5.5.2 Romberg Integration

Although the Trapezoidal rule is the easiest Newton-Cotes formula to apply, it lacks the degree of accuracy generally required. **Romberg Integration** is a method that has wide application because it improves the approximation fairly rapidly. Romberg integration is mostly designed for cases where the function to be integrated is known. This is because knowledge of the function permits the evaluation required for the initial implementations of the Trapezoidal rule.

Let f(x) be known either explicitly or as a tabulation of equispaced data:

x	x_0	x_1	x_2	...	x_n
f(x)	f_0	f_1	f_2	...	f_n

The first step in Romberg's method is to define a series of sums: I_{11}, I_{12}, I_{13}, ..., where

$$I_{11} = \frac{1}{2}(f_0 + f_n);\ h' = \frac{(b-a)}{n}, \quad \text{where } n = 1.$$

$$I_{12} = \left[I_{11} + f\left(a + \frac{h'}{2}\right)\right]$$

$$I_{13} = \left[I_{12} + f\left(a + \frac{h'}{4}\right) + f\left(a + \frac{3h'}{4}\right)\right]$$

$$I_{14} = \left[I_{13} + f\left(a + \frac{h'}{8}\right) + f\left(a + \frac{3h'}{8}\right) + f\left(a + \frac{5h'}{8}\right) + f\left(a + \frac{7h'}{8}\right)\right]$$

f(x) → a

→ h

From these sums, various other values T_{11}, T_{12}, T_{13}, ..., are computed using the following relations:

$$T_{11} = h' I_{11}$$

$$T_{12} = \frac{h'}{2} I_{12}$$

$$T_{13} = \frac{h'}{4} I_{13}$$

$$T_{14} = \frac{h'}{8} I_{14}, \text{ and so on.}$$

$$T_{15} = \frac{h'}{16} I_{15}$$

Note: h is the difference between consecutive values of x, but h' is the difference between the upper and lower limits of the integral.

Romerg's table is as follows:

Integration Sums	Calculation of Approximations			
I_{11}	T_{11}			
		T_{22}		
I_{12}	T_{12}		T_{33}	
		T_{23}		T_{44}
I_{13}	T_{13}		T_{34}	
		T_{24}		
I_{14}	T_{14}			

j= COL
K=ROW

With the values of T_{11}, T_{12}, ..., we compute the first-order Romberg integration as follows:

$$T_{22} = T_{12} + \frac{1}{3}(T_{12} - T_{11})$$

$$T_{23} = T_{13} + \frac{1}{3}(T_{13} - T_{12})$$

$$T_{24} = T_{14} + \frac{1}{3}(T_{14} - T_{13})$$

We now compute the second-order Romberg integration:

$$T_{33} = T_{23} + \frac{1}{15}(T_{23} - T_{22})$$

$$T_{34} = T_{24} + \frac{1}{15}(T_{24} - T_{23})$$

j= 2

Calculation of third-order Romberg integration:

$$T_{44} = T_{34} + \frac{1}{63}(T_{34} - T_{33}), \text{ etc.}$$

K=3

General formula to calculate various values in the table is,

23 2 24 23

$$T_{j+1,K+1} = T_{j,K+1} + \frac{1}{4^j - 1}\left[T_{j,K+1} - T_{jK}\right] \quad \ldots (5.37)$$

34 4 2 15

The procedure continues until the difference between two successive values on the diagonal agree to the desired accuracy. In each column, the bottom number is hopefully the most accurate number. Trapezoidal and Simpson's rules are sometimes inadequate for problem contexts where high efficiency and low errors are needed.

Romberg method is one technique that is designed to obviate these shortcomings. It has been reported in literature that the error in column k of the Romberg table diminishes by about a factor of $\boxed{\frac{1}{4^{k+1}}}$ as one progresses down its rows. The algorithm is clear, although the justification is quite hard.

Finally, one might feel that accuracy of these integration formulas can be increased using higher-order formulas until sufficient accuracy is obtained. However, there are two reasons why this strategy might fail; namely, that the function may not be adequately approximated by a polynomial, in which case the truncation error becomes large, or, that the formulas may be subject to excessive rounding error. One interesting experience about the usage of formulas with an even number of strips is that they not only give zero error for polynomials upto degree n but also for polynomials of degree n + 1. In view of this extra accuracy an even-order formulas would normally be used. However, the exception to this is the Trapezoidal rule, which is valuable because of its simplicity. The other point of significance is that the error depends on a derivative of the function to be integrated.

To summarize there is clearly a major gain in efficiency in using methods which are higher order than the Trapezoidal rule, such as Simpson's rule and especially Romberg integration. All in all, Romberg integration is a powerful but quite simple method, which we recommend for general use. For a given number of intervals, it is much more accurate than the Trapezoidal rule, and quite a bit more accurate than Simpson's rule, but does not need any more function evaluations.

Example 5 (a) Using Romberg integration method, evaluate the integral: $\int_1^{2.6} \frac{dx}{x}$. Let n = 8. $x_n = x_8$

(b) Write a computer program to implement the above procedure.

Solution (a) Tabulated values are as follows:

$$h = \frac{2.6-1}{8} = 0.2$$

Since, $a = 1$, $b = 2.6$ and $n = 8$, the functional values are calculated as below: x_4 $x_n = b$

$x_0 = a$

x	1.0	1.2	1.4	1.6	1.8	2.0	2.2	2.4	2.6
f(x)	1.000	0.833	0.714	0.625	0.556	0.500	0.455	0.417	0.385

f_0 f_4 f_8

Calculations of I_{11}, I_{12}, I_{13} and I_{14}.

$$I_{11} = \frac{1}{2}(f_0 + f_8)$$

$$= \frac{1}{2}(1.00 + 0.385) = 0.6925$$

g(x) f(1.8)

$$h' = \frac{(b-a)}{n} = 2.6 - 1 = 1.6 \text{ [since } n = 1]$$

1'

$$T_{11} = h'\, I_{11} = 1.6 \times 0.6925 = 1.1080$$

$$I_{12} = \left[I_{11} + f\left(a + \frac{h'}{2}\right)\right]$$

f(1.8)

$$= [I_{11} + f_4] = 0.6925 + 0.556 = 1.2485$$

$$T_{12} = \frac{h'}{2} I_{12} = \frac{1.6}{2} \times 1.2485 = 0.9988$$

1.4 2.2

$$I_{13} = \left[I_{12} + f\left(a + \frac{h'}{4}\right) + f\left(a + \frac{3h'}{4}\right)\right]$$

$$= [I_{11} + f_2 + f_6]$$

$$= 1.2485 + 0.714 + 9.455 = 2.4175$$

$$T_{13} = \frac{h'}{4} I_{13} = \frac{1.6}{4} \times 2.4175 = 0.9670$$

$$I_{14} = \left[I_{13} + f\left(a + \frac{h'}{8}\right) + f\left(a + \frac{3h'}{8}\right) + f\left(a + \frac{5h'}{8}\right) + f\left(a + \frac{7h'}{8}\right)\right]$$

$$= [I_{13} + f_1 + f_3 + f_5 + f_7]$$

$$= 2.4175 + 0.833 + 0.625 + 0.500 + 0.417 = 4.7925$$

$$T_{14} = \frac{h'}{8} I_{14} = \frac{1.6}{8} \times 4.7925 = 0.9585$$

Calculating other values in the table:

$$T_{22} = T_{12} + \frac{1}{3}(T_{12} - T_{11})$$

$$= 0.9988 + \frac{1}{3}(0.9988 - 1.1080)$$

$$= 0.9988 - 0.0364 = 0.9624$$

$$T_{23} = T_{13} + \frac{1}{3}(T_{13} - T_{12})$$

$$= 0.9670 + \frac{1}{3}(0.9670 - 0.9988)$$

$$= 0.9670 - 0.0106 = 0.9564$$

$$T_{24} = T_{14} + \frac{1}{3}(T_{14} - T_{13})$$

$$= 0.9585 + \frac{1}{3}(0.9585 - 0.9670)$$

$$= 0.9557$$

$$T_{33} = T_{23} + \frac{1}{15}(T_{23} - T_{22})$$

$$= 0.9564 + \frac{1}{15}(0.9564 - 0.9624)$$

$$= 0.9564 - 0.0004 = 0.9560$$

$$T_{34} = T_{24} + \frac{1}{15}(T_{24} - T_{23})$$

$$= 0.9557 + \frac{1}{15}(0.9557 - 0.9564) = 0.9557$$

$$T_{44} = T_{34} + \frac{1}{63}(T_{34} - T_{33})$$

$$= 0.9557 + \frac{1}{63}(0.9557 - 0.9560)$$

$$= 0.9557 - 0.0000 = 0.9557$$

Displaying these values in tabular form, we have,

Interval	Trapezoidal Sums	Romberg Values		
		First-Order	Second-Order	Third-Order
1	1.1080			
		0.9624		
2	0.9988		0.9560	
		0.9564		0.9557
4	0.9670		0.9557	
		0.9557		
8	0.9585			

We note that the final result, 0.9557, is accurate upto 4 dp.

It is often useful to have predetermined a specific value for n and instead modify the algorithm slightly to allow the procedure to continue until a value of n is found that satisfied $|T_{n,n} - T_{n-1,n-1}| < \in$, for a given tolerance $\in$.