Chapter 9

Eigenvalues and Eigenvectors

9.1 INTRODUCTION

In this chapter, we study some basics of computing **eigenvalues** and **eigenvectors**. They play a prominent role in the study of differential equations and in many applications in engineering and physical sciences.

Let A be a square matrix, $[a_{ij}]_{n\times n}$. We shall investigate the problem of finding, λ, and non-trivial vector $x_{1\times n}$ (A vector is non-trivial if its all components are not equal to zero), such that,

$$Ax = \lambda x \qquad \text{... (i)}$$

or $\quad (Ax - \lambda x) = 0$

or $\quad (A - \lambda I)\, x = 0 \qquad \text{... (ii)}$

It is known that a solution x ($\neq 0$) exists provided

$$\det\,(A - \lambda) = 0 \qquad \text{... (iii)}$$

More explicitly,

$$\begin{pmatrix} a_{11}-\lambda & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22}-\lambda & a_{23} & \cdots & a_{2n} \\ a_{31} & a_{32} & a_{33}-\lambda & \cdots & a_{3n} \\ \vdots & \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & a_{n3} & \cdots & a_{nn}-\lambda \end{pmatrix} = 0 \qquad \text{... (iv)}$$

If we are to expand the above determinant, we obtain an nth degree polynomial in λ:

$$\det\,(A - \lambda I) = (-1)^n \lambda^n + a_{n-1}\lambda^{n-1} + a_{n-2}\lambda^{n-2} + \cdots + a_1 \lambda + a_0$$

The above polynomial called the **characteristic polynomial** of A. Each value of λ which satisfies (iv), yields a system of homogeneous equations of equation (ii). Thus the problem, of finding the values of λ for which (ii) possesses non-trivial solutions is the same as finding the roots of the characteristic polynomial:

$$(-1)^n \lambda^n + a_{n-1}\lambda^{n-1} + a_{n-2}\lambda^{n-2} + \cdots + a_1 \lambda + a_0 = 0 \qquad \text{... (v)}$$

Here, (v) is called the **characteristic equation** (also sometimes called the **secular equation**) whose roots are $\lambda_1, \lambda_2, \ldots, \lambda_n$ and are called the **eigenvalues** (or **latent roots**) of A and x is called the **eigenvector** (or **latent vector**) of A corresponding to λ. These roots can be **distinct** (i.e., $\lambda_1 \neq \lambda_2 \neq \ldots \neq \lambda_n$), or **complex** or **repeated.** In the case of multiple roots, say a p-fold root λ_j, the problem is more involved.

9.2 METHODS TO SOLVE EIGENVALUE PROBLEMS

The eigenvalue problem reduces to the problem of finding the roots of the characteristic equation, det $(A - \lambda I) = 0$. This can be done directly by expanding the determinant in power of λ if A is of order 3×3 or less. As the size of the given matrix grows; this method rapidly becomes inefficient and time-consuming. However, for particular cases, for instance, for sparse matrices, it may still be quite useful.

If the given matrix is a real symmetric matrix, all its roots are real and Newton's method may be used to find the roots of the characteristic equation (See Section 255). If the given matrix is real but not symmetric, there may be complex roots of the characteristic equation. If $\lambda = a + i b$ is a root, then $\lambda = a - i b$ is also a root of the characteristic equation and corresponding to these two complex roots is the real quadratic factor $\lambda^2 - 2a\lambda + a^2 + b^2 = 0$. It is necessary therefore to first seek quadratic factor, if the given matrix is real and non-symmetric.

To solve eigenvalue problem (determination of eigenvalues and the corresponding eigenvectors) has grown into an extensive special area of numerical methods. The methods developed for this purpose are numerous and it is not possible to describe them one by one or even summarize them comprehensively in this book. To keep our study to a reasonable length, we restrict our attention to the following three methods:

- General method
- Leverrier-Faddeev method
- Power method

Let us describe the above methods one buy one.

9.2.1 General Method

This is a simple method and we illustrate it by the following two examples:

Example 1 Find the eigenvalues and the corresponding eigenvectors of the following matrix:

$$A = \begin{vmatrix} 2 & 1 & -1 \\ 0 & -2 & -2 \\ 1 & 1 & 0 \end{vmatrix}$$

Solution **Characteristic Polynomial**

$$\det(A - \lambda I) = \det\left(\begin{vmatrix} 2-\lambda & 1 & -1 \\ 0 & -2-\lambda & -2 \\ 1 & 1 & -\lambda \end{vmatrix}\right)$$

Expanding the determinant:

$$\det(A - \lambda I) = (2-\lambda)\begin{vmatrix} -2-\lambda & -2 \\ 1 & -\lambda \end{vmatrix} - 0\begin{vmatrix} 1 & -1 \\ 1 & -\lambda \end{vmatrix} + 1\begin{vmatrix} 1 & -1 \\ -2-\lambda & -2 \end{vmatrix}$$

$$= (2-\lambda)[2\lambda + \lambda^2 + 2] + 0 + [-2-2-\lambda]$$

$$= 4\lambda + 2\lambda^2 + 4 - 2\lambda^2 - \lambda^3 - 2\lambda - 4 - \lambda$$

$$= -\lambda^3 + \lambda$$

Roots of the Characteristic Equation

$$-\lambda^3 + \lambda = 0$$

or $\lambda^3 - \lambda = 0$

or $\lambda(\lambda^2 - 1) = 0$

$\lambda_1 = 0$, $\lambda_2 = 1$ and $\lambda_3 = -1$

So the eigenvalues are $0, 1, -1$.

The set of all eigenvalues of matrix A, usually denoted by the symbol $\sigma(A)$, is called the spectrum of A.

The eigenvectors corresponding to the above eigenvalues may now be calculated.

(i) When $\lambda_1 = 0$, eigenvector, $x^{(1)} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$

$$\therefore (A - \lambda I) = \begin{bmatrix} 2 & 1 & -1 \\ 0 & -2 & -2 \\ 1 & 1 & 0 \end{bmatrix}$$

$$(A - \lambda I)x = \begin{bmatrix} 2 & 1 & -1 \\ 0 & -2 & -2 \\ 1 & 1 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Solving:

$$\begin{aligned} 2x + y - z &= 0 \\ -2y - 2z &= 0 \\ x + y &= 0 \end{aligned}$$

Thus, from the last two equations, we get

$$x = -y = z$$

$$x^{(1)} = \begin{bmatrix} x \\ -x \\ x \end{bmatrix}$$

Since eigenvalues are of arbitrary length, we are free to choose one component. So, we may choose any **non-zero** value of x.

Let us use x = 1 and we get

When $\lambda_1 = 0$, $x^{(1)} = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} = [1 \ \ -1 \ \ 1]^T$

(ii) When $\lambda_2 = 1$, eigenvector, $x^{(2)} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$

$$\therefore (A - \lambda I) = \begin{bmatrix} 1 & 1 & -1 \\ 0 & -3 & -2 \\ 1 & 1 & -1 \end{bmatrix}$$

$$(A - \lambda I)x = \begin{bmatrix} 1 & 1 & -1 \\ 0 & -3 & -2 \\ 1 & 1 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Solving:

$$\begin{aligned} x + y - z &= 0 \\ -3y - 2z &= 0 \\ x + y - z &= 0 \end{aligned}$$

These equations yield:

$$y = \frac{-2}{3}z$$

$$x = -y + z$$

$$= +\frac{2}{3}z + z = \frac{5}{3}z$$

$$\therefore \quad x^{(2)} = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} \frac{5}{3}z \\ -\frac{2}{3}z \\ z \end{bmatrix}$$

Letting z = 1, we get

$$\text{When } \lambda_2 = 0, \; x^{(2)} = \begin{bmatrix} \frac{5}{3} \\ -\frac{2}{3} \\ 1 \end{bmatrix} = \begin{bmatrix} \frac{5}{3} & \frac{-2}{3} & 1 \end{bmatrix}^T$$

(iii) When $\lambda_3 = -1$, eigenvector, $x^{(3)} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$

$$\therefore (A-\lambda I) = \begin{bmatrix} 3 & 1 & -1 \\ 0 & -1 & -2 \\ 1 & 1 & 1 \end{bmatrix}$$

$$(A-\lambda I)x = \begin{bmatrix} 3 & 1 & -1 \\ 0 & -1 & -2 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{aligned} 3x + y - z &= 0 \\ -y - 2z &= 0 \\ x + y - z &= 0 \end{aligned}$$

Solving:

$$y = -2z$$

$$x = -y - z$$

$$= +2z - z = z$$

$$x^{(3)} = \begin{bmatrix} z \\ -2z \\ z \end{bmatrix}$$

Letting $z = 1$, we get

When $\lambda_3 = -1$, $x^{(3)} = \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix} = [1 \quad -2 \quad 1]^T$

Answers:

When $\lambda_1 = 0$, $x^{(1)} = [1 \quad -1 \quad 1]^T$

When $\lambda_2 = 0$, $x^{(2)} = \left[\frac{5}{3} \quad \frac{-2}{3} \quad 1\right]^T$

When $\lambda_3 = -1$, $x^{(3)} = [1 \quad -2 \quad 1]^T$

Example 2 Find the eigenvalues and their corresponding eigenvectors from the following matrix:

$$A = \begin{bmatrix} 2 & 0 & 0 \\ 2 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix}$$

Solution **Characteristic Polynomial**

$$\det(A-\lambda I) = \det\left(\begin{bmatrix} 2-\lambda & 0 & 0 \\ 2 & 2-\lambda & 1 \\ 1 & 1 & 2-\lambda \end{bmatrix}\right)$$

$$= (2-\lambda)\begin{bmatrix} 2-\lambda & 1 \\ 1 & 2-\lambda \end{bmatrix} - 0\begin{bmatrix} 2 & 1 \\ 1 & 2-\lambda \end{bmatrix} + 0\begin{bmatrix} 2 & 2-\lambda \\ 1 & 1 \end{bmatrix}$$

$$= (2-\lambda)[(2-\lambda)^2 - 1] - 0 + 0$$

$$= (2-\lambda)[4 + \lambda^2 - 4\lambda - 1]$$

$$= (2-\lambda)[\lambda^2 - 4\lambda + 3]$$

$$= -\lambda^3 + 6\lambda^2 - 11\lambda + 6$$

Characteristic Polynomial

$$\lambda^3 - 6\lambda^2 + 11\lambda - 6 = 0$$

Factorizing to get eigenvalues:

$$(\lambda-1)(\lambda-2)(\lambda-3) = 0$$

$$\therefore\ \lambda = 1, 2, 3$$

(i) When $\lambda_1 = 1$, eigenvector, $x^{(1)} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$

$$\therefore \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Solving:

$$\begin{aligned} x &= 0 \\ 2x + y + z &= 0 \\ x + y + z &= 0 \end{aligned}$$

Thus, $x = 0$, $z = -y$

$$x^{(1)} = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ y \\ -y \end{bmatrix}$$

Letting $y = 1$, we get

$$\therefore \quad x^{(1)} = \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}$$

When $\lambda_1 = 1,\ x^{(1)} = [0 \quad 1 \quad -1]^T$

(ii) When $\lambda_2 = 2;\ x^{(2)} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$

$$\begin{bmatrix} 0 & 0 & 0 \\ 2 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$2x + z = 0$$
$$x + y = 0$$

$x = -y;$ or $y = -x$

$$z = -2x$$

$$x^{(2)} = \begin{bmatrix} x \\ -x \\ -2x \end{bmatrix};\ \text{let } x = 1$$

$$= \begin{bmatrix} 1 \\ -1 \\ -2 \end{bmatrix}$$

When $\lambda_2 = 2,\ x^{(2)} = [1 \quad -1 \quad -2]^T$

(iii) When $\lambda_3 = 3;\ x^{(3)} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$

$$\begin{bmatrix} -1 & 0 & 0 \\ 2 & -1 & 1 \\ 1 & 1 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{aligned} -x &= 0 \\ 2x - y + z &= 0 \\ x + y + z &= 0 \end{aligned}$$

$$x = 0$$

$$z = y$$

$$x^{(3)} = \begin{bmatrix} 0 \\ y \\ y \end{bmatrix}; \text{ put } y = 1$$

$$= \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$$

When $\lambda_3 = 3,\ x^{(3)} = \begin{bmatrix} 0 & 1 & 1 \end{bmatrix}^T$

Answers:

When $\lambda_1 = 1,\ x^{(1)} = \begin{bmatrix} 0 & 1 & -1 \end{bmatrix}^T$

When $\lambda_2 = 2,\ x^{(2)} = \begin{bmatrix} 0 & -1 & -2 \end{bmatrix}^T$

When $\lambda_3 = 3,\ x^{(3)} = \begin{bmatrix} 0 & 1 & 1 \end{bmatrix}^T$

Some Remarks

It is important to remember that following points in using this method:

a) Eigenvalues and eigenvectors can be real as well as complex valued.

b) The dimension of the eigenspace corresponding to an eigenvalue is less than or equal to the multiplicity of that eigenvalue.

c) The method used above is suitable for 2×2 and 3×3 matrices. Eigenvalues and eigenvectors of larger matrices are often computed using some other techniques described in the later sections.

d) However, this method is not suitable for computer.

9.2.2 Leverrier-Faddeev Method

The **Leverrier-Faddeev method** is used to find all eigenvalues and the corresponding eigenvectors. It is a more efficient method as compared to previously discussed and it can be easily computerized.

It uses the trace and proceeds as follows:

Let $A_1 = A$ (where A is the given matrix). Also, $P_1 = \text{trace}(A) = \sum_{i=1}^{n} a_{ii}$

Let $A_2 = A(A_1 - P_1 I)$; $P_2 = \frac{1}{2}\,\text{trace}(A_2)$

Let $A_3 = A(A_2 - P_2 I)$; $P_3 = \frac{1}{3}\,\text{trace}(A_3)$

$\vdots$

Let $A_n = A(A_{n-1} - P_{n-1} I)$; $P_n = \frac{1}{n}\,\text{trace}(A_n)$

The numbers $P_1, P_2, \ldots, P_n$ are required coefficients in the characteristic equation. Then, $\lambda^n - P_1\lambda^{n-1} - P_2\lambda^{n-2} - \cdots - P_n = 0$.

Solve it for $\lambda_1, \lambda_2, \lambda_3, \ldots, \lambda_n$.

Check: From the last step, $A_n - P_n I = 0$

Before solving a numerical example, we will introduce the following terminology:

- Trace and determinant of a matrix
- Inverse of a matrix
- Spectral radius

(a) Trace and Determinant of a Matrix

The sum of diagonal elements of a square matrix is called the **trace** of the matrix and equals the sum of its eigenvalues.

Let $A = [a_{ij}]_{n\times n}$ be an nth order non-singular square matrix, then the trace of A is

$$\text{tr}(A) = \sum_{i=1} a_{ij}\text{; sum of the diagonal elements.}$$

$$\text{Also tr}(A) = \sum_{i=1}^{n} \lambda_i\text{; sum of } \lambda\text{'s.}$$

Determinant of the matrix A:

$$\det(A) = \lambda_1, \lambda_2, \lambda_3, \ldots, \lambda_i$$

$$= \prod_{i=1}^{n} \lambda_i$$

It means that the product of the eigenvalues of a square matrix is equal to the determinant of that matrix.

(b) Inverse

This method can also be used for finding the inverse of A which is given by:

$$A^{-1} = \frac{1}{P_n}[A_{n-1} - P_{n-1} I]$$

(c) Spectral Radius

The spectral radius of a square matrix A is the largest absolute eigenvalue. It is denoted by $\delta(A)$.

$$\delta(A) = \max |\lambda_i|;\ 1 \le i \le n.$$

Example 3 (a) Determine the eigenvalues for the following matrix:

$$A = \begin{bmatrix} 3 & 2 & 4 \\ 2 & 0 & 2 \\ 4 & 2 & 3 \end{bmatrix}$$

(b) Find also the inverse, trace, determinant and spectral radius of A.

Solution

(a) Let $A_1 = \begin{bmatrix} 3 & 2 & 4 \\ 2 & 0 & 2 \\ 4 & 2 & 3 \end{bmatrix};\ P_1 = tr(A_1) = \sum a_{ii} = 3+0+3 = 6$

$$A_2 = A(A_1 - P_1 I)$$

$$= \begin{bmatrix} 3 & 2 & 4 \\ 2 & 0 & 2 \\ 4 & 2 & 3 \end{bmatrix} \left(\begin{bmatrix} 3 & 2 & 4 \\ 2 & 0 & 2 \\ 4 & 2 & 3 \end{bmatrix} - 6 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \right)$$

$$= \begin{bmatrix} 3 & 2 & 4 \\ 2 & 0 & 2 \\ 4 & 2 & 3 \end{bmatrix} \begin{bmatrix} -3 & 2 & 4 \\ 2 & -6 & 3 \\ 4 & 2 & -3 \end{bmatrix} = \begin{bmatrix} 11 & 2 & 4 \\ 2 & 8 & 2 \\ 4 & 2 & 11 \end{bmatrix}$$

$$P_2 = \frac{1}{2} tr(A_2)$$

$$= \frac{1}{2}[11+8+11] = \frac{1}{2} \times 30 = 15$$

$$A_3 = A(A_2 - P_2 I)$$

$$= \begin{bmatrix} 3 & 2 & 4 \\ 2 & 0 & 2 \\ 4 & 2 & 3 \end{bmatrix} \left(\begin{bmatrix} 3 & 2 & 4 \\ 2 & 8 & 2 \\ 4 & 2 & 11 \end{bmatrix} - 15 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \right)$$

$$= \begin{bmatrix} 3 & 2 & 4 \\ 2 & 0 & 2 \\ 4 & 2 & 3 \end{bmatrix} \begin{bmatrix} -4 & 2 & 4 \\ 2 & -7 & 3 \\ 4 & 2 & -4 \end{bmatrix}$$

$$= \begin{bmatrix} 8 & 0 & 0 \\ 0 & 8 & 0 \\ 0 & 0 & 8 \end{bmatrix}$$

$$P_3 = \frac{1}{3} \operatorname{tr}(A_3)$$

$$= \frac{1}{3}[8 + 8 + 8] = 8$$

Check: $A_3 - P_3 I = 0$

$$\begin{bmatrix} 8 & 0 & 0 \\ 0 & 8 & 0 \\ 0 & 0 & 8 \end{bmatrix} - 8 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = 0$$

Characteristic Polynomial

$$\lambda^3 - 6\lambda^2 - 15\lambda - 8 = 0$$

Factorizing :

$$(\lambda + 1)(\lambda^2 - 7\lambda - 8) = 0$$

$$(\lambda + 1)(\lambda + 1)(\lambda - 8) = 0$$

$$\lambda = -1, -1, 8$$

(b) **Inverse:** A^{-1}

$$A^{-1} = \frac{1}{P_n}[A_{n-1} - P_{n-1} I]$$

$$= \frac{1}{8} \left(\begin{bmatrix} 11 & 2 & 4 \\ 2 & 8 & 2 \\ 4 & 2 & 11 \end{bmatrix} - 15 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \right)$$

$$= \frac{1}{8}\begin{bmatrix} -4 & 2 & 4 \\ 2 & -7 & 3 \\ 4 & 2 & -4 \end{bmatrix}$$

$$= \begin{bmatrix} -\frac{1}{2} & \frac{1}{4} & \frac{1}{2} \\ \frac{1}{4} & -\frac{7}{8} & \frac{1}{4} \\ \frac{1}{2} & \frac{1}{4} & -\frac{1}{2} \end{bmatrix}$$

Trace of A :

$$\text{tr}(A) = \sum_{i=1}^{n} \lambda_i$$

$$= -1 - 1 + 8 = 6$$

Determinant of A :

$$\det(A) = \prod_{i=1}^{n} \lambda_i$$

$$= -1 \times -1 \times 8 = 8$$

Spectral radius of A :

$$\delta(\lambda_i) = 8$$

9.2.3 Power Method

It is the simplest iterative procedure for determining the **largest** (or **principal**) eigenvalue and the corresponding eigenvector of a matrix. It is easy to apply and is probably the most widely used method. The eigenvalue having the greatest absolute value is called the **dominant eigenvalue.** This method is used because in many applications only the dominant eigenvalue of a matrix is needed. Power method fails if there is no dominant eigenvalue.

Assume eigenvalues of an $n \times n$ matrix are arranged to be

$$|\lambda_1| > |\lambda_2| \geq |\lambda_3| \geq \ldots \geq |\lambda_n| \geq 0$$

The process proceeds as follows:

Let $x^{(1)}$ be any non-zero vector and define a sequence of vectors,

$$x^{(2)}, x^{(3)}, x^{(4)}, \ldots, x^{(n)}$$

by the recursive relation:

$$x^{(r+1)} = A x^{(r)}$$

Then as $r \to \infty$, $x^{(r)} \to$ (multiple of) $q^{(1)}$. It is important to know that for finding the roots of a polynomial equation of degree ≥ 4 is not a simple task, it has to be carried out using iterative methods.

Proof Express $x^{(1)}$ in terms of eigenvectors:

$$x^{(1)} = \alpha_1 q^{(1)} + \alpha_2 q^{(2)} + \ldots + \alpha_n q^{(n)} \qquad \text{Linear independence.}$$

Then,

$$x^{(2)} = A[\alpha_1 q^{(1)} + \alpha_2 q^{(2)} + \ldots + \alpha_n q^{(n)}] \qquad (\text{since } Aq = \lambda q)$$

$$= \lambda_1 \alpha_1 q^{(1)} + \lambda_2 \alpha_2 q^{(2)} + \ldots + \lambda_n \alpha_n q^{(n)}$$

$$x^{(3)} = A[\lambda_1 \alpha_1 q^{(1)} + \lambda_2 \alpha_2 q^{(2)} + \ldots + \lambda_n \alpha_n q^{(n)}]$$

$$= \alpha_1 \lambda_1^2 q^{(1)} + \alpha_2 \lambda_2^2 q^{(2)} + \ldots + \alpha_n \lambda_n^2 q^{(n)}$$

$$\vdots$$

$$x^{(r)} = \alpha_1 \lambda_1^{(r-1)} q^{(1)} + \alpha_2 \lambda_2^{(r-1)} q^{(2)} + \ldots + \alpha_n \lambda_n^{(r-1)} q^{(n)}$$

So, if r is large enough, $|\lambda_1^{r-1}| \gg |\lambda_2^{r-1}|$, and so, $x^{(r)} \simeq \alpha_1 \lambda_1^{r-1} q^{(1)}$.

In practice, we usually scale down at each iteration by dividing $x^{(r)}$ by its largest element:

i.e. $\quad y^{(r+1)} = A x^{(r)}$

Then as $r \to \infty$, $x^{(r)} \to q^{(1)}$ and ratio of $y^{(r+1)}$ to $x^{(r)} \to \lambda_1$.

This iterative method will converge if the largest eigenvalue is real and is not a multiple root. Convergence is most rapid when the ratio of the largest eigenvlue to the next largest eigenvalue is large.

Computing the smallest eigenvalue of a matrix

The eigenvalue of smallest magnitude of a matrix is the same as the inverse (reciprocal) of the dominant eigenvalue of the inverse of the matrix. Since most applications of eigenvalues need the eigenvalue of smallest magnitude, the inverse matrix is often solved for its dominant eigenvalue. This is why the dominant eigenvalue is so important.

In order to find the smallest eigenvalue of a matrix, we apply the principle that the reciprocals of eigenvalues of a matrix are the eigenvalues of the inverse of the matrix. That is, if λ is an eigenvalue of A, then

$$A^{-1} X = \frac{1}{\lambda^2}$$

Therefore, taking the inverse of A and then using the iteration we have just described will give the largest eigenvalue of the inverse of A. The reciprocal of this value will then be the smallest eigenvalue of A.

Let us now illustrate this method by the following two examples.

Example 4 Given the following square matrix:

$$\begin{bmatrix} 5 & 0 & 1 \\ 0 & -1 & 0 \\ 1 & 0 & 5 \end{bmatrix}$$

Find its dominant eigenvalue and its corresponding eigenvector using power method. Try the initial guess as: $x^{(1)} = [1 \quad 1 \quad 1]^T$.

Solution

Given $y^{(r+1)} = A\,x^{(r)}$

Let r = 1, then $y^{(2)} = A\,x^{(1)}$

$x^{(2)} = \frac{1}{6}\,y^{(2)} = \frac{1}{6}\begin{bmatrix} 6 \\ -1 \\ 6 \end{bmatrix} = \begin{bmatrix} 1 \\ -.167 \\ 1 \end{bmatrix}$	$y^{(2)} = A\,x^{(1)} = \begin{bmatrix} 5 & 0 & 1 \\ 0 & -1 & 0 \\ 1 & 0 & 5 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 6 \\ -1 \\ \textcircled{6} \end{bmatrix}$ ↑ Largest value
$x^{(3)} = \frac{1}{6}\,y^{(3)} = \frac{1}{6}\begin{bmatrix} 6 \\ -.167 \\ 6 \end{bmatrix} = \begin{bmatrix} 1 \\ -.0278 \\ 1 \end{bmatrix}$	$y^{(3)} = A\,x^{(2)} = \begin{bmatrix} 5 & 0 & 1 \\ 0 & -1 & 0 \\ 1 & 0 & 5 \end{bmatrix}\begin{bmatrix} 1 \\ -.167 \\ 1 \end{bmatrix} = \begin{bmatrix} 6 \\ -.167 \\ \textcircled{6} \end{bmatrix}$
$x^{(4)} = \frac{1}{6}\,y^{(4)} = \frac{1}{6}\begin{bmatrix} 6 \\ -.0278 \\ 6 \end{bmatrix} = \begin{bmatrix} 1 \\ -.0046 \\ 1 \end{bmatrix}$	$y^{(4)} = A\,x^{(3)} = \begin{bmatrix} 5 & 0 & 1 \\ 0 & -1 & 0 \\ 1 & 0 & 5 \end{bmatrix}\begin{bmatrix} 1 \\ .0278 \\ 1 \end{bmatrix} = \begin{bmatrix} 6 \\ -.0278 \\ 6 \end{bmatrix}$
$x^{(5)} = \frac{1}{6}\,y^{(5)} = \frac{1}{6}\begin{bmatrix} 6 \\ .0046 \\ 6 \end{bmatrix} = \begin{bmatrix} 1 \\ .0008 \\ 1 \end{bmatrix}$	$y^{(5)} = A\,x^{(4)} = \begin{bmatrix} 5 & 0 & 1 \\ 0 & -1 & 0 \\ 1 & 0 & 5 \end{bmatrix}\begin{bmatrix} 1 \\ -.0046 \\ 1 \end{bmatrix} = \begin{bmatrix} 6 \\ .0046 \\ 6 \end{bmatrix}$

$x^{(6)}=\frac{1}{6}y^{(6)}=\frac{1}{6}\begin{bmatrix}6\\-.0008\\6\end{bmatrix}=\begin{bmatrix}1\\-.0001\\1\end{bmatrix}$	$y^{(6)}=Ax^{(5)}=\begin{bmatrix}5&0&1\\0&-1&0\\1&0&5\end{bmatrix}\begin{bmatrix}1\\-.0008\\1\end{bmatrix}=\begin{bmatrix}6\\-.0008\\6\end{bmatrix}$
$x^{(7)}=\frac{1}{6}y^{(7)}=\frac{1}{6}\begin{bmatrix}6\\.0001\\6\end{bmatrix}=\begin{bmatrix}1\\0\\1\end{bmatrix}$	$y^{(7)}=Ax^{(6)}=\begin{bmatrix}5&0&1\\0&-1&0\\1&0&5\end{bmatrix}\begin{bmatrix}1\\-.0001\\1\end{bmatrix}=\begin{bmatrix}6\\.0001\\6\end{bmatrix}$
$x^{(8)}=\frac{1}{6}y^{(8)}=\frac{1}{6}\begin{bmatrix}6\\0\\6\end{bmatrix}=\begin{bmatrix}1\\0\\1\end{bmatrix}$	$y^{(8)}=Ax^{(7)}=\begin{bmatrix}5&0&1\\0&-1&0\\1&0&5\end{bmatrix}\begin{bmatrix}1\\0\\1\end{bmatrix}=\begin{bmatrix}6\\0\\6\end{bmatrix}$

At this stage, $x^{(8)} = x^{(7)}$

Thus, $\lambda_1 \approx 6$

$$q^{(1)} = \begin{bmatrix}1\\0\\1\end{bmatrix}$$

True answer : $\lambda_1 = 6;\ \lambda_2 = 4;\ \lambda_3 = -1$

Corresponding eigenvectors :

For $\lambda_1 = 6$, $\quad q^{(1)} = \begin{bmatrix}1\\0\\1\end{bmatrix}$,

For $\lambda_2 = 4$, $\quad q^{(2)} = \begin{bmatrix}-1\\0\\1\end{bmatrix}$,

For $\lambda_3 = -1$, $\quad q^{(3)} = \begin{bmatrix}0\\1\\0\end{bmatrix}$.

Example 5: Use power method to determine the dominant eigenvalue and its corresponding eigenvector of the following matrix:

$$A = \begin{bmatrix}1&0&-1\\1&2&1\\2&2&3\end{bmatrix}$$

Use $x^{(1)} = [1 \ \ 1 \ \ 1]^T$.

Write computer program to implement power method.

Solution: Let $y^{(2)} = A x^{(1)}$ and $x^{(1)} = [1 \ \ 1 \ \ 1]^T$

$x^{(2)} = \frac{1}{7} y^{(2)} = \frac{1}{7}\begin{bmatrix}0\\4\\7\end{bmatrix} = \begin{bmatrix}0\\.57\\1\end{bmatrix}$	$y^{(2)} = A x^{(1)} = \begin{bmatrix}1 & 0 & -1\\1 & 2 & 1\\2 & 2 & 3\end{bmatrix}\begin{bmatrix}1\\1\\1\end{bmatrix} = \begin{bmatrix}0\\4\\ \textcircled{7}\end{bmatrix}$
$x^{(3)} = \frac{1}{4.14} y^{(3)} = \frac{1}{4.14}\begin{bmatrix}-1\\2.14\\4.14\end{bmatrix} = \begin{bmatrix}-.24\\.52\\1\end{bmatrix}$	$y^{(3)} = \begin{bmatrix}1 & 0 & -1\\1 & 2 & 1\\2 & 2 & 3\end{bmatrix}\begin{bmatrix}0\\.57\\1\end{bmatrix} = \begin{bmatrix}-1\\2.14\\4.14\end{bmatrix}$
$x^{(4)} = \frac{1}{3.56}\begin{bmatrix}-1.24\\1.80\\3.56\end{bmatrix} = \begin{bmatrix}-0.35\\.51\\1\end{bmatrix}$	$y^{(4)} = A x^{(3)} = \begin{bmatrix}1 & 0 & -1\\1 & 2 & 1\\2 & 2 & 3\end{bmatrix}\begin{bmatrix}-.24\\.52\\1\end{bmatrix} = \begin{bmatrix}-1.24\\1.80\\3.56\end{bmatrix}$
$\vdots$	$\vdots \qquad \vdots$
$x^{(10)} = \frac{1}{3.04}\begin{bmatrix}-1.48\\1.52\\3.04\end{bmatrix} = \begin{bmatrix}-.49\\.50\\1\end{bmatrix}$	$y^{(10)} = \begin{bmatrix}-1.48\\1.52\\3.04\end{bmatrix}$
$x^{(11)} = \frac{1}{3.02}\begin{bmatrix}-1.49\\1.51\\3.02\end{bmatrix} = \begin{bmatrix}-0.49\\0.50\\1\end{bmatrix}$	$y^{(11)} = \begin{bmatrix}-1.49\\1.51\\3.02\end{bmatrix}$

$$x^{(10)} = x^{(11)}$$

$$\therefore \lambda_1 = 3$$

$$q^{(1)} \approx \begin{bmatrix}-0.49\\0.50\\1\end{bmatrix}$$

True answer : $\lambda_1 = 3;\ q^{(1)} = \begin{bmatrix} -.50 \\ .50 \\ 1 \end{bmatrix}$

This program has been taken from the following website:

http://www.net.pk/mtshome/appNumericalAnalysis.html

Computer Program **Power Method**

```
#include <iostream.h>
#include <math.h>
#include <fstream.h>
#include <string.h>

void read_n(int&,ifstream&);
void read_array(double**, int, ifstream&);
void write_array(double**, int);
void deallocate_arrary(double**,int);
void mult(double**, double*, double*,int);
double fun(double*,int);
double vectornorm(double*,int);

void main( )
{
        int n,i,k,M;
        double** array;
        double* x;
        double* y;
        double r,vn,tol,rold;

        M=1000;
        tol=0.0;
        ifstream arrayin;
        arrayin.open("array.dat");

        read_n(n,arrayin);
        x=new double[n];
        y=new double[n];

        x[0]=-1.0;
        x[1]=1.0;
```

```
x[2]=1.0;

for(i=0;i<n;i++)
{
            y[i]=0.0;
}

//allocate array
array=new double*[n];
for (i=0;i<n;i++)
{
            array[i] = new double[n];
}

read_ array(array, n, arrayin);
arrayin.close( );

//array now read into file
cout<<"Original Matrix:"<<endl;
write_array(array,n);

k=0;
rold=1.0;
r=10.0;
while(k<M && fabs(rold-r)>tol)
{
      cout<<"k= "<<k<<" x="<<x[0]<<" "<<x[1]<<" "<<x[2]<<"r="
      <<r<<endl;
      mult(array,x,y,n);
      rold=r;
      r=fun(y,n)/fun(x,n);
      vn=vectornorm(y,n);
      for(i=0;i<n;i++)x[i]=y[i]/vn;
      k++;
}
cout<<"k= "<<k<<" x="<<x[0]<<" "<<x[1]<<" "<<x[2]<<"r="
<<r<<endl;
deallocate_array(array,n);
delete x,y;
cout<<"Press ENTER to end"<<endl;
cin.get( );
return 0;
```

```
void read_n(int &n, ifstream & arrayin)
{
        char temp;
        //read in number of rows
        n=int(arrayin.get( ) )-int('0');
        temp=arrayin.get( );    //get next character
        while(temp != '      '  && temp != '\n')
        {
                n=n*10+int(temp)-int('0');
                temp=arrayin.get( );
        }
}

void read_array(double** array, int n, ifstream &arrayin)
{
    double tempd,div;
    bool divflag;
    char temp;
    int i,j;
    //read in array from file
    for(i=0;i<n;i++)
    {
        for(j=0;j<n;j++)
        {
            div=1;
            divflag=false;
            tempd=0.0;
            if(!arrayin.eof( ) ) temp=arrayin.get( ); //get next character
            while(temp != ' ' && temp != '\n' && !arrayin.eof( ) )
            {
                if (temp== '-')
                {
                    div*=-1;
                    temp=arrayin.get( );
                }
                if (tem== '.')
                {
                    temp=arrayin.get( );
                    divflag=true;
                }
                else
```

```
                {
                        if (divflag==true)
                        {
                            div=div*10.0;
                        }
                        tempd=tempd*10+int(temp)-int('0');
                        if(!arrayin.eof( ) )
                        {
                            temp=arrayin.get( );
                        }
                }
            }
            array[i][j]=tempd/div;
        }
        while(!arrayin.eof( ) && temp != '\n')
        {
            temp=arrayin.get( );
        }
    }
    //array[i][j] is now the ith row jth column element of the array retrun;
}
void deallocate_array(double** array, int n)
{
        // deallocate array
        int i;
        for (i=0;i<n;i++)
        {
          deleat[ ] array[i];
        }
        deleat[ ] array;
}

void write_array(double** array,int n)
{
        int i,j;
        for(i=0;i<n;i++)
        {
                for(j=0;j<n;j++)
                {
                    cout<<array[i][j]<< " ";
                }
                cout<<endl;
        }
```

```
        return;
}

void mult(double** A,double* x,double* y,int n)
{
            for (int i=0;i<n;i++)
            {
                  y[i]=0;
                  for (int j=0;j<n;j++)
                  {
                      y[i]=y[i]+A[i][j]*x[j];
                  }
            }
            return;
}

double fun(double* x,int n)
{
      //cout<<x[0]<< " "<<[1]<<" "<<x[2]<<endl;
      return x[1];
}

double vectornorm(double* x,int n)
{
      double answer=x[0];
      for(int i=1;i<n;i++)
      {
          if(abs(x[i])>=abs(answer))
          {
                  answer=x[i];
          }
      }
      return answer;
}
```

9.3 MATRIX DEFLATION

There are different methods for finding subsequent eigenvalues of a matrix, we will discuss only one of these, i.e. the **deflation method** which is a straightforward approach.

Suppose we have applied the power method to a matrix A and have obtained its largest eigenvalue λ_1 and corresponding eigenvector $q^{(1)}$. We now require to find the eigenvalue λ_2, to do so, λ_1 must be removed by a process called **deflation**. Deflation