Chapter 8

Linear Systems of Equations

8.1 BASIC CONCEPTS

Consider a set of m simultaneous linear algebraic equations in n unknown, x_1, $x_2, \ldots, x_n$:

$$\left.\begin{aligned} a_{11} x_1 + a_{12} x_2 + a_{13} x_3 + \cdots + a_{1n} x_n &= b_1 \\ a_{21} x_1 + a_{22} x_2 + a_{23} x_3 + \cdots + a_{2n} x_n &= b_2 \\ a_{31} x_1 + a_{32} x_2 + a_{33} x_3 + \cdots + a_{3n} x_n &= b_3 \\ &\vdots \\ a_{m1} x_1 + a_{m2} x_2 + a_{m3} x_3 + \cdots + a_{mn} x_n &= b_m \end{aligned}\right\} \qquad \ldots (8.1)$$

In a more compact notation, the above equations can be rewritten as:

$$\sum_{j=1}^{n} a_{ij} x_j = b_i \text{; for } i = 1, 2, \ldots, m.$$

Three type of quantities occur here:

(a) The unknowns, $x_1, x_2, \ldots, x_n$.

(b) The coefficients a_{ij}, where $i = 1, 2, \ldots, m$ and $j = 1, 2, \ldots, n$

(a) The right hand sides, $b_1, b_2, \ldots, b_m$.

Equations (8.1) can also be written in the matrix notation, as,

$$Ax = b \qquad \ldots (8.2)$$

$$\text{where } A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ a_{31} & a_{32} & \cdots & a_{3n} \\ \cdots & \cdots & \cdots & \cdots \\ & & \vdots & \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} ; \; x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_n \end{bmatrix} ; \; b = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ \vdots \\ b_m \end{bmatrix}$$

A is a rectangular matrix having m rows and n columns, x and b are column vectors.

Problems of this type occur in almost all disciplines. Our aim is to develop methods, which can solve such problems and are easily implemented on a digital computer.

8.2 METHODS TO SOLVE A SYSTEM OF LINEAR EQUATIONS

Various methods have been devised to solve systems of linear equations. This shows that no single method is best suited to all situations. These methods should be judged on the basis of their speed and accuracy. Speed is of importance in solving large systems because of the large volume of computations involved and accuracy is necessary because of the round off errors involved in performing these computations.

The methods for solving systems of linear equations can be classified as:

i) Direct methods

ii) Indirect (iterative) methods

By a **direct method**, we mean a method which calculates the required solution without any initial or intermediate approximations in a finite number of steps. Amongst the direct methods, we will describe the following:

a) Cramer's rule and its modified form

b) Gaussian elimination method and its variations

c) Triangular decomposition method

d) Solution of tridiagonal system of equations

An **indirect method** starts with an initial sequence of approximations and proceeds by calculating a sequence of further approximations, which eventually gives the solution as accurately as desired. The most commonly used methods in this category are:

a) Jacobi's method

b) Gauss-Seidel method

Even when a direct method does exist, an iterative method may be preferable because it is more efficient or more stable.

8.3 CRAMER'S RULE AND ITS MODIFIED FORM

According to Cramer's rule, the system (8.2) can be solved using,

$x_2 =$

$$x_r = \frac{\det(A_r)}{\det(A)} \qquad \ldots (8.3)$$

where the determinant $\det(A_r)$ is exactly the same as det(A), except the rth column of det(A), has been replaced by the column of constants $b_1, b_2, \ldots, b_m$.

Let us illustrate this method using the following example.

$A = \begin{bmatrix} a_{11} & a_{12} & a_3 \\ a_{21} & & \end{bmatrix}$

Example 1 Solve the following system of equations:

$$7x_1 + 6x_2 + 3x_3 = 19$$
$$3x_1 + 2x_2 - x_3 = 7$$
$$x_1 + 4x_2 + 2x_3 = -2$$

Solution From the given system of equations, we obtain,

$$A = \begin{bmatrix} 7 & 6 & 3 \\ 3 & 2 & -1 \\ 1 & 4 & 2 \end{bmatrix}; \quad x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}; \quad b = \begin{bmatrix} 19 \\ 7 \\ -2 \end{bmatrix}$$

$$\det(A) = \det\left[\begin{vmatrix} 7 & 6 & 3 \\ 3 & 2 & -1 \\ 1 & 4 & 2 \end{vmatrix}\right]$$

$$= 7\left[\begin{vmatrix} 2 & -1 \\ 4 & 2 \end{vmatrix}\right] - 6\left[\begin{vmatrix} 3 & -1 \\ 1 & 2 \end{vmatrix}\right] + 3\left[\begin{vmatrix} 3 & 2 \\ 1 & 4 \end{vmatrix}\right]$$

$$= 7(4+4) - 6(6+1) + 3(12-2)$$

$$= 7 \times 8 - 6 \times 7 + 3 \times 10$$

$$= 56 - 42 + 30 = 44$$

$$\det(A_1) = \det\left[\begin{vmatrix} 19 & 6 & 3 \\ 7 & 2 & -1 \\ -1 & 4 & 2 \end{vmatrix}\right] = 176$$

$$\det(A_2) = \det\left[\begin{vmatrix} 7 & 19 & 3 \\ 3 & 7 & -1 \\ 1 & -2 & 2 \end{vmatrix}\right] = -88$$

$$\det(A_3) = \det\left[\begin{vmatrix} 7 & 6 & 19 \\ 3 & 2 & 7 \\ 1 & 4 & -2 \end{vmatrix}\right] = 44$$

$$x_1 = \frac{\det(A_1)}{\det(A)} = \frac{176}{44} = 4$$

$$x_2 = \frac{\det(A_2)}{\det(A)} = \frac{-88}{44} = -2$$

$$x_3 = \frac{\det(A_3)}{\det(A)} = \frac{44}{44} = 1$$

The solution of the equations:

$$x_1 = 4,\ x_2 = -2,\ x_3 = 1$$

A

Alternative Method

Pre-multiplying both sides of (8.2) by the inverse of matrix A (i.e., A^{-1}), we get,

$$\boxed{A^{-1}A}x = A^{-1}b$$

Ax = b

$$1 . x = A^{-1}b$$

$$\boxed{x = A^{-1}b} \qquad \ldots (8.4)$$

where $A^{-1} = \dfrac{\operatorname{adj}(A)}{\det(A)}$ and adj(A) is the adjoint of the matrix A. Cramer's rule, of course, is identical to the formula (8.4).

The adjoint (or adjugate) of a square matrix A is the transpose of the matrix obtained by replacing each element of A by its cofactor. It is written as,

$$\operatorname{adj}(A) = [A_{ij}]^T = [A_{ji}]$$

Example 2 Given the following system of equations:

$$x_1 + x_2 - x_3 = 10$$
$$x_1 - 2x_2 + 3x_3 = -4$$
$$x_1 + x_2 + 2x_3 = 10$$

a) Find the determinant, adjoint and inverse of A.

b) Solve also the system of equations

Solution

$$A = \begin{bmatrix} 2 & 1 & -1 \\ 1 & -2 & 3 \\ 1 & 1 & 2 \end{bmatrix};\quad x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix};\quad b = \begin{bmatrix} 10 \\ -4 \\ 10 \end{bmatrix}$$

a) $$\det(A) = \det\left[\begin{vmatrix} 2 & 1 & -1 \\ 1 & -2 & 3 \\ 1 & 1 & 2 \end{vmatrix}\right] = -16$$

$$\text{Minor } M_{11} \text{ of } a_{11} = \det\left[\begin{vmatrix} -2 & 3 \\ 1 & 2 \end{vmatrix}\right] = -7$$

$$\text{Cofactor } A_{11} \text{ of } a_{11} = (-1)^{1+1} M_{11} = -7$$

$$\text{Minor } M_{12} \text{ of } a_{12} = \det\left[\begin{vmatrix} 1 & 3 \\ 1 & 2 \end{vmatrix}\right] = -1$$

$$\text{Cofactor } A_{12} \text{ of } a_{12} = (-1)^{1+2} M_{12} = 1$$

Similarly, other cofactors are computed and written as below:

$$A_{13} = 3$$

$$A_{21} = -3;\quad A_{22} = 5;\quad A_{23} = -1$$

$$A_{31} = 1;\quad A_{32} = -7;\quad A_{33} = -5$$

$$\operatorname{adj}(A) = \begin{bmatrix} -7 & 1 & 3 \\ -3 & 5 & -1 \\ 1 & -7 & -5 \end{bmatrix}^T = \begin{bmatrix} -7 & -3 & 1 \\ 1 & 5 & -7 \\ 3 & -1 & -5 \end{bmatrix}$$

$$A^{-1} = \frac{\operatorname{adj}(A)}{\det(A)} = \frac{-1}{16}\begin{bmatrix} -7 & -3 & 1 \\ 1 & 5 & -7 \\ 3 & -1 & -5 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{7}{16} & \frac{3}{16} & \frac{-1}{16} \\ \frac{-1}{16} & \frac{-5}{16} & \frac{7}{16} \\ \frac{-3}{16} & \frac{1}{16} & \frac{5}{16} \end{bmatrix} \quad \frac{1}{16}$$

b) Solution of Equations

Using the formula, $x = A^{-1}b$, we get,

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} \frac{7}{16} & \frac{3}{16} & \frac{-1}{16} \\ \frac{-1}{16} & \frac{-5}{16} & \frac{7}{16} \\ \frac{-3}{16} & \frac{1}{16} & \frac{5}{16} \end{bmatrix} \times \begin{bmatrix} 10 \\ -4 \\ 10 \end{bmatrix} = \begin{array}{cc} \begin{bmatrix} 3 \\ 5 \\ 1 \end{bmatrix} & \begin{array}{l} x_1 \\ x_2 \\ x_3 \end{array} \end{array}$$

On simplification, we get,

$$x_1 = 3, \ x_2 = 5, \ x_3 = 1$$

Some remarks on the above methods:

If the number of equations in a problem is small (i.e., three or four), we may use Cramer's rule safely, but if the problem involves more equations and unknowns, we have to be careful. Suppose a problem has n equations and the same number of unknowns and, if we have to use determinants, then $[(n^2 - 1)n! + n]$ multiplications are required to solve the system of equations by Cramer's rule. For large n, $n^2 \times n!$ is a good estimate of the number of multiplications.

We regard determinants as a useful tool in developing theory, but in practical, solving numerical analysis, we should disregard them. Some other methods, which are better than Cramer's rule, should be used. They do not require computation of determinants and cofactors. These methods are discussed in the subsequent sections and can be used for any number of equations.

8.4 GAUSSIAN ELIMINATION METHODS

The Gaussian elimination method reduces a system of linear equations to a simpler form. The method works in two stages:

- **Forward stage**

  This stage is concerned with the manipulation of equations in order to eliminate some unknowns from the equations and produce an upper triangular system.

- **Backward (or Back substitution) stage**

  This stage is concerned with the actual solution of the equations and uses the back substitution process on the reduced upper triangular system.

We shall describe this method by considering the following system of four equations, for the sake of convenience and simplicity: