Poisson Distribution

It is named after a french mathematician sime'on Denis Poisson (1781 – 1840) who published its derivation 1837.

Definition:-

It is a probability density function that is often used as a mathematical model of the number of outcomes obtained in a suitable interval of time. Its mean is equal to its varience.

- $\boxed{n}$ is very large $n \geq 20$ • $\boxed{P}$ is very Small $P < 0.05$

Function:-

μ | mean, average $\boxed{\mu = np}$

$$f(x) = \frac{e^{-\mu}\mu^x}{x!} \qquad x = 0, 1, 2, 3 \ldots \infty \qquad e = 2.71828$$

Parameter:- $\boxed{n}$ total number $\boxed{P}$ Probability

'μ' is the parameter of poisson distribution.

Logitimate Property:-

$$\sum_{x=0}^{\infty} f(x) = 1$$

L.H.S

$$\sum_{x=0}^{\infty} f(x) = \sum_{x=0}^{\infty} \frac{e^{-\mu}\mu^x}{x!}$$

Date:

$e^{\theta} = 1 + \theta + \frac{\theta}{2!} + \frac{\theta}{3!} - - - \frac{\theta^{\theta}}{\theta!} + -$

$$= e^{-\mu} \sum_{x=0}^{\infty} \frac{\mu^x}{x!} = e^{-\mu}\left[1 + \mu + \frac{\mu^2}{2!} + \frac{\mu^3}{3!} + \cdots\cdots\right]$$

$$= e^{-\mu} e^{\mu}$$

$$= e^{-\mu+\mu}$$

$$= e^{0} \qquad e^0 = 1$$

$$\sum_{x=0}^{\infty} f(x) = 1$$

$\boxed{\text{Proved}}$

(*). **MEAN**

$$E(x) = \mu$$

$E(x) = \sum_{x=0}^{\infty} x f$

Derivation of Mean:-

$$E(X) = \sum_{x=0}^{\infty} x f(x)$$

$$= \sum_{x=0}^{\infty} x \cdot \frac{e^{-\mu}\mu^x}{x!}$$

$$= \sum_{x=0}^{\infty} \cancel{x} \cdot \frac{e^{-\mu}\mu^{x-1+1}}{\cancel{x}(x-1)!}$$

$$= \sum_{x=0}^{\infty} \frac{e^{-\mu}\mu^{x-1}\mu^{1}}{(x-1)!}$$

$$= \mu \cdot e^{-\mu} \sum_{x=0}^{\infty} \frac{\mu^{x-1}}{(x-1)!}$$

$\therefore$ Since the first term in Summation being zero is omitted.

Date:

$$= \mu e^{-\mu} \sum_{x=1}^{\infty} \frac{\mu^{x-1}}{(x-1)!} \qquad \therefore e^{\mu} = \frac{\mu^{x-1}}{(x-1)!}$$

Using Exponential Series:-

$$= \mu e^{-\mu} e^{\mu}$$

$\boxed{E(X) = \mu}$ Proved.

(*) VARIANCE

$$Var(x) = \mu$$

Derivation of Variance:-

$$Var(X) = E(X^2) - [\underbrace{E(X)}_{\mu}]^2 \quad \text{—— (1)}$$

Now;

$$E(X^2) = E(x^2 - x + x)$$

$$= E(x(x-1) + x)$$

$$= E(x) + E(x(x-1))$$

$$= \mu + \sum_{x=0}^{\infty} x(x-1) \frac{e^{-\mu} \mu^x}{x!}$$

$$= \mu + e^{-\mu} \sum_{x=0}^{\infty} \frac{\cancel{x(x-1)} \mu^x}{\cancel{x(x-1)}(x-2)!}$$

$$= \mu + e^{-\mu} \sum_{x=0}^{\infty} \frac{\mu^{x-2+2}}{(x-2)!}$$

(x will start at 2 as the first two terms in $\sum$ are zero)

$$= \mu + e^{-\mu} \mu^2 \sum_{x=2}^{\infty} \frac{\mu^{x-2}}{(x-2)!}$$

Using Exponential Series:-

$$= \mu + e^{-\mu} \mu^2 e^{\mu}$$

$$E(x^2) = \mu + \mu^2$$

Put in (1)

$$Var(x) = E(x^2) - [E(x)]^2$$

$$= \mu + \mu^2 - (\mu)^2$$

$$= \mu + \cancel{\mu^2} - \cancel{\mu^2}$$

$$\boxed{Var(x) = \mu}$$

Proved

MGF :- Moment Generating Function

Let X be a discrete random Variable with a Poisson distribution with Parameter μ $\therefore \mu \in \mathbb{R}$

Then the MGF of X is

$$M_o(t) = e^{\mu(e^t - 1)}$$

Proof :-

From the definition of Poisson dist. X has Pmf

$$f(x) = \frac{\mu^x e^{-\mu}}{x!}$$

JUNA NOT

Date: ____________

Now;

$$M_o(t) = E(e^{tx})$$

$$= \sum e^{tx} f(x)$$

$$= \sum_{x=0}^{\infty} e^{tx} \frac{\mu^x e^{-\mu}}{x!}$$

$$= e^{-\mu} \sum_{x=0}^{\infty} \frac{(\mu e^t)^x}{x!}$$

Using Exponential Series:-

$$= e^{-\mu} e^{\mu e^t} \qquad \therefore e^{\mu} = \frac{(\mu)^x}{x!}$$

$$= e^{-\mu + \mu e^t}$$

$$M_o(t) = e^{\mu(e^t - 1)}$$

PROVED !

(*) HIGHER MOMENTS

RAW Moments OR Moments About Origi

$$\mu_1' = \frac{d}{dt} e^{\mu(e^t-1)} \Big|_{t=0}$$

$$= \frac{d}{dt} e^{-\mu + \mu e^t} \Big|_{t=0}$$

$$\mu_1' = e^{-\mu} \mu e^t e^{\mu e^t} \Big|_{t=0} \qquad \text{——— (1)}$$

Put $t=0$

$$= e^{-\mu} \mu e^0 e^{\mu e^0}$$

$$= e^{-\mu} \mu(1) e^{\mu(1)}$$

$$= e^{-\mu+\mu}\,\mu$$

$$\boxed{\mu_1' = \mu}$$

Now;

Differentiate equ (1) w.r.t 't'.

$$\mu_2' = \frac{d}{dt}\left(e^{-\mu}\mu e^t e^{\mu e^t}\right)\Big|_{t=0}$$

$$= (e^{-\mu}\mu)\frac{d}{dt}\, e^{\mu e^t + t}\Big|_{t=0}$$

$$\mu_2' = e^{-\mu}\mu\left[e^{\mu e^t + t}\left(\mu e^t + 1\right)\right]\Big|_{t=0} \quad \text{———} (2)$$

Put $t=0$

$$= e^{-\mu}\mu\left[e^{\mu}\left(\mu+1\right)\right]$$

$$= e^{-\mu+\mu}\,\mu(\mu+1)$$

$$\boxed{\mu_2' = \mu^2 + \mu}$$

Now;

Differentiate eq (2) w.r.t 't'.

$$\mu_3' = \mu e^{-\mu}\frac{d}{dt}\left[e^{\mu e^t + t}\left(\mu e^t + 1\right)\right]\Big|_{t=0}$$

$$\mu_3' = \mu e^{-\mu}\left[e^{\mu e^t}(\mu e^t+1)^2 + \mu e^{\mu e^t+2t}\right]\Big|_{t=0} \quad \text{—③}$$

Put $t=0$

$$\mu_3' = \mu e^{-\mu}\left[e^{\mu}(\mu+1)^2 + \mu e^{\mu}\right]$$

$$= \mu e^{-\mu+\mu}\left[(\mu+1)^2 + \mu\right]$$

$$= \mu(\mu+1)^2 + \mu^2$$

$$= \mu^3 + \mu + 2\mu^2 + \mu^2$$

$$\boxed{\mu_3' = \mu^3 + 3\mu^2 + \mu}$$

<u>Similarly:-</u>

$$\boxed{\mu_4' = \mu^4 + 6\mu^3 + 7\mu^2 + \mu}$$

(★) Moments About ~~Place~~ Mean

$$\mu_1 = \mu_1' - \mu_1' = 0$$

$$\mu_2 = \mu_2' - (\mu_1')^2 = \cancel{\mu^2} + \mu - \cancel{\mu^2} = \mu$$

$$\mu_3 = \mu_3' - 3(\mu_1'\mu_2') + 2\mu_1'^3$$

$$= \mu^3 + 3\cancel{\mu^2} + \mu - 3\mu^3 - 3\cancel{\mu^2} + 2\mu^3$$

$$= 3\cancel{\mu^3} - 3\cancel{\mu^3} + \mu = \mu$$

$$\mu_4 = \mu_4' + 6\mu_1'^2 \mu_2' - 4\mu_1'\mu_3' - 3\mu_1'^4$$

$$= \mu^4 + 6\mu^3 + 7\mu^2 + \mu + 6\mu^4 + 6\mu^3 - 4\mu^4 - 12\mu^3 - 4\mu^2 - 3\mu^4$$

$$= \mu^4 + 6\mu^4 - 4\mu^4 - 3\mu^4 + 6\mu^3 + 6\mu^3 - 12\mu^3 + 7\mu^2 - 4\mu^2 + \mu$$

$$= \cancel{7\mu^4} - \cancel{7\mu^4} + \cancel{12\mu^3} - \cancel{12\mu^3} + 7\mu^2 - 4\mu^2 + \mu$$

$$\boxed{\mu_4 = 3\mu^2 + \mu}$$

(*) **SKEWNESS**

$$\beta_1 = \frac{\mu_3^2}{\mu_2^3}$$

$$= \frac{\mu^2}{\mu^3}$$

$$\boxed{\beta_1 = \frac{1}{\mu}}$$

* **KURTOSIS**

$$\beta_2 = \frac{\mu_4}{\mu_2^2}$$

$$= \frac{3\mu^2 + \mu}{\mu^2}$$

$$\boxed{\beta_2 - 3 + \frac{1}{\mu}}$$

If X is a Poisson random Variable with parameter $\mu = 2$; find the probabilities for $x = 0,1,2,3$ or more.

Here the Poisson distribution is

$$P(x;2) = \frac{e^{-2}(2)^x}{2!} \quad (x = 0,1,2,\ldots)$$

The desired probabilities for $x = 0,1,2,3$ or more are computed as below:

$$P(X=0) = P(0;2) = e^{-2}$$
$$= 0.135335$$

$$P(X=1) = P(1;2) = \frac{e^{-2}\cdot 2}{1!}$$
$$= 2(0.135335)$$
$$= 0.27067$$

$$P(X=2) = P(2;2) = \frac{e^{-2}(2)^2}{2!}$$
$$= \frac{4}{2}(0.135335)$$
$$= 0.27067,$$

$$P(X \geq 3) = 1 - P(X<3)$$
$$= 1 - [P(X=0) + P(X=1) + P(X=2)]$$
$$= 1 - [0.135335 + 0.27067 + 0.27067]$$
$$= 0.323325$$