pi).$$

This can be seen vectorially (Fig. 8) by noting that a point z traversing the circle $|z - z_0| = R$ once in the counterclockwise direction corresponds to the sum of the fixed vector z_0 and a vector of length R whose angle of inclination θ varies from $\theta = 0$ to $\theta = 2\pi$.

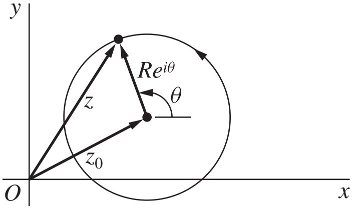

FIGURE 8

8. PRODUCTS AND POWERS IN EXPONENTIAL FORM

Simple trigonometry tells us that $e^{i\theta}$ has the familiar additive property of the exponential function in calculus:

$$\begin{aligned} e^{i\theta_1}e^{i\theta_2} &= (\cos\theta_1 + i\sin\theta_1)(\cos\theta_2 + i\sin\theta_2) \\ &= (\cos\theta_1\cos\theta_2 - \sin\theta_1\sin\theta_2) + i(\sin\theta_1\cos\theta_2 + \cos\theta_1\sin\theta_2) \\ &= \cos(\theta_1+\theta_2) + i\sin(\theta_1+\theta_2) = e^{i(\theta_1+\theta_2)}. \end{aligned}$$

Thus, if $z_1 = r_1e^{i\theta_1}$ and $z_2 = r_2e^{i\theta_2}$, the product z_1z_2 has the exponential form

$$z_1z_2 = r_1e^{i\theta_1}r_2e^{i\theta_2} = r_1r_2e^{i\theta_1}e^{i\theta_2} = (r_1r_2)e^{i(\theta_1+\theta_2)}. \tag{1}$$

Furthermore,

$$\frac{z_1}{z_2} = \frac{r_1e^{i\theta_1}}{r_2e^{i\theta_2}} = \frac{r_1}{r_2}\cdot\frac{e^{i\theta_1}e^{-i\theta_2}}{e^{i\theta_2}e^{-i\theta_2}} = \frac{r_1}{r_2}\cdot\frac{e^{i(\theta_1-\theta_2)}}{e^{i0}} = \frac{r_1}{r_2}e^{i(\theta_1-\theta_2)}. \tag{2}$$

Note how it follows from expression (2) that the inverse of any nonzero complex number $z = re^{i\theta}$ is

$$z^{-1} = \frac{1}{z} = \frac{1e^{i0}}{re^{i\theta}} = \frac{1}{r}e^{i(0-\theta)} = \frac{1}{r}e^{-i\theta}. \tag{3}$$

Expressions (1), (2), and (3) are, of course, easily remembered by applying the usual algebraic rules for real numbers and e^x.

Another important result that can be obtained formally by applying rules for real numbers to $z = re^{i\theta}$ is

$$z^n = r^ne^{in\theta} \qquad (n = 0, \pm1, \pm2, \ldots). \tag{4}$$

It is easily verified for positive values of n by mathematical induction. To be specific, we first note that it becomes $z = re^{i\theta}$ when $n = 1$. Next, we assume that it is valid when $n = m$, where m is any positive integer. In view of expression (1) for the product of two nonzero complex numbers in exponential form, it is then valid for $n = m + 1$:

$$z^{m+1} = z^mz = r^me^{im\theta}re^{i\theta} = (r^mr)e^{i(m\theta+\theta)} = r^{m+1}e^{i(m+1)\theta}.$$

Expression (4) is thus verified when n is a positive integer. It also holds when $n = 0$, with the convention that $z^0 = 1$. If $n = -1, -2, \ldots$, on the other hand, we define z^n in terms of the multiplicative inverse of z by writing

$$z^n = (z^{-1})^m \quad \text{where} \quad m = -n = 1, 2, \ldots .$$

Then, since equation (4) is valid for positive integers, it follows from the exponential form (3) of z^{-1} that

$$z^n = \left[\frac{1}{r}e^{i(-\theta)}\right]^m = \left(\frac{1}{r}\right)^m e^{im(-\theta)} = \left(\frac{1}{r}\right)^{-n} e^{i(-n)(-\theta)} = r^ne^{in\theta} \qquad (n = -1, -2, \ldots).$$

Expression (4) is now established for all integral powers.

Expression (4) can be useful in finding powers of complex numbers even when they are given in rectangular form and the result is desired in that form.

EXAMPLE 1. In order to put $(-1+i)^7$ in rectangular form, write

$$(-1+i)^7 = (\sqrt{2}\, e^{i3\pi/4})^7 = 2^{7/2} e^{i\,21\pi/4} = (2^3 e^{i5\pi})(2^{1/2} e^{i\,\pi/4}).$$

Because

$$2^3 e^{i5\pi} = (8)(-1) = -\,8$$

and

$$2^{1/2} e^{i\,\pi/4} = \sqrt{2}\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right) = \sqrt{2}\left(\frac{1}{\sqrt{2}} + \frac{i}{\sqrt{2}}\right) = 1 + i,$$

we arrive at the desired result: $(-1+i)^7 = -\,8\,(1+i)$.

Finally, we observe that if $r = 1$, equation (4) becomes

$$(e^{i\theta})^n = e^{in\theta} \quad (n = 0, \pm 1, \pm 2, \ldots). \tag{5}$$

When written in the form

$$(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta \quad (n = 0, \pm 1, \pm 2, \ldots), \tag{6}$$

this is known as ***de Moivre's formula***. The following example uses a special case of it.

EXAMPLE 2. Formula (6) with $n = 2$ tells us that

$$(\cos\theta + i\sin\theta)^2 = \cos 2\theta + i\sin 2\theta,$$

or

$$\cos^2\theta - \sin^2\theta + i2\sin\theta\cos\theta = \cos 2\theta + i\sin 2\theta.$$

By equating real parts and then imaginary parts here, we have the familiar trigonometric identities

$$\cos 2\theta = \cos^2\theta - \sin^2\theta, \qquad \sin 2\theta = 2\sin\theta\cos\theta.$$

(See also Exercises 10 and 11, Sec. 9.)

9. ARGUMENTS OF PRODUCTS AND QUOTIENTS

If $z_1 = r_1 e^{i\theta_1}$ and $z_2 = r_2 e^{i\theta_2}$, the expression

$$z_1 z_2 = (r_1 r_2) e^{i(\theta_1+\theta_2)} \tag{1}$$

in Sec. 8 can be used to obtain an important identity involving arguments:

$$\arg(z_1 z_2) = \arg z_1 + \arg z_2. \tag{2}$$