⇒ Topic:-

⇒ "Hessian Determinant":-

⇒ Definition:-

A Hessian $|H|$ is a determinent composed of all second order partial derivatives, with 2nd order direct partial on principal diagonal cross partial the other diagonal.

⇒ Mathematical form:-

$$|H| = \begin{vmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{vmatrix}$$

or

$$|H| = \begin{vmatrix} \frac{\partial^2 f}{\partial x^2} & \frac{\partial^2 f}{\partial x_y} \\ \frac{\partial^2 f}{\partial x_y} & \frac{\partial^2 f}{\partial y^2} \end{vmatrix}$$

⇒ Mirrors of Hessian determinant:-

There are two miners of hessian determinant.

i): **1st Principal Minor:-**

The first element of Hession determinant is principal minor.

$\Rightarrow$ It is represented by $|H_1|$.

$\Rightarrow$ So, $|H_1| = f_{xx}$

ii): **2nd Principal Minor:-**

The first 2x2 determinant is called the 2nd principal minor.

$\Rightarrow$ It is denoted by $|H_2|$.

$\Rightarrow$ So, $|H_2| = \begin{vmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{vmatrix}$

$\Rightarrow$ **Positive definate:-**

If $|H_1| > 0$ and $|H_2| > 0$; then Hessian determinant is positive definate.

$\therefore$ $f_{xx} > 0$ and $f_{xx} f_{yy} - f_{xy} f_{yx} > 0$

then determinant is +ve definate.

$\Rightarrow$ **Negative definate:-**

If first principal minor $|H_1| < 0$ and 2nd principal minor $|H_2| < 0$ then determinant is called negative definate.

⇒ If one of the condition $|H_1| < 0$ or $|H_2| < 0$, anyone of these is zero, then it will also negative definate.

⇒ Significance or use of Hessian in economics:-

In economics, Hessian determinant is used for the optimization of function.

For optimization, we need two steps,

⇒ Find first order condition i.e FOC.

⇒ then, find second order condition

Hessian is used to find 2nd order condition i.e SOC

There are two conditions are given to satisfy SOC.

i): If $Z_{xx} Z_{yy} > 0$ and

$$Z_{xx} Z_{yy} > (Z_{yy})^2$$

Then, function is minimized function.

ii). If $Z_{xx}\ Z_{yy} < 0$ and

$Z_{xx}\ Z_{yy} > (Z_{yy})^2$

then, function is maximized function.

⇒ Higher order Hessian:-

If $y = f(x_1, x_2, x_3)$ then Hessian become 3rd order which is

$$|H| = \begin{vmatrix} y_{11} & y_{12} & y_{13} \\ y_{21} & y_{22} & y_{23} \\ y_{31} & y_{32} & y_{33} \end{vmatrix}$$

Hence, $|H_1| = Y_{11}$

And

$$|H_2| = \begin{vmatrix} y_{11} & y_{12} \\ y_{21} & y_{22} \end{vmatrix}$$

and Now

$$|H_3| = |H|$$

Now, if $|H_1| > 0$, $|H_2| > 0$ and $|H_3| > 0$ then Hessian is positive definate.

And if any one is negative i.e $|H_1| < 0$ or $|H_2| < 0$ or

$|H_3| < 0$ or all are less than zero then Hessian is negative definate.

⇒ For optimization of function, If all three minors are greater than zero than function is minimum. and if any of them is less than zero then function is maximum.

⇒ **Question:-**

The function $A = 5x^2 - 8x - 2xy - 6y + 4y^2$ use Hessian determinant to optmix this function.

⇒ **Solution:-**

$$A = 5x^2 - 8x - 2xy - 6y + 4y^2 \Rightarrow ①$$

Taking derivative of eq ① with respect to "x".

$$\Rightarrow A_x = 10x - 8 - 2y \rightarrow ②$$

then, taking derivative of eq ① with respect to "y".

$$\Rightarrow A_y = 2x - 6 + 8y \Rightarrow ③$$

Now,

Set the 1st order partial derivative equal to zero.

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$A_x = 0$

then, equation ② becomes,

$10x - 8 - 2y = 0 \Rightarrow$ ④

When,

$A_y = 0$

then, equation ③ becomes,

$2x - 6 + 8y = 0 \Rightarrow$ ⑤

Multiply eq ⑤ by ⑤, we get

$10x - 30 + 40y = 0$

then,

$$\begin{aligned} \cancel{10x} - 30 + 40y &= 0 \\ \cancel{10x} - 8 - 2y &= 0 \\ - \quad + \quad + \\ \hline -22 + 42y &= 0 \end{aligned}$$

$42y = 22$

$\boxed{y = \frac{11}{21}}$

Putting the value of y in equation ④, we get

$10x - 8 - 2\left(\frac{11}{21}\right) = 0$

$10x - 8 - \frac{22}{21} = 0$

$10x = \frac{168 + 22}{21}$

$x = \frac{190}{21 \times 10}$

$\boxed{x = \frac{19}{21}}$

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So, Critical point is $\left(\frac{19}{21}, \frac{11}{21}\right)$

Then, Again derivative of eq ②
with respect to "x".

$$A_{xx} = \frac{d}{dx}(10x - 8 - 2y)$$

$$\boxed{A_{xx} = 10}$$

Then, Taking derivative of eq ③
with respect to "y".

$$A_{xy} = \frac{d}{dy}(10x - 8 - 2y)$$

$$\boxed{A_{xy} = -2}$$

Then, Taking derivative of eq ③
with respect to "y".

$$A_{yy} = \frac{d}{dy}(2x - 6 + 8y)$$

$$\boxed{A_{yy} = 8}$$

Then, taking derivative of eq ③
with respect to "x".

$$A_{yx} = \frac{d}{dx}(2x - 6 + 8y)$$

$$\boxed{A_{yx} = 2}$$

Using Hessian determinant:-

$$|H| = \begin{vmatrix} A_{xx} & A_{xy} \\ A_{yx} & A_{yy} \end{vmatrix}$$

$$|H| = \begin{vmatrix} 10 & -2 \\ 2 & 8 \end{vmatrix}$$

$$|H| = 84 > 0$$

$$|H_1| = 10 > 0$$

$$|H| = |H_2| = 84 > 0$$

Since, H_1 and H_2 both are positive definite. So, Hessian determinant is positive definite.

So, A is minimum at critical values. For finding minimum value of function.

Put $x = \frac{19}{21}$ $y = \frac{11}{21}$ in original function.

$$A = 5x^2 - 8x - 2xy - 6y + 4y^2$$

$$A = 5\left(\frac{19}{21}\right)^2 - 8\left(\frac{19}{21}\right) - 2\left(\frac{19}{21}\right)\left(\frac{11}{21}\right) - 6\left(\frac{11}{21}\right) + 4\left(\frac{11}{21}\right)^2$$

$$A = \frac{1805}{441} - \frac{152}{21} - \frac{418}{441} - \frac{66}{21} + \frac{484}{441}$$

$$A = \frac{1805 - 3192 - 418 - 1386 + 484}{441}$$

$$\boxed{A = -\frac{2707}{441}}$$