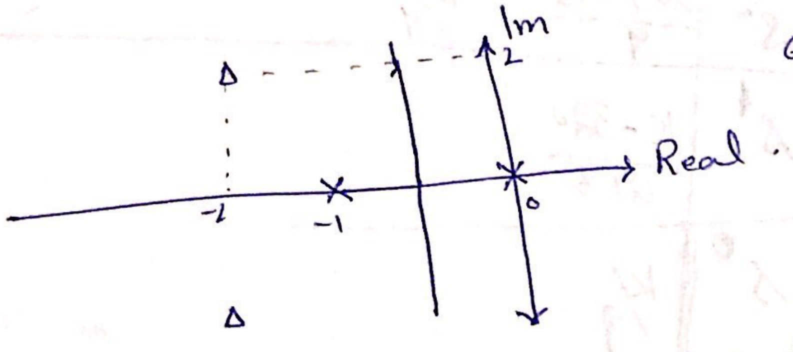
Design Via Root Locus

Assume the RL of a sys.

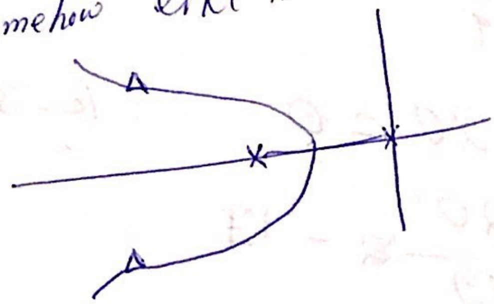


$G(s) = \frac{1}{s(s+1)}$

and assume Δ represent desired pole locations.

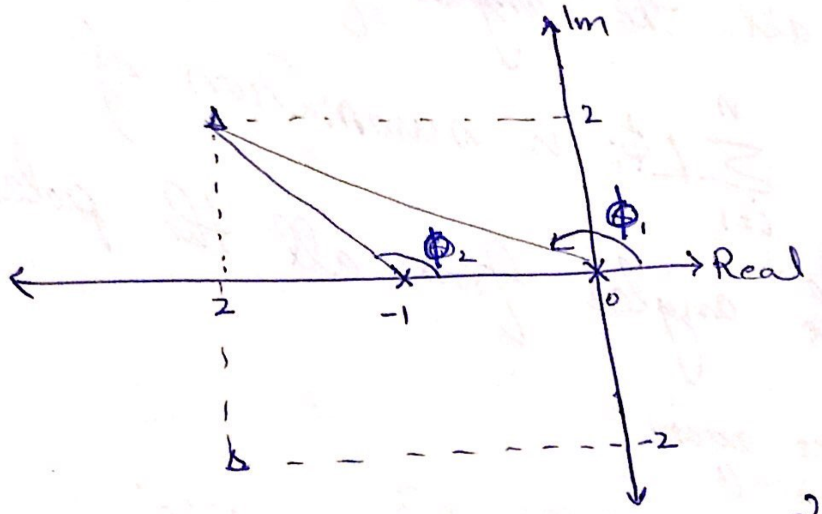
⇒ Notice that to achieve Δ, the locus must pass through these pts.

i.e somehow like this



⇒ Now recall that if these pts. are to be on the locus, they must satisfy the ① angle condition & ② Magnitude condition.

① Satisfying Phase condition



$$\phi_1 = -\left[180^\circ - \tan^{-1}\left(\tfrac{2}{2}\right)\right] = -\left[180^\circ - 45^\circ\right] = -135^\circ$$

$$\phi_2 = -\left[180^\circ - \tan^{-1}\left(\tfrac{2}{1}\right)\right] = -\left[180^\circ - 63.4^\circ\right] \simeq -115^\circ$$

θ_Δ = $-135^\circ - 115^\circ = -250^\circ$

Phase at $-2+2j$

However, we need θ_Δ to be -180° according to the phase condition.

So an additional ($-180^\circ + 250^\circ = 70^\circ$) of phase is required for Δ to be on the locus.

⇒ The formula is

$$\theta_{req} + \left[\sum_{i=1}^{m} \angle\psi_i - \sum_{i=1}^{n} \angle\phi_i\right] = -180^\circ$$

where $\sum_{i=1}^{m} \angle\psi_i$ is summation of all the angles from zeros and $\sum_{i=1}^{n} \angle\phi_i$ is summation of all the angles from all the poles.

=> So once again

$$\theta_{req} \cancel{+} [0^\circ - \phi_1 - \phi_2] = -180^\circ$$

θ_{req} ↓ required phase

0° ↓ ∵ there are no zeros

$$\theta_{req} + [0^\circ - 135^\circ - 115^\circ] = -180^\circ$$

$$\theta_{req} - 250^\circ = -180^\circ$$

$$\theta_{req} = -180^\circ + 250^\circ = 70^\circ$$

$$\boxed{\theta_{req} = 70^\circ}$$

So we need to have an additional 70° for Δ $(-2 \pm 2j)$ to be on the locus.

$\Rightarrow$ To do this let us add a dynamic controller.

Let $\quad K(s) = K_p + K_d s$

$$= K_p\left[1 + \frac{K_d}{K_p} s\right] \times$$

or

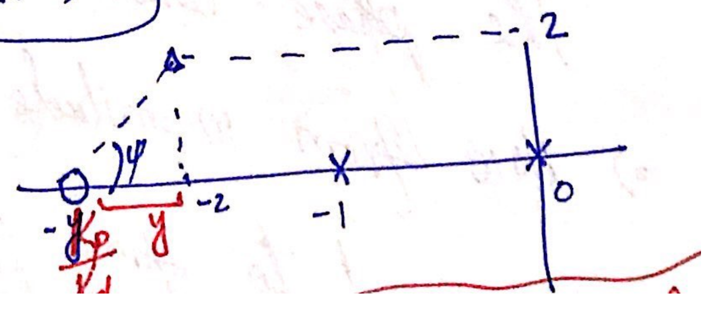
$$= \underbrace{K_d\left(s + \frac{K_p}{K_d}\right)}_{\downarrow}$$

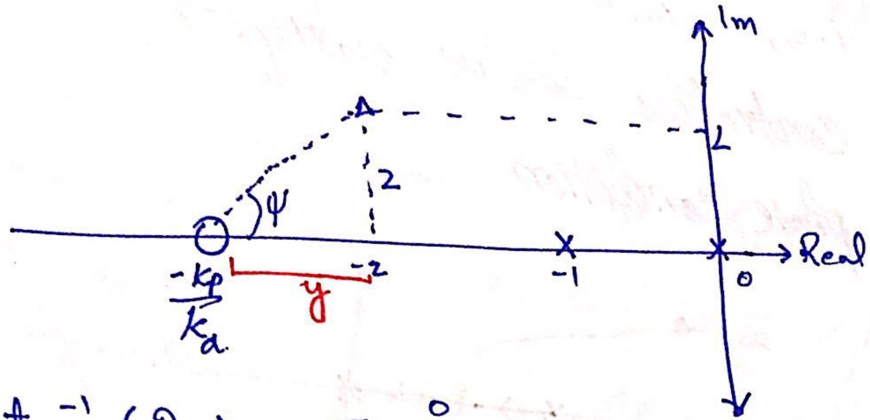
Now this compensator zero has to provide the additional 70° phase.

$$\cancel{\text{i.e. } \tan^{-1}\left(\frac{K_p/K_d}{y}\right) = 70^\circ}$$

$$\cancel{\frac{K_p/K_d}{y} = \tan(70^\circ)}$$

$$\cancel{y = \frac{K_p/K_d}{\tan(70^\circ)}}$$

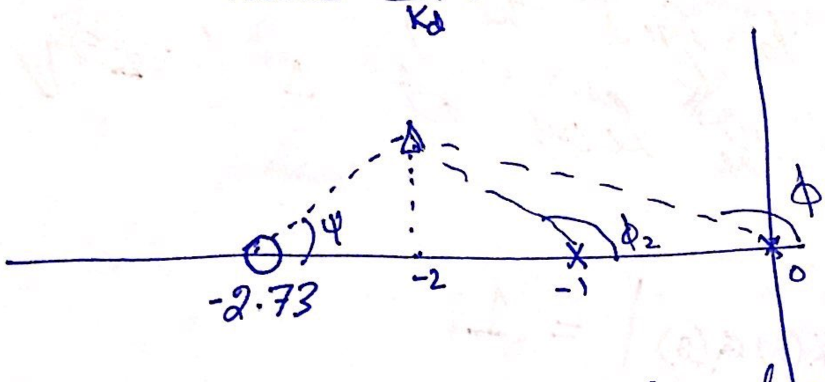




$$\tan^{-1}\left(\frac{2}{y}\right) = 70^\circ$$

$$\frac{2}{y} = \tan 70^\circ \Rightarrow y = \frac{2}{\tan 70^\circ} = 0.73$$

hence $\frac{K_p}{K_d} = +2 + 0.73 = +2.73$



⇒ So the controller zero is placed at

$S = -2.73$

⇒ Now watch again

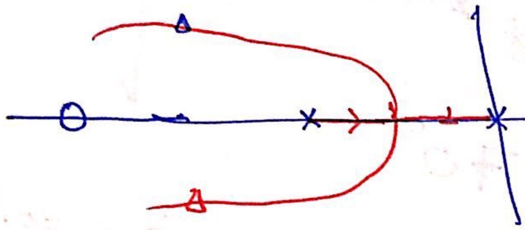
$$L(s) = \frac{K_d\left(S + \frac{K_p}{K_d}\right)}{S(S+1)}$$

$$\theta_{req} + \left[\overset{\psi}{70^\circ} - \overset{\phi_1}{135^\circ} - \overset{\phi_2}{115^\circ}\right] = -180^\circ$$

$$\theta_{req} + \left[-180^\circ\right] = 180^\circ$$

$$\theta_{req} = 0^\circ$$

i.e, with the use of PD controller we've satisfied the phase condition

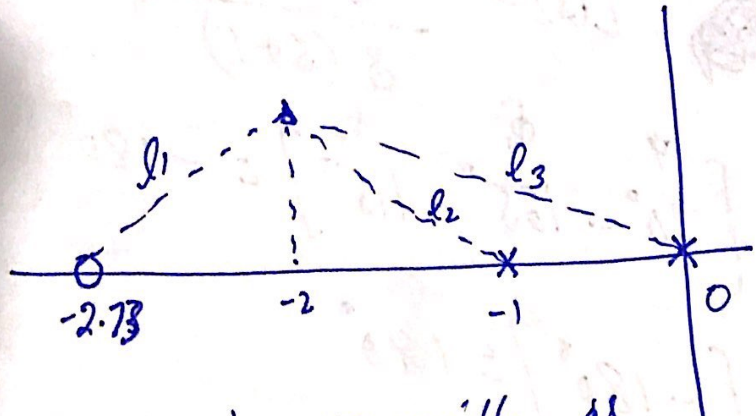


i.e., Now the locus passes through Δ

⇒ Now from magnitude condition we need to find the gain which stops the locus at Δ $(-2 \pm 2j)$

$$\left| K(s)\,G(s) \right|_{\Delta} = 1$$

$$\left| \frac{K_d\,(s+2.73)}{s(s+1)} \right|_{\Delta} = 1 \rightarrow (1)$$



Eq (1) can be re-written as

$$\left| K_d \frac{l_1}{l_2 l_3} \right| = 1 \rightarrow (2)$$

$$l_1 = \sqrt{2^2 + 0.13^2} = 2.13$$

$$l_2 = \sqrt{2^2 + 1^2} = 2.24$$

$$l_3 = \sqrt{2^2 + 2^2} = 2.83$$

put in eqn. (2)

$$\left| K_d \frac{2.13}{2.24 \times 2.83} \right| = 1$$

$$\left| K_d \frac{2.13}{6.34} \right| = 1$$

$$K_d \times 0.34 = 1$$

$$\boxed{K_d = 2.94}$$

Verify:→ $1 + K(s)G(s) = 0$

$$1 + K_d\left(s + \frac{K_p}{K_d}\right) \Big/ s(s+1) = 0$$

$$s^2 + s + 2.94s + 8.03 = 0 \qquad \because K_d = 2.94, \quad \frac{K_p}{K_d} = 2.73$$

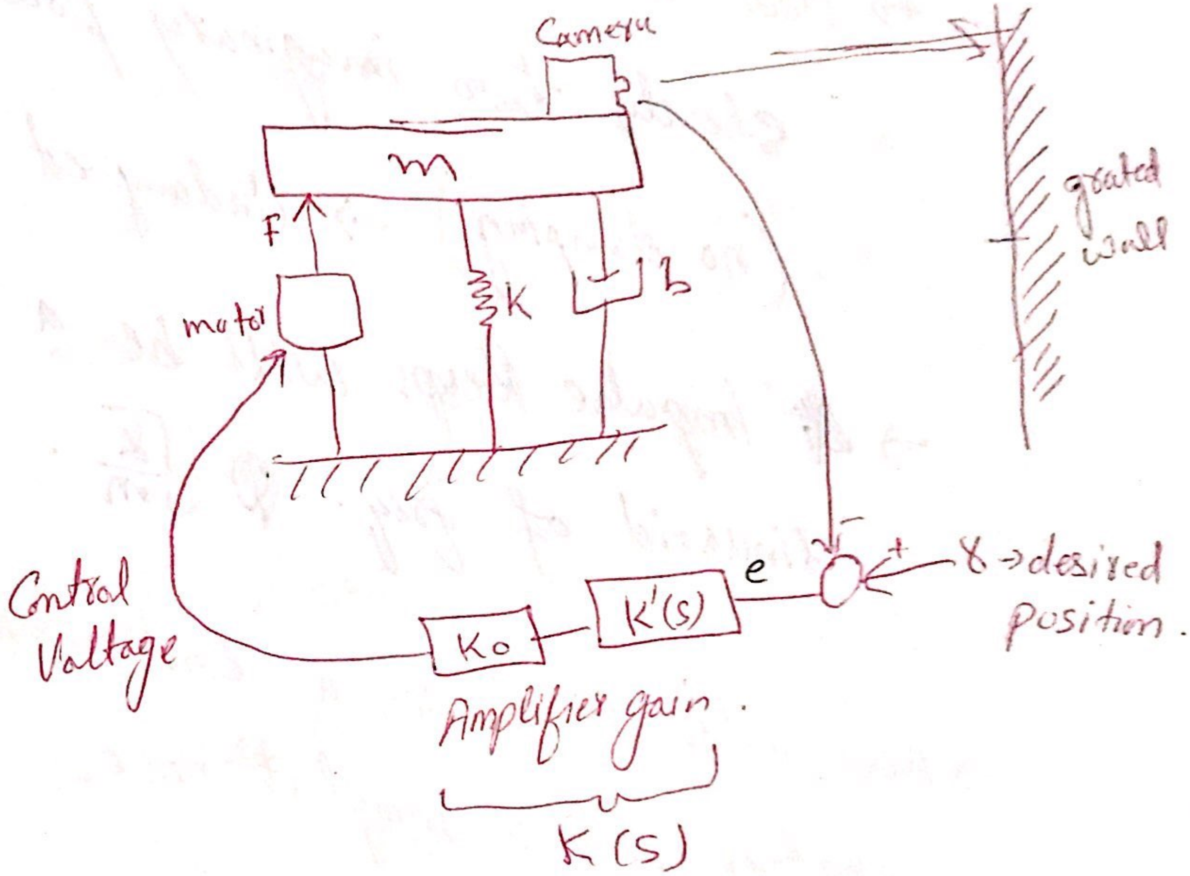
$$s^2 + (3.94s) + 8.03 = 0$$

$$s_{1,2} = \frac{-3.94 \pm \sqrt{3.94^2 - 4 \times 8.03}}{2} = \frac{-3.94 \pm j\sqrt{16.6}}{2}$$

$$= -1.93 \pm j\,2.04$$

The minor deviation is bcz we approximated many values

Design Example 1



We want to control the mass position s.t the c/L has a damping of 0.707 ⇒ $\theta = 45°$

$\theta = \sin^{-1}\zeta$

EOMs $F = m\ddot{x} + b\dot{x} + kx$

$$\frac{X(s)}{F(s)} = \frac{1}{ms^2 + bs + k} \Rightarrow \frac{1/m}{s^2 + \frac{b}{m}s + \frac{k}{m}}$$

if we assume that there is no damping $b \simeq 0$ then.

$$\frac{X(s)}{P(s)} = G(s) = \frac{1/m}{s^2 + \frac{K}{m}}$$

→ How will the impulse Response be??

clearly two imaginary poles

(no damping) → Undamped

→ Impulse Resps will be a

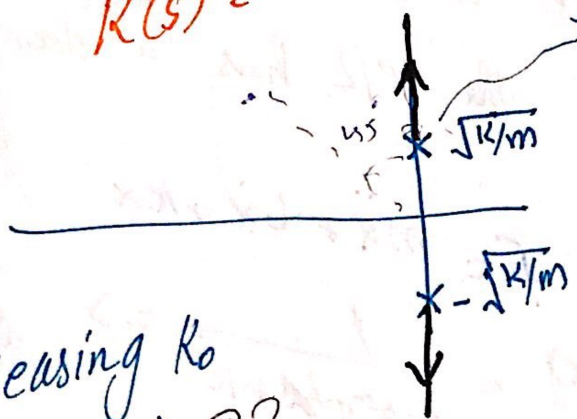
sinusoid of freq. $\sqrt{\frac{K}{m}}$.

→ Now we've to design a controller.

starting with a simplest case.

Proportional Controller.

$K(s) = K_0$



$-\phi_d + 0° - 90° = -180°$

$-\phi_d = -90°$

$\phi_d = 90°$

Can we achieve $\xi = 0.707$ with increasing or decreasing K_0 (the proportional gain)??

The damping is not increasing at all so proportional controller is not sufficient

What do we want ??

We actually want the locus to be pulled to the left half plane.

How we do this ??

A property of the R/L is that the zeros attract the locus towards themselves & the poles push the locus away. So if we want to pull the locus to the left half plane we need to have a zero somewhere in the LHP.

So let us try PD control

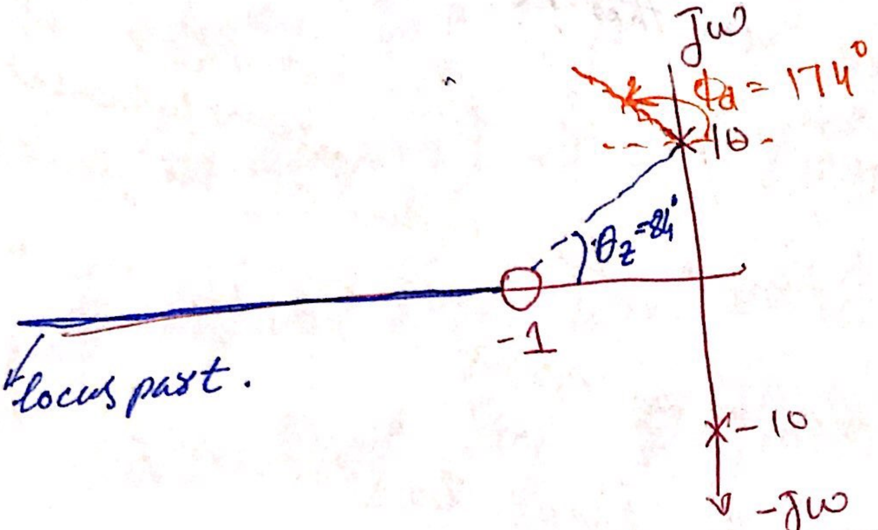
$$K(s) = K_p + K_d s = K_d\left(s + \frac{K_p}{K_d}\right)$$

$$\Rightarrow G(s)K(s) = \frac{K_d}{m}\,\frac{(s + K_p/K_d)}{s^2 + K/m} = K_o L(s)$$

where $K_o = \dfrac{K_d}{m}$

Let us assume

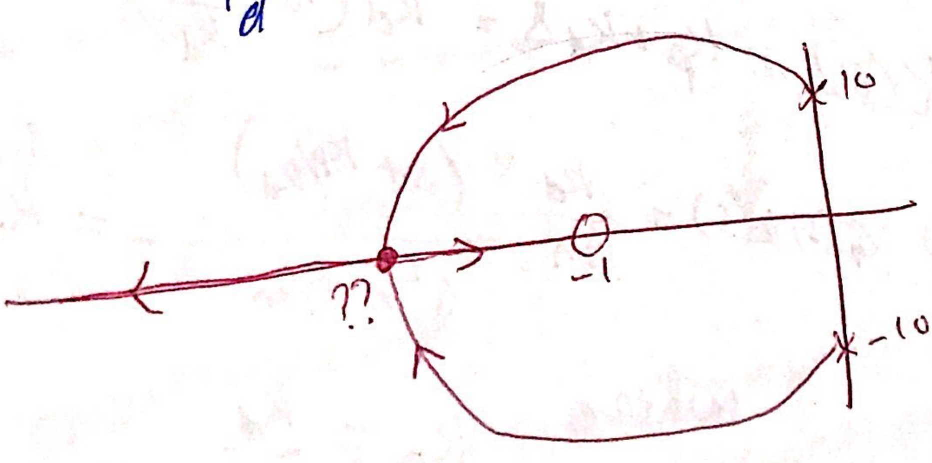
$$\frac{K_p}{K_d} = 1 \quad \text{and} \quad \frac{k}{m} = 100$$



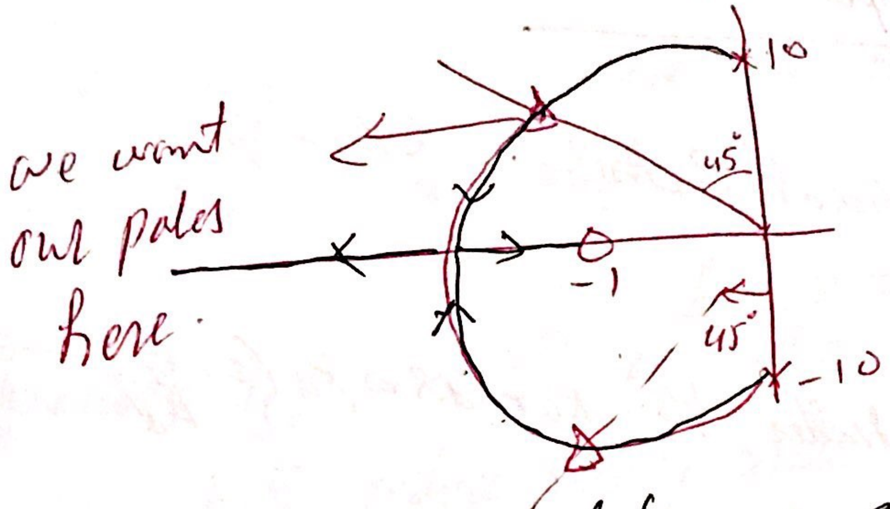
① One pole will go to ∞ and the other will go to zero at -1.

② $\phi_{dep} = ?$ $\quad -\phi_d + 84° - 90° = -180°$

$$\phi_d = 174°$$



Remember we want $\zeta = 0.707 \Rightarrow \theta = 45^\circ$.



The magnitude condition and phase conditions will be satisfied on the locus.

Break away

$L(s) = \frac{s+1}{s^2+100}$

$b\frac{da}{ds} - a\frac{db}{ds} = 0$

$(s^2+100) - (s+1)(2s) = 0$

$s^2+100 - 2s^2 - 2s = 0$

~~$3s^2 + 2s + 100$~~

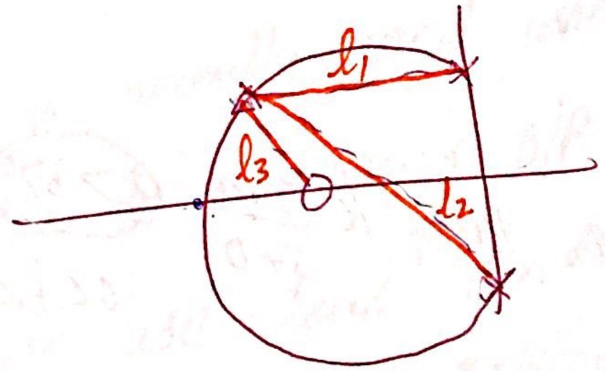
$-s^2 - 2s + 100 = 0$

$s^2 + 2s - 100 = 0$

$\frac{-2 \pm \sqrt{4+400}}{2}$

$-1 \pm \sqrt{101}$, -11.05, 9.05

$$\left| \frac{K_d}{m} \frac{s + \frac{K_p}{K_d}}{s^2 + K/m} \right|_{s=\Delta} = 1$$

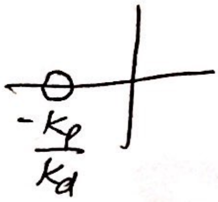


the $\left| \frac{K_d}{m} \frac{l_3}{l_1 l_2} \right| = 1 \quad K_d = \frac{m l_1 l_2}{l_3}$

Some Simple Controllers

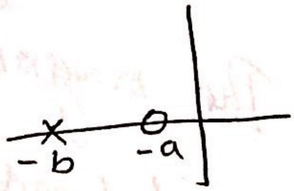
① Proportional Controller: $K(s) = K_p$

② P-D Controller $K(s) = K_p + K_d s \Rightarrow K_d\left(s + \frac{K_p}{K_d}\right)$



Similar effect

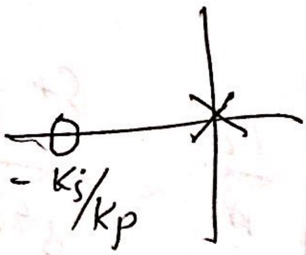
③ Lead Controller $K(s) = K\frac{s+a}{s+b}$, $0 < a < b$



④ P-I Controller $K(s) = K_p + \frac{K_i}{s}$

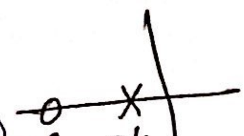
$$= \frac{K_p s + K_i}{s}$$

$$= K_p\left(\frac{s + K_i/K_p}{s}\right)$$



$\int e\,dt$

⑤ Lag Controller $K(s) = K\frac{s+a}{s+b}$ $a > b > 0$



$0 < b < a$

We'll focus on these.

→ Control ② and ③ provide similar effect on the R/L
i.e they both increase the damping and pull the locus to the left.

→ Controllers ④ and ⑤ are also similar (have similar effect)
i.e. both reduce steady state tracking error.

→ [Poles push the locus
Zeros pull " "]

→ Designing a controller in R/L mean shaping the locus s.t some desired poles become a part of the locus

→ PD controller is normally used when two sensors are available for measuring o/p & its derivative otherwise we use lead controller
∵ $K_c(s)_{PD} = K_p + K_d s$ is improper (only 0 no pole)
So it can't be realized practically.

→ If we've two sensors measuring y and $\dot{y}$ then.

$$K(s) + K_p\, y + K_d\, \dot{y}$$

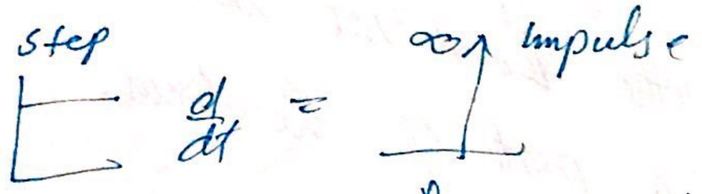
but if we do not have $\dot{y}$ measurement then we've to take derivative of y and pure derivative is not possible.

e.g ① if $y = \cos\omega t$

$$\dot{y} = -\omega \sin\omega t$$

↓ increasing ω will saturate $\dot{y}$.

②



how can we implement ∞ practically.

→ So we normally use an approximation of PD control which is "lead".

PI Vs lag

step signals. If exact 0 e_{ss} is required for ref: tracking or dist: rejection we use PI control otherwise we use a lag controller.

$$E_{ss}\Big|_{R_{step}} = \frac{1}{1+G(0)K(0)}$$

$$E_{ss}\Big|_{d_{step}} = \frac{-1}{\frac{1}{G(0)} + K(0)}$$

$E_{ss} \downarrow$ $K(0) \uparrow$

$$K(0)_{PI} = \infty$$

$$K(0)_{lag} = \frac{a}{b}$$

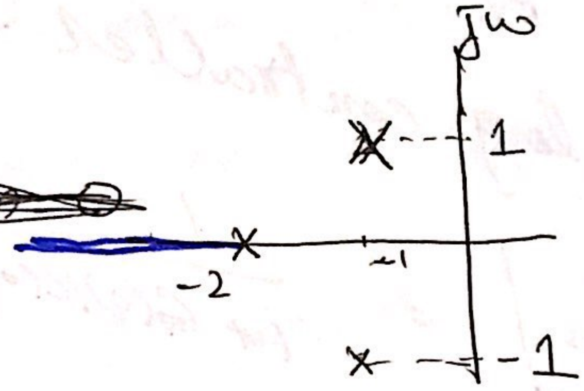
So if e_{ss} exact 0 is not required then lag is a better option bcz pure integrators are problematic they tend to windups tend to saturate.

→ For practical application we normally preffer to use lag controllers insteads.

Examp: Draw the Root locus

$$G(s) = \frac{1}{(s+2)(s^2+2s+2)} \qquad -1 \pm j$$

①



②

$n = 3, \; m = 0$

no. of asymptotes $= n - m = 3$

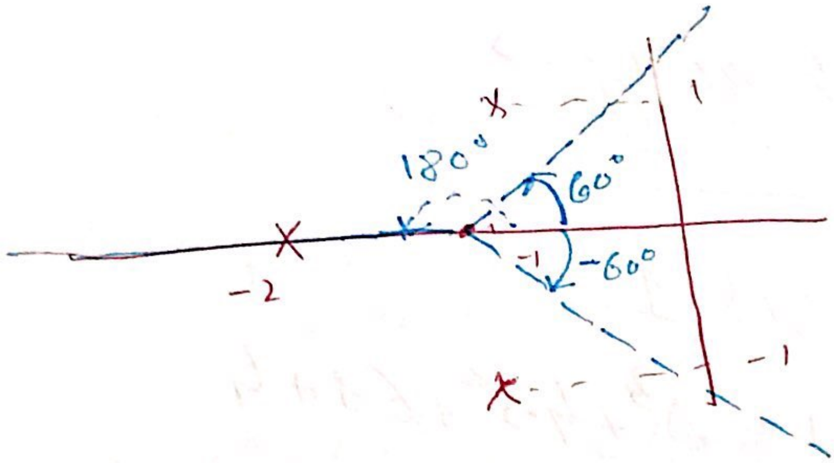
$$\alpha = \frac{-2 - 1 + \cancel{j} - 1 - \cancel{j}}{3} = \frac{-4}{3} = -1.33$$

$$\phi_i = \frac{180^\circ + 360(i-1)}{n-m}$$

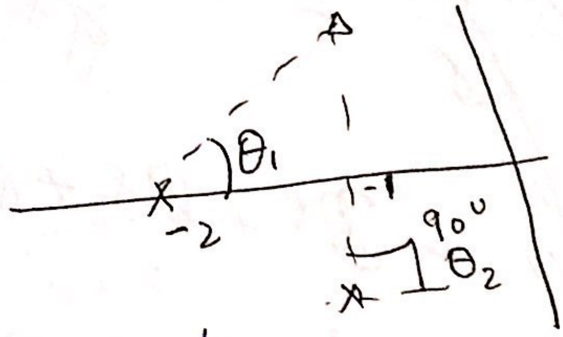
$$\phi_1 = \frac{180^\circ}{3} = 60^\circ$$

$$\phi_2 = \frac{180^\circ + 360^\circ}{3} = 180^\circ$$

$$\phi_3 = \frac{180+(360\times 2)}{3} = 300^\circ = -60^\circ$$



③ $\phi_d = ?$

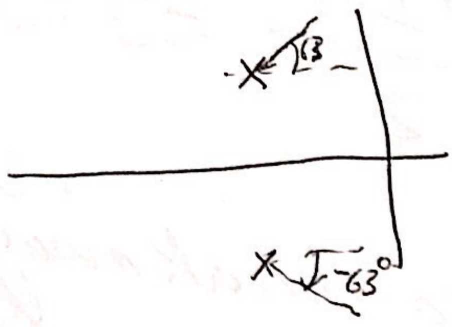


$$-\phi_d + 0^\circ - 90^\circ - \tan^{-1}\left(\frac{1}{2}\right) = -180^\circ$$

$$-\phi_d - 90^\circ - \cancel{26.57}^\circ = -180^\circ$$

$$\phi_d = 180^\circ - 90^\circ - \cancel{26.56}^\circ$$

$$= \cancel{63.43}^\circ$$



Scanned with CamScanner

(4) Break away :->

$$a(s) = 1$$

$$b(s) = s^3 + 4s^2 + 6s + 4$$

$$b(s)\frac{da(s)}{ds} - a(s)\frac{db(s)}{ds} = 0$$

$$0 - 3s^2 + 8s + 6 = 0$$

$$s^2 - 4s - 3 = 0$$

$$r_1, r_2 = \frac{4 \pm \sqrt{16 - 12}}{2}$$

$$\frac{4 \pm 4}{2} = \boxed{r_1 = 4 , r_2 = 0}$$

Both invalid

so no breakaway point exist.

⑤ $j\omega$-crossing:

$$1 + GK = 0$$

$$1 + \frac{K}{(s+2)(s^2+2s+2)}$$

$$\Rightarrow s^3 + 4s^2 + 6s + 4 + K$$

Routhian array:

s^3	1	6
s^2	4	$4+K$
s	$\frac{24-4-K}{4}$	
s^0	$4+K$	

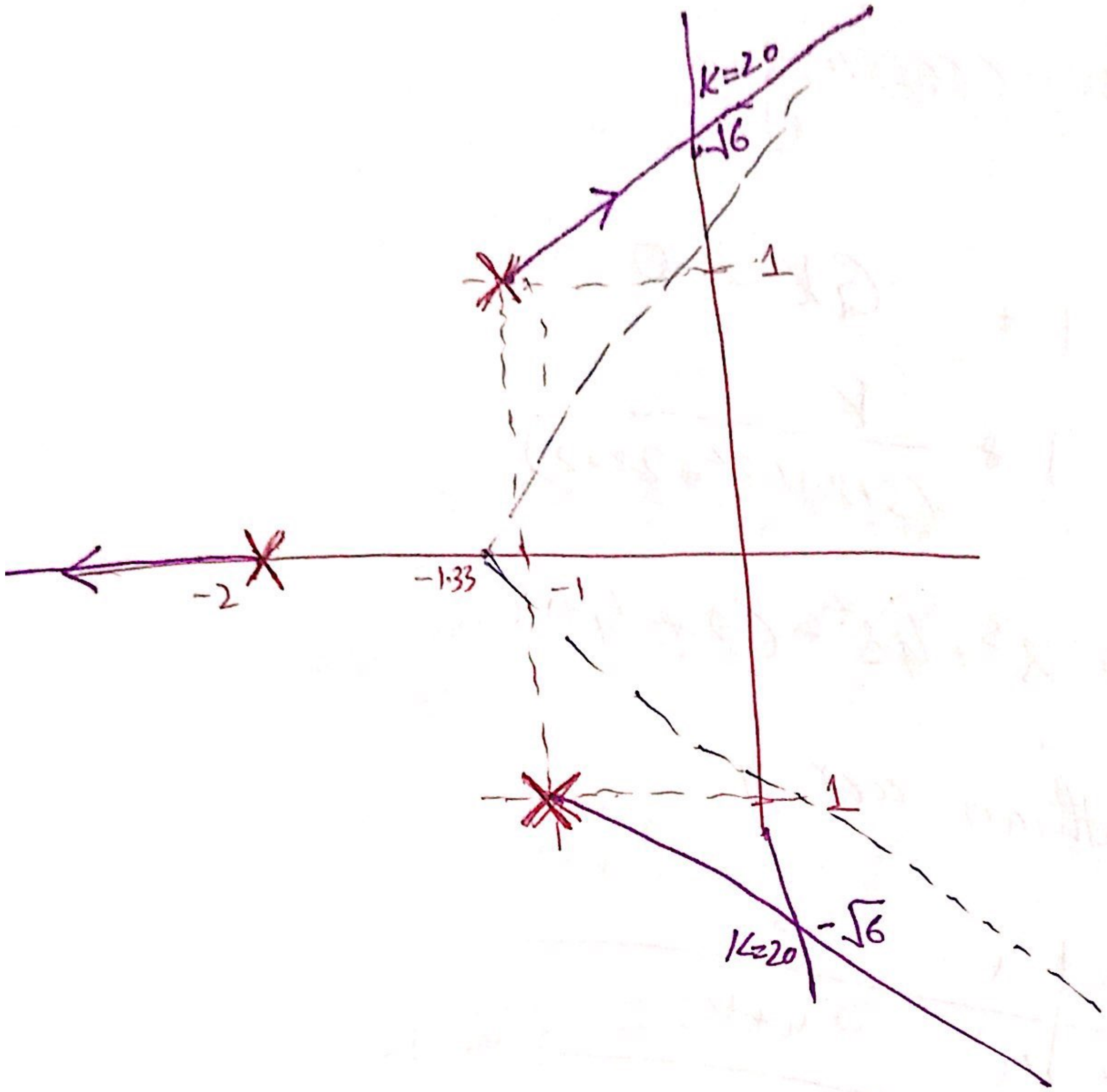
$$\frac{24-4-K}{4} = 0$$

$$20 - K = 0 \Rightarrow K = 20$$

$$4s^2 + 4 + 20 = 0$$

$$4s^2 + 24 = 0$$

$$s^2 = -6 \qquad s_{1,2} = \pm j\sqrt{6}$$



Jw-crossing an alternate Method

Find a solution to $1 + K\,L(j\omega) = 0$

$$1 + K L(s) = 1 + \frac{K}{s^3 + 4s^2 + 6s + 4} = 0 \qquad s^3 + 4s^2 + 6s + 4 + K = 0$$

$$1 + K L(j\omega) = -j\omega^3 - 4\omega^2 + 6j\omega + 4 + K = 0$$

$$= j(-\omega^3 + 6\omega) + (-4\omega^2 + 4 + K) = 0$$

↳ this complex number is zero iff real = 0 and imag = 0

4-20

Put

$$6\omega - \omega^3 = 0$$

$$\omega(6-\omega^2) = 0$$

$$\omega = 0$$

$$\boxed{\omega = \pm\sqrt{6}}$$

for finding corresponding K (gain)

$$-4\omega^2 = -4 - K$$

$$\omega^2 = \frac{K+4}{4}$$

As $\omega = \sqrt{6}$

$$6 \times 4 - 4 = K$$

$$24 - 4 = K$$

$$\boxed{K = 20}$$