Uniform Distribution:

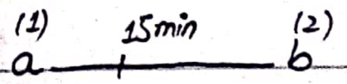
A continuous random variable X is said to follow a uniform distribution with parameters a and b can be written as.

$$X \sim u[a, b]$$

If it's probability density function is constant with a finite interval a, b and zero outside this interval $[a, b]$.

Example:

If buses arrive at a given bus stop every 15 min and you arrive at the bus stop at a random time then the time you wait for the next bus to arrive could be described by a uniform distribution over the interval 0 to 15.



$\because a = 0$
$b = 15$

P.d.f

$$f(x) = \frac{1}{b-a} \; ; \quad a \leq x \leq b$$

Properties:

i) The total area under the curve for a uniform distribution is unity.

ii) The mean of uniform distribution is

$$\frac{a+b}{2}$$

iii) The variance of uniform distribution is $\frac{(b-a)^2}{12}$.

iv) The harmonic mean of uniform distribution is $\frac{b-a}{\ln(b)-\ln(a)}$

v) The median of uniform distribution is $\frac{a+b}{2}$ and is equal to mean.

(Harmonic Mean:- Reciprocal of the A.M of the reciprocal of the values)

property:-

$$\text{Area} = \int_a^b f(x)dx$$

$$= \int_a^b \frac{1}{b-a} dx$$

$$= \frac{1}{b-a}\int_a^b 1dx \Rightarrow \frac{1}{b-a} \, x\Big|_a^b$$

$$= \frac{1}{\cancel{b-a}} \cancel{(b-a)} = 1 \text{ (proved)}$$

MEAN:-

$$\text{Mean} = E(x) = \int_{-\infty}^{\infty} x f(x) dx$$

$$= x\int_a^b \frac{1}{b-a} dx$$

$$= \frac{1}{b-a}\int_a^b x dx$$

$$= \frac{1}{b-a} \cdot \left. \frac{x^2}{2} \right|_a^b$$

$$= \frac{1}{2(b-a)} (b^2-a^2) \Rightarrow \frac{(b-a)(b+a)}{2}$$

$$= \frac{b+a}{2}$$

Hence proved

Valience:

$$Var(u) = \frac{(b-a)^2}{12}$$

$$Var(x) = E(x^2) - [E(x)]^2 \quad \text{(i)}$$

$$E(x^2) = \int_a^b x^2 \cdot \frac{1}{b-a} dx$$

$$= \frac{1}{b-a} \int_a^b x^2 dx \Rightarrow \frac{1}{b-a} \left. \frac{x^3}{3} \right|_a^b$$

$$E(x^2) = \frac{1}{3(b-a)} (b^3-a^3)$$

$$= \frac{(b-a)(a^2+ab+b^2)}{3(b-a)}$$

$$= \frac{a^2+ab+b^2}{3}$$

Using in (1)

$$= \frac{a^2+ab+b^2}{3} - \frac{(a+b)^2}{4}$$

$$= \frac{4(a^2+ab+b^2) - 3(a+b)^2}{12}$$

$$= \frac{4a^2+4ab+4b^2-3(a^2+b^2+2ab)}{12}$$

$$= \frac{4a^2+4ab+4b^2-3a^2-6ab-3b^2}{12}$$

$$= \frac{a^2+b^2-2ab}{12}$$

$$Var(u) = \frac{(a-b)^2}{12}$$

Hence proved

Harmonic Mean :-

$$H.M = \int_{-\infty}^{\infty} \frac{1}{E(1/x)} dx$$

$$E(1/x) = \int_a^b 1/x \, f(x) dx$$

$$= \int_a^b 1/x \cdot \left(\frac{1}{b-a}\right) dx$$

$$= \frac{1}{b-a} \int_a^b 1/x \, dx$$

$$= \frac{1}{b-a} \left[\ln(x)\right]_a^b$$

$$= \frac{1}{b-a} \left[\ln(b) - \ln(a)\right]$$

$$E(1/x) = \frac{\ln(b)-\ln(a)}{b-a}$$

$$H.M = \frac{1}{E(1/x)} = \frac{1}{\frac{\ln b-\ln a}{b-a}}$$

$$H.M = \frac{b-a}{\ln b-\ln a} \quad \text{(proved)}$$

MEDIAN:

$$\int_a^m f(x)\,dx = \frac{1}{2}$$

$$\int_a^m \frac{1}{b-a}\,dx = 1/2$$

$$\frac{1}{b-a}\int_a^m dx = 1/2$$

$$\frac{1}{b-a}\, x \Big|_a^m = 1/2$$

$$\frac{m-a}{b-a} = 1/2$$

$$2(m-a) = b-a$$

$$m-a = \frac{b-a}{2}$$

$$m = \frac{b-a}{2} + a$$

$$m = \frac{b-a+2a}{2}$$

$$m = \frac{a+b}{2}$$ Hence proved

Note

Moment about mean So mean=0

Moment about origin So origin=0

"MOMENT GENERATING FUNCTION OF UNIFORM DISTRIBUTION:"

$$M_x(t) = E(e^{tx})$$

$$= \int_{-\infty}^{\infty} e^{tx} f(x)dx$$

$$= \int_{a}^{b} e^{tx} \cdot \frac{1}{b-a} dx$$

$$= \frac{1}{b-a} \int_{a}^{b} e^{tx} dx$$

$$= \frac{1}{(b-a)} \cdot \frac{e^{tx}}{t} \Big|_{a}^{b}$$

$$= \frac{1}{t(b-a)} (e^{tb} - e^{ta})$$

$$= \frac{e^{tb} - e^{ta}}{t(b-a)}$$

Characteristic Functions:-
(CF)

$$M_x(t) = E(e^{itx})$$

$$= \int_{-\infty}^{+\infty} e^{itx} \frac{1}{b-a} dx$$

$$= \frac{1}{b-a} \int_{a}^{b} e^{itx} dx = \frac{1}{b-a} \cdot \left[\frac{e^{itx}}{t}\right]_{a}^{b}$$

$$= \frac{1}{t(b-a)} (e^{itb} - e^{ita})$$

$$= \frac{e^{itb} - e^{ita}}{t(b-a)}$$

Moment about origin for Uniform Distribution:

$$\mu_r = E(x)^r \qquad \because U_1 = \frac{\Sigma(x-U)^1}{n} \qquad \because U_2 = \frac{\Sigma(x-U)^2}{n}$$

$$= \int_{-\infty}^{+\infty} x^r f(x)dx$$

$$= \int_a^b x^r \frac{1}{b-a} dx$$

$$= \frac{1}{b-a}\int_a^b x^r dx$$

$$= \frac{1}{b-a} \left. \frac{x^{r+1}}{r+1} \right|_a^b$$

$$= \frac{1}{(b-a)(r+1)} \cdot (b^{r+1} - a^{r+1})$$

$$\mu_r = \frac{(b^{r+1} - a^{r+1})}{(b-a)/r+1)}$$

$\because r = 1$

$$U_1 = \frac{(b^2 - a^2)}{2(b-a)}$$

$$U_1 = \frac{(b-a)(b+a)}{2(b-a)}$$

$$U_1 = \frac{b+a}{2}$$

$$U_2' = \frac{b^3 - a^3}{3(b-a)}$$

$$= \frac{(b-a)(a^2+ab+b^2)}{3(b-a)}$$

$$u_2' = \frac{a^2+ab+b^2}{3}$$

$$\boxed{u_2 = u_2' + (u_1')^2}$$

THEOREM:

Show that odd order moments about mean are zero and even order moment about mean is

$$U_{2r} = \frac{2(b-a)^{2r+1}}{2^{2r+1}(b-a)(2r+1)}$$

Proof:

$$U_r = E(x-\mu)^r = \int_{-\infty}^{+\infty} (x-\mu)^r f(x)\,dx$$

$$= \int_a^b \left(x - \frac{a+b}{2}\right)^r \frac{1}{b-a}\,dx$$

$$= \frac{1}{b-a}\int_a^b \left[x - \frac{(a+b)}{2}\right]^r dx$$

$$= \frac{1}{b-a}\left[\frac{\left(x-\frac{a+b}{2}\right)^{r+1}}{r+1}\right]_a^b$$

$$= \frac{1}{(b-a)(r+1)}\left[\left(x-\frac{a+b}{2}\right)^{r+1}\right]_a^b$$

$$= \frac{1}{(b-a)(r+1)}\left[\left(b-\frac{a+b}{2}\right)^{r+1} - \left(a-\frac{a+b}{2}\right)^{r+1}\right]$$

$$= \frac{1}{(b-a)(\lambda+1)}\left[\left(\frac{2b-a-b}{2^{\lambda+1}}\right)^{\lambda+1} - \left(\frac{2a-a-b}{2}\right)^{\lambda+1}\right]$$

$$u_\lambda = \frac{1}{(b-a)(\lambda+1)}\left[(b-a)^{\lambda+1}\cdot\frac{1}{2^{\lambda+1}} - \frac{(a-b)^{\lambda+1}}{2^{\lambda+1}}\right]$$

$$= \frac{1}{2^{\lambda+1}(b-a)(\lambda+1)}\left[(b-a)^{\lambda+1} - (a-b)^{\lambda+1}\right]$$

For odd order put $\lambda = 2\lambda+1$

$$u_{2\lambda+1} = \frac{1}{2^{2\lambda+1+1}(b-a)(2\lambda+1+1)}\left[(b-a)^{2\lambda+1+1} - (a-b)^{2\lambda+1+1}\right]$$

$$= \frac{1}{2^{2\lambda+2}(b-a)(2\lambda+2)}\left[(b-a)^{2\lambda+2} - (a-b)^{2\lambda+2}\right]$$

$$u_{2\lambda+1} = \frac{0}{2^{2\lambda+2}(b-a)(2\lambda+2)} = 0$$

For even put $\lambda = 2\lambda$

$$u_{2\lambda} = \frac{1}{2^{2\lambda+1}(b-a)(2\lambda+1)}\left[(b-a)^{2\lambda+1} - (a-b)^{2\lambda+1}\right]$$

$$= \frac{(b-a)^{2\lambda+1} - (a-b)^{2\lambda+1}}{2^{2\lambda+1}(b-a)(2\lambda+1)}$$

$$u_{2\lambda} = \frac{(a-b)^{2\lambda+1} - \left[-(b-a)^{2\lambda+1}\right]}{2^{2\lambda+1}(b-a)(2\lambda+1)}$$

$$= \frac{(b-a)^{2r+1} + (b-a)^{2r+1}}{2^{2r+1}(b-a)(2r+1)}$$

$$\mu_{2r} = \frac{2(b-a)^{2r+1}}{2^{r+1}(b-a)(2r+1)}$$

Hence proved

Note:-
even order moment always positive.
odd order moment always negative.

Now we have to find Skewness and Kurtosis using μ_2.

Skewness:-

$$B_1 = \frac{(\mu_3)^2}{(\mu_2)^3}$$

* μ_3 is negative So skewness is negative
* μ_3 is positive So skewness is positive.

Kurtosis:-

$$B_2 = \frac{\mu_4}{(\mu_2)^2}$$

Skewness of uniform distribution:-
Uniform distribution on rectangular distribution is always Symmetric.

Kurtosis of UNIFORM Distribution:

$$\mu_{2r} = \frac{2(b-a)^{2r+1}}{2^{2r+1}(b-a)(2r+1)}$$

$\because 2r=2$

$r=1$

$$\mu_2 = \frac{2(b-a)^{2(1)+1}}{2^{2+1}(b-a)(2+1)}$$

$$\mu_2 = \frac{2(b-a)^3}{2^3(3)(b-a)}$$

$$= \frac{(b-a)^3}{12(b-a)}$$

$$\mu_2 = \frac{(b-a)^2}{12}$$

Now for $r=2$

$$\mu_4 = \frac{2(b-a)^5}{2^5(b-a)(5)}$$

$$= \frac{2(b-a)^5}{32(5)(b-a)}$$

$$\mu_4 = \frac{2(b-a)^4}{80 \cdot 2} \Rightarrow \frac{(b-a)^4}{80}$$

$$B = \frac{(b-a)^4}{80} \div \left[\frac{(b-a)^2}{12}\right]^2$$

$$= \frac{(b-a)^4}{80} \times \frac{(12)^2}{(b-a)^4}$$

$$= \frac{144}{80} = 1.8$$

So platykurtic.

MOMENTS :-

The r^{th} moment of a random variable μ about the mean also called the r^{th} central moment, is defined as

$$\mu_r = E\left[(X-\mu)^r\right]$$

where $r = 0, 1, 2 \ldots$

It follows that $\mu_0 = 1$, $\mu_1 = 0$ and $\mu_2 = \sigma^2$

Such That

$$\mu_r = \sum_{j=1}^{n} (x_j - \mu)^r f(x_j) = \sum (x-\mu)^r f(x)$$

$$\mu_r = \sum (x-\mu)^r f(x) \quad \text{(discrete variable)}$$

$$= \int_{-\infty}^{\infty} (x-\mu)^r f(x)\,dx \quad \text{(Continous Variable)}$$

The r^{th} moment of X about the origin is defined as

$$\mu_r = E(X^r)$$

where $r = 0, 1, 2 \ldots$

$$\mu_r' = \sum_{j=1}^{n} x_j^r f(x_j) = \sum x^r f(x)$$

$$\mu_r' = \sum x^r f(x)$$

Discrete Variable.

$$\mu_r' = \sum x^r f(x)$$

Continuous variable

$$\mu_r' = \int_{-\infty}^{\infty} x^r f(x)\,dx$$

The relationship between this moment is given by

$$\mu_r = \mu_r' - \binom{r}{1}\mu_{r-1}'\mu_1' + \cdots + (-1)^j\binom{r}{j}\mu_{r-j}'\mu_1'^j + \cdots + (-1)^r \mu_1'^r$$

As a special case using $\mu_1' = \mu$ and $\mu_0 = 1$ we have

$$\mu_2 = \mu_2' - \mu_1'^2 = \mu_2' - \mu^2$$

$$\mu_3 = \mu_3' - 3\mu_2'\mu_1' + 2\mu_1'^3 = \mu_3' - 3\mu_2'\mu + 2\mu^3$$

$$\mu_4 = \mu_4' - 4\mu_3'\mu_1' + 6\mu_2'\mu_1'^2 - 3\mu_1'^4$$

$$= \mu_4' - 4\mu_3'\mu + 6\mu_2'\mu^2 - 2\mu^4$$

Moment Coefficient of Skewness and kurtoses :-

This is defined as

For Skewness

$$\alpha_3 = \sqrt{\beta_1} = \frac{\mu_2}{\mu_2^{3/2}} = \frac{\mu_3}{\sigma^3}$$

For Kurtoses

$$\beta_2 = \frac{\mu_1}{\mu_2^2} = \frac{\mu_4}{\sigma^4}$$

Example

Find the first four moments about the mean

i) The moment of skewness and kurtosis for the probability distribution.

Solution

The first four moment about mean

$$\mu_1 = E(X-\mu) = \Sigma(X-\mu)f(x)$$

$$\mu_2 = E(X-\mu)^2 = \Sigma(X-\mu)^2 f(x)$$

$$\mu_3 = E(X-\mu)^3 = \Sigma(X-\mu)^3 f(x)$$

$$\mu_4 = E(X-\mu)^4 = \Sigma(X-\mu)^4 f(x)$$

By previous moment is computed

$$\mu_1 = 0$$

$$\mu_2 = \Sigma(X-\mu)^2 f(x) = 6/8 = 3/4$$

$$\mu_3 = 0$$

$$\mu_4 = \Sigma(X-\mu)^4 f(x) = 1.3125$$

The first four moment about origin

$$\mu_1 = E(X) = \Sigma x f(x) = 12/8 = 1.5$$

$$\mu_2 = E(X^2) = \Sigma x^2 f(x) = 24/8 = 3$$

$$\mu_3 = E(X^3) = \Sigma x^3 f(x) = 58/8 = 6.75$$

$$\mu_4 = E(X^4) = \Sigma x^4 f(x) = 16.5$$

Moments about the mean are

$$\mu_1 = 0, \quad \mu_2 = \mu_2' - \mu_1'^2 = 3-(1.5)^2 = 0.75$$

$$\mu_3 = \mu_3' - 3\mu_2' \mu_1' + 2\mu_1'^3$$

$$= 6.75 - 3(3)(1.5) + 2(1.5)^3 = 0$$

$$\mu_4 = \mu_4' - 4\mu_3'\mu_1' + 6\mu_2'\mu_1'^2 - 3\mu_1'^4$$

$$= 16.5 - 4(6.75)(1.5) + 6(3)(1.5)^2 - 3(1.5)^4$$

$$= 1.3125$$

Moment Co-efficient of Skewness.

$$\alpha_3 = \sqrt{\beta_1} = \frac{\mu_3}{\mu_2^{3/2}} = \frac{0}{\sqrt{(0.75)^3}} = 0$$

Moment Co-efficient of Kurtosis

$$\alpha_4 = \beta_2 = \frac{\mu_4}{\mu_2^2} = \frac{1.3125}{(0.75)^2} = 2.33$$

Example:-

Find first four moments

i) about the origin

ii) about the Mean

where R.V X with Density function

$$f(x) = \begin{cases} 4x(9-x^2)/81 & 0 \leq x \leq 3 \\ 0 & \text{otherwise} \end{cases}$$

Solution

$$\mu_1' = E(X) = \int x f(x) dx$$

$$= \int_0^3 x \frac{4x(9-x^2)}{81} dx$$

$$= \frac{4}{81}\left[\int_0^3 9x^2 dx - \int_0^3 x^4 dx\right]$$

$$= \frac{4}{81}\left[9 \cdot \frac{x^3}{3}\Big|_0^3 - \frac{x^5}{5}\Big|_0^3\right]$$

$$= \frac{4}{81}\left[3(3)^3 - \frac{(3)^5}{5}\right]$$

$$= \frac{4}{81}\left[81 - \frac{243}{5}\right] = \frac{4}{81}\left(\frac{405-243}{5}\right)$$

$$= \frac{4}{81}\left(\frac{162}{5}\right) \Rightarrow \frac{648}{405} = \frac{8}{5}$$

$$\mu_2 = \int x^2 f(x)\,dx \Rightarrow \int_0^3 x^2 . \frac{x4}{81}(9-x^2)\,dx$$

$$= \frac{4}{81}\left[\int_0^3 9x^3\,dx - \int_0^3 x^5\,dx\right]$$

$$\mu_2 = \frac{4}{81}\left[9 . \frac{x^4}{4}\Big|_0^3 - \frac{x^6}{6}\Big|_0^3\right]$$

$$= \frac{4}{81}\left[\frac{9(3)^4}{4} - \frac{(3)^6}{6}\right]$$

$$= \frac{4}{81}\left[182.25 - 121.5\right]$$

$$\boxed{\mu_2 = 3}$$

$$\mu_3 = \int x^3 f(x)\,dx$$

$$= \frac{4}{81}\int_0^3 x^3 x(9-x^2)\,dx$$

$$\mu_3 = \frac{4}{81}\left[\int_0^3 9x^4\,dx - \int_0^3 x^6\,dx\right]$$

$$= \frac{4}{81}\left[9 . \frac{(3)^5}{5} - \frac{(3)^7}{7}\right]$$

$$\boxed{\mu_3' = \frac{216}{35}}$$

$$\mu_4' = E(X^4) = \int_0^3 x^4 f(x)dx$$

$$= \frac{4}{81}\int_0^3 x^5 (9-x^2)dx$$

$$= \frac{4}{81}\left[9.\frac{x^6}{6} - \frac{x^8}{8}\right]_0^3$$

$$= \frac{4}{81}\left[\frac{9(3)^6}{3} - \frac{(3)^8}{8}\right]$$

$$\boxed{\mu_4' = \frac{27}{2}}$$

ii) <u>Moment about Mean</u>

$$\mu_1 = 0 \qquad \mu_2 = 3 - \left(\frac{8}{5}\right)^2 = \frac{11}{25}$$

$$\mu_3 = \frac{216}{35} - 9\left(\frac{8}{5}\right) + 2\left(\frac{8}{5}\right)^3 = \frac{-32}{875}$$

$$\mu_4 = \frac{27}{2} - 4\left(\frac{216}{35}\right)\left(\frac{8}{5}\right) + 18\left(\frac{8}{5}\right)^2 - 3\left(\frac{8}{5}\right)^4$$

$$= \frac{3693}{8750}$$

i) <u>Moment co-efficient of Skewness</u>

$$\alpha_1 = \sqrt{\beta_1} = \frac{-32/875}{\sqrt{(11/25)^3}} = -0.1253$$

Moment Co-efficient of Kurtosis :-

$$\alpha_4 = \beta_2 = \frac{\mu_4}{\mu_2^2} = \frac{3693/8750}{(11/25)^2}$$

$$= \frac{2308125}{105875} = 2.18$$

$$\boxed{\beta_2 = 2.18}$$