CONSTRAINED OPTIMIZATION

Introduction: **(BZU: 2013) (GCUF: 2015)**

In chapter 7 we explained the concepts of maxima and minima with the help of Ist order and 2nd order conditions. But the mathematicians also present the situation of maximum and minimum subject to some constraint. Moreover, in such like functions, there exist two independent variables like x and y. Such functions are in the form $z = f(x, y)$ and $g = f(x, y)$ etc. Therefore, to explain constrained optimization we have to get assistance from Partial derivatives etc. Accordingly, to explain constrained optimization we use two methods: (1) Mathematical method I, (2) Lagrangian multiplier method. We discuss them in turn.

1. Mathematical Method I——— Substitution Method

It is explained as: If we have a basic function $f(x, y)$ and constraint function $g(x, y)$ we will keep the constraint function equal to zero to find the equation for x or y. Then a new function will be formed by putting the equation for x or y in basic function. Then we will differentiate it and keep it equal to zero. This is the first order condition for optimum. As a result, we will get the value of y or x. Then such obtained values of x and y will be inducted in the constraint function to satisfy it. Finally, we take second derivative, if it is positive the function will be minimum according to sufficient condition, if it is negative, the function will be maximum according to sufficient condition. They are explained with examples:

Example 1. Basic function: $f(x, y) = 5x^2 + 6y^2 - xy$

Constrained function: $g(x, y) = x + 2y - 24$

we find the values of x and y where the constrained function is maximized or minimized.

Finding value of x with constraint function: $x + 2y = 24 \Rightarrow x = 24 - 2y$.

Putting the value of x in basic function and calling it G (y) $\Rightarrow f(x, y) = 5x^2 + 6y^2 - xy$

$G(y) = 5(24 - 2y)^2 + 6y^2 - (24 - 2y)y \Rightarrow G(y) = 2880 - 480y + 20y^2 + 6y^2 - 24y + 2y^2$

$\Rightarrow 2880 - 504y + 28y^2 \Rightarrow G(y) = 28y^2 - 504y + 880$

Keeping Ist derivative equal to zero

$$\frac{dG}{dy} = 56y - 504 = 0 \Rightarrow y = 9 \text{ and } \frac{d^2G}{dy^2} = 56 > 0$$

Putting value of $y = 9$ in x, $x = 24 - 2(9) = 6$. Then at $x = 6$ and $y = 9$ the 2nd derivative is +ve. Thus the constrained function is minimized. Moreover, the constrained function is also satisfied: $x + 2y = 24 \Rightarrow 6 + 2(9) = 24$.

The value of objective function is: $G(y) = 2880 - 504(9) + 28(9)^2 = 612$

Example 2: Basic function: $f(x, y) = 12xy - 3y^2 - x^2$. Constraint function: $g(x, y) = x + y = 16$

We find the values of x and y where the function is maximized or minimized.

Finding the value of x from constraint function and putting it in basic function. We call it G (y)

$x + y = 16 \Rightarrow x = 16 - y$

$G(y) = 12y(16 - y) - 3y^2 - (16 - y)^2 \Rightarrow G(y) = 192y - 12y^2 - 3y^2 - 256 + 32y - y^2$

$Gy = -16y^2 + 224y - 56$

$$\frac{dG}{dy} = -32y + 224 = 0 \Rightarrow \quad -32y = -224 \Rightarrow y = 7$$

Putting y = 7 in x, $\quad x = 16 - 7 = 9$

$$\frac{dG}{dy} = -32y + 224 \quad , \quad \frac{d^2G}{dy^2} = -32 < 0$$

Thus at x = 9 and y = 7, the 2nd derivative is negative. Hence the constraint function is maximized. Moreover the constraint function is satisfied i.e. $x + y = 16 \Rightarrow 9 + 7 = 16$. While the value of objective function is as :

$$Gy = -16y^2 + 224y - 56 \Rightarrow Gy = -16(7)^2 + 224(7) - 56 = 728$$

Mathematical Method II – —— Lagrange Multiplier Method **(UOP: 2011-A)**

It is explained as: If the function is Z = f (x, y) or f (x, y), while the constraint function is g (x, y). The constraint function is kept equal to zero and multiplied with Lagrangian Multiplier (λ). This new function is added (or subtracted) to the basic function to yield us following objective function to be optimized: F (x, y, λ) = f (x, y) ± λg (x, y). This function is known as 'Lagrangian Function'. Afterwards it will be partially differentiated w.r.t. x, y and λ and kept equal to zero. as : Fx = Fy = Fλ = 0. This is called necessary condition of optimization. The so obtained partial derivatives are solved with simultaneous equations method to get the value of x, y and λ.

By putting such values of x and y, the constraint function is satisfied. While putting the values of x, y and λ the value of Lagrange function is obtained.

But the mathematicians present the 'Sufficient or 2nd order condition' regarding Maxima as:

2fxy (gx) (gy) – fxx (gy)² – fyy (gx)² > 0

Whereas the sufficient or 2nd order condition regarding Minima is as:

2fxy (gx) (gy) – fxx (gy)² – fyy (gx)² < 0

Where fxy = the 2nd partial derivative of x, w.r.t. y.
fxx – the 2nd partial derivative of x, w.r.t. x.
fyy = the 2nd partial derivative of y, w.r.t. y. fyx – the 2nd partial derivative of y, w.r.t. x.
gx – the 1st partial derivative of g, w.r.t. x. gy – the 1st partial derivative of g, w.r.t. y.

This concept will be discussed in coming pages.

The Proof of above conditions can be obtained from the mathematical concept of 'Bordered Hessian Determinant'. Its standard form is as:

$$|\overline{H}| = \begin{vmatrix} 0 & gx & gy \\ gx & fxx & fxy \\ gy & fyx & fyy \end{vmatrix} = 0[(f_{xx})(f_{yy}) - (f_{xy})(f_{yx})] - (g_x)[(g_x)(f_{yy}) - (g_y)(f_{xy})]$$

$$+(g_y)[(g_x)(f_{yx}) - (g_y)(f_{xx})]$$

$$= 0 - g_x[(g_x)f_{yy} - (g_y)f_{xy}] + g_y[(g_x)f_{yx} - (g_y)f_{xx}]$$

$$= -(g_x)^2 f_{yy} + (g_x)(g_y) f_{xy} + (g_x)(g_y) f_{yx} - (g_y)^2 f_{xx} = 2 g_x g_y f_{xy} - (g_y)^2 f_{xx} - (g_x)^2 f_{yy}$$

whereas summing $g_x g_y f_{xy}$ and $g_y g_x f_{yx}$ we get $2g_x g_y f_{xy}$.

Now we explain such all by using the examples:

Example 1: By using the necessary and sufficient condition find the values of x. y and λ where the constrained function is maximized or minimized: (UOPR: 2007, 2008) (UOH: 2006)

Basic function: $f(x, y) = 5x^2 - 6y^2 - xy$, Constraint function: $x + 2y = 24$

Keeping constraint function equal to zero and multiplying it with λ. Adding it in basic function, taking partial derivatives w.r.t. x, y and λ and keeping them equal to zero.

$g(x, y) = x + 2y - 24 = 0, \quad \lambda(x + 2y - 24) = 0, \ \lambda x + \lambda 2y - \lambda 24 = 0$

$F(x, y, \lambda) = 5x^2 + 6y^2 - xy + (\lambda x + \lambda 2y - \lambda 24)$

$F(x, y, \lambda) = 5x^2 + 6y^2 - xy + \lambda x + \lambda 2y - \lambda 24$

$\frac{\partial F}{\partial x} = f_x = 10x - y + \lambda = 0$ ...... (1)

$\frac{\partial F}{\partial y} = f_y = 12y - x + 2\lambda = 0$ ...... (2)

$\frac{\partial F}{\partial \lambda} = f_\lambda = x + 2y - 24 = 0$ ...... (3)

> **Solve as Example 1:**
> $Z = f(x, y) = 10x^2 + 12y^2 + 4$
> subject to constraint
> $2x + 4y = 48$, Whether it is maxima or minima.
> (UOH: 2008)

Finding values of x, y and λ. Putting values of x and y in constraint. Putting values of x, y and λ in objective function. $x = 24 - 2y$. $10(24 - 2y) - y + \lambda = 0 \Rightarrow 240 - 20y - y + \lambda = 0$
$240 - 21y + \lambda = 0 \Rightarrow \lambda = -240 + 21y$

$12y - x + 2\lambda = 0 \quad 12y - (24 - 2y) + 2\lambda = 0 \Rightarrow 12y - 24 + 2y + 2\lambda = 0$

$14y - 24 + 2\lambda = 0 \Rightarrow 2\lambda = 24 - 14y \Rightarrow \lambda = 12 - 7y$

$-240 + 21y = 12 - 7y \Rightarrow 28y = 252 \Rightarrow y = 9$

$x = 24 - 2(9) = 6, \lambda = 12 - 7(9) = 12 - 63 = -51 \quad x + 2y = 24 \Rightarrow 6 + 2(9) = 24 \Rightarrow 24 = 24$

$F(x,y,\lambda) = 5x^2 + 6y^2 - xy + \lambda x + \lambda 2y - \lambda 24 = 5(6)^2 + 6(9)^2 - (6)(9) - 51(6) - 51(2)(9) - 51(24) = -1836$

Now we check 2nd order conditions. For this purpose we take second partial derivatives of the objective function and first partial derivatives of constraint function.

$\frac{\partial F}{\partial x} = f_x = 10x - y + \lambda$

$\frac{\partial^2 F}{\partial x^2} = f_{xx} = 10$

$g(x, y) = x + 2y - 24 = 0$

$g_x = \frac{\partial g}{\partial x} = 1, \; g_y = \frac{\partial g}{\partial y} = 2$

$\frac{\partial F}{\partial y} = f_y = 12y - x + 2\lambda$

$\frac{\partial^2 F}{\partial y^2} = f_{yy} = 12$

$\frac{\partial}{\partial y}\left(\frac{\partial F}{\partial x}\right) = \frac{\partial^2 F}{\partial y \partial x} = f_{xy} = -1$

Putting these values in the above mentioned condition: $2f_{xy}(g_x)(g_y) - f_{xx}(g_y)^2 - f_{yy}(g_x)^2$

$= 2(-1)(1)(2) - (10)(2)^2 - (12)(1)^2 = -4 - 40 - 12 \Rightarrow -56 < 0$

Thus we find that at $x = 6$, $y = 9$ and $\lambda = -51$, both the necessary and sufficient condition of constrained minima are met, as $-56 < 0$. Thus, there is minima here.

Example 2: We find the values of x, y and λ where constraint function is maximized or minimized. Basic function: $f(x, y) = 12xy - 3y^2 - x^2$. Constraint function: $g(x, y) = x + y = 16$. Keeping $g(x, y) = 0$, multiplying with λ and adding in basic function. Differentiating partially w.r.t. x, y and λ and keeping them equal to zero.

Finding the values of x, y and λ. $g(x, y) = x + y - 16 = 0$

$\lambda(x + y - 16) = 0 \Rightarrow \lambda x + \lambda y - \lambda 16 = 0$

$F(x, y, \lambda) = 12xy - 3y^2 - x^2 + (\lambda x + \lambda y - \lambda 16)$

$F(x, y, \lambda) = 12xy - 3y^2 - x^2 + \lambda x + \lambda y - \lambda 16$

$f_x = \frac{\partial F}{\partial x} = 12y - 2x + \lambda = 0$ ...... (1)

$f_y = \frac{\partial F}{\partial y} = 12x - 6y + \lambda = 0$ ...... (2)

$f_\lambda = \frac{\partial F}{\partial \lambda} = x + y - 16 = 0$ ...... (3) $\boxed{x = 16 - y}$

> Solve as Example 2: If $f(x, y) = x^3 - y^2 + xy + 5x$ find the extreme values of the function subject to $x - 2y = 0$.
> (UOH: 2011)

$12y - x + 2\lambda = 0 \Rightarrow 12y - 2(16 - y) + \lambda = 0 \Rightarrow 12y - 32 + 2y + \lambda = 0$

$14y - 32 + \lambda = 0 \Rightarrow -14y + 32 = \lambda$

$12x - 6y + \lambda = 0 \Rightarrow 12(16 - y) - 6y + \lambda = 0$

$192 - 12y - 6y + \lambda = 0 \Rightarrow 192 - 18y + \lambda = 0 \Rightarrow \lambda = -192 + 18y$

$$-14y - 32 = 18y - 192 \quad \Rightarrow -14y - 18y = -192 - 32$$

$$-32y = -224 \quad \Rightarrow y = 7. \quad x = 16 - y = 16 - 7 = 9$$

$\lambda = -14y + 32 = -14(7) + 32 = -66$ and $\lambda = 18y - 192 = 18(7) - 192 = -66$

Putting values of x and y in constraint function. Putting the values of x, y and λ in objective function. $x + y = 16 \Rightarrow 9 + 7 = 16$

$F(x, y, \lambda) = 12xy - 3y^2 - x^2 + \lambda x + \lambda y - 16 = 12(9)(7) - 3(7)^2 - (9)^2 - 66(9) - 66(7) - 16 = -544$

Thus at $x = 9$, $y = 7$ and $\lambda = -66$, the first order condition of optimization i.e. $f_x = f_y = f_\lambda = 0$ are met. Now we calculate second order condition to confirm maxima or minima at such values. Therefore, we take 2nd partial derivatives of objective function and Ist partial derivatives of constraint function.

$$\frac{\partial F}{\partial x} = f_x = 12y - 2x + \lambda$$

$$\frac{\partial^2 F}{\partial x^2} = f_{xx} = -2$$

$$g(x, y) = x + y - 16 = 0$$

$$g_x = \frac{\partial g}{\partial x} = 1, \quad g_y = \frac{\partial g}{\partial y} = 1$$

$$\frac{\partial f}{\partial y} = 12x - 6y + \lambda$$

$$\frac{\partial^2 F}{\partial y^2} = f_{yy} = -6$$

$$\frac{\partial}{\partial y}\left(\frac{\partial F}{\partial x}\right) = \frac{\partial^2 F}{\partial y \partial x} = f_{xy} = 12$$

Putting the values of 2nd partials of objective function and Ist partials of constraint function in the sufficient condition equation. $2f_{xy}(g_x)(g_y) - f_{xx}(g_y)^2 - f_{yy}(g_x)^2$

$$= 2(12)(1)(1) - (-2)(1)^2 - (-6)(1)^2 = 24 - (-2) - (-6) \quad \Rightarrow 32 > 0$$

Thus we find that following second order condition the answer is greater than zero i.e., $32 > 0$. It means that $x = 9$ and $y = 7$, it is constrained maxima in the light of sufficient conditions.

ECONOMIC APPLICATION OF CONSTRAINED OPTIMIZATION (BZU: 2014)

We have to face so many cases in Economics where Maximization or Minimization is attached with some constraint. As, (1) a consumer wants to maximize his satisfaction subject to his income constraint (2) a producer wants to minimize his cost of production subject to his output constraint (3) a firm wants to maximize her output subject to cost constraint, (4) a firm wants to maximize the produce of two goods, subject to constraint. Such all will be explained by using Substitution Method and Lagrangian Multiplier Method.

minimized in the light of necessary and sufficient conditions.

4. Minimization of Joint Cost, Subject to the Condition that Firm is Producing Two Goods

Lagrangian Multiplier method is followed to solve the example.

Cost function $C = 5x^2 - 2xy - 3y^2 + 800$. Quota constraint: $x + y = 39 \Rightarrow 39 - x - y = 0$

Multiplying the quota constraint with Lagrangian multiplier and adding it in cost function

$\lambda(39 - x - y) = 0$ and $C = 5x^2 + 2xy + 3y^2 + 800 + \lambda(39 - x - y)$

Taking Ist partial derivative of x, y and λ.

$$\frac{\partial C}{\partial x} = C_x = 10x + 2y - \lambda = 0 \quad \ldots\ldots (1)$$

$$\frac{\partial C}{\partial y} = C_y = 2x + 6y - \lambda = 0 \quad \ldots\ldots (2)$$

$$\frac{\partial C}{\partial \lambda} = C_\lambda = 39 - x - y = 0 \quad \ldots\ldots (3)$$

Solving (1) for λ: $\lambda = 10x + 2y$. Putting value of λ in (2), $2x - 6y - (10x + 2y) = 0$

$2x + 6y - 10x - 2y = 0 \Rightarrow -8x + 4y = 0$ ...... (4), Multiplying (3) by 8,

$$312 - 8x - 8y = 0$$
$$-8x + 4y = 0$$
$$+ \quad -$$

$312 - 12y = 0 \Rightarrow 312 = 12y \Rightarrow y = 26$

Putting the value of $y = 26$ in quota constraint: $39 - x - 26 = 0 \Rightarrow 39 - 26 = x \Rightarrow x = 13$

Putting in the value of x and λ finding the value of putting it in equation of x.

$10x + 2y - \lambda = 0 \Rightarrow \lambda = 10x + 2y = 10(13) + 2(26) = 182$

Taking partial 2nd derivative of cost function and Ist partial derivatives of production function

$C_x = 10x + 2y - \lambda$,	$C_{xx} = \frac{\partial^2 C}{\partial x^2} = 10$
$C_y = 2x + 6y - \lambda$,	$C_{yy} = \frac{\partial^2 C}{\partial y^2} = 6$
$C_{xy} = 2$,	$C_{yx} = 2$

$$g(x,y) = 39 - x - y = 0 = \frac{\partial g}{\partial x} = gx = -1, \quad \frac{\partial g}{\partial y} = gy = -1$$

According to sufficient condition of cost minimization: $2C_{xy}(g_x)(g_y) - C_{xx}(g_y)^2 - C_{yy}(g_x)^2 < 0$

$= 2(2)(-1)(-1) - (10)(-1)^2 - (6)(-1)^2 = -12 < 0$

Thus it proved that if firm produces $y = 26$ and $x = 13$, its production quota is met, as $x + y = 39$ $\Rightarrow 13 + 26 = 39$. Producing such quantities of x and y the joint costs are minimized in the light of necessary and sufficient conditions.

5. Maximization of Profits Subject to the Condition that Firm is Producing Two Goods

Example 1: The profit function and maximum output capacity of a firm are given, how many quantities of x and y the firm should produce that its profits be maximized.

The given profit function $\pi = 80x - 2x^2 - xy - 3y^2 + 100y$

The production efficiency constraint: $x + y = 12$

The same above example is solved with Lagrangian Multiplier method.

The given profit function of firm : $\pi = 80x - 2x^2 - xy - 3y^2 + 100y$

Production constraint of two goods : $x + y = 12 \Rightarrow 12 - x - y = 0$

Forming the Lagrangian function: $\pi = 80x - 2x^2 - xy - 3y^2 + 100y + \lambda(12 - x - y)$

Taking partial derivatives with respect to x, y and λ and keeping equal to zero.

$$\frac{\partial \pi}{\partial x} = \pi_x = 80 - 4x - y - \lambda = 0 \quad \text{... (1)}$$

$$\frac{\partial \pi}{\partial y} = \pi_y = -x - 6y + 100 - \lambda = 0 \quad \text{... (2)}$$

$$\frac{\partial \pi}{\partial \lambda} = \pi_\lambda = 12 - x - y = 0 \quad \text{... (3)}$$

Solving (1) for λ and putting value of λ in (2) : $\lambda = 80 - 4x - y$

$-x - 6y + 100 - (80 - 4x - y) = 0$

$-x - 6y + 100 - 80 + 4x + y = 0 \Rightarrow -5y + 3x + 20 = 0$ ... (4)

$3x - 5y = -20$ .... (4) and $x + y = 12$ .... (3)

Solving with simultaneous equations. For this, Eq.(3) is multiplied by 3.

$$\begin{array}{rcrcr} 3x & - & 5y & = & -20 \\ 3x & + & 3y & = & 36 \\ - & & - & & - \\ \hline & & -8y & = & -56 \end{array}$$

$-8y = -56 \Rightarrow y = 7$, then $x = 12 - y = 12 - 7 = 5$

Putting values of x and y in 'λ' $\Rightarrow \lambda = 80 - 4x - y = 80 - 4(5) - 7 = 53$

Putting values of x and y in π function.

$\pi = 80(5) - 2(5)^2 - (5)(7) - 3(7)^2 + 100(7) = 868$

Putting x and y in constraint function : $x + y = 12 \Rightarrow 5 + 7 = 12$

The second order conditions are checked:

$\pi_x = 80 - 4x - y - \lambda$, $\pi_{xx} = -4$, $\pi_{xy} = -1$, $\pi_y = -x - 6y + 100 - \lambda$, $\pi_{yy} = -6$, $\pi_{yx} = -1$

$g_x = \frac{\partial}{\partial x}(12 - x - y) = -1$, $\quad g_y = \frac{\partial}{\partial y}(12 - x - y) = -1$

Thus according to sufficient profit maximization condition: $2\pi_{xy}(g_x)(g_y) - \pi_{xx}(g_y)^2 - \pi_{yy}(g_x)^2 > 0$

$$= 2(-1)(-1)(-1) - (-4)(-1)^2 - (-6)(-1)^2 > 0 \Rightarrow 8 > 0$$

Thus we find that if firm produces 7 units of y and 5 units of x its productive capacity is met. Moreover, firm's profits are maximized on the ground of necessary and sufficient conditions.