$$V(s) = \frac{F_0}{s(ms+b)} + \frac{mV_0}{ms+b}$$

$$= \frac{F_0}{b}\left\{\frac{b/m}{s(s+b/m)} + \frac{V_0}{s+b/m}\right\}$$

Taking $\mathcal{L}^{-1}$ we get

$$\boxed{V(t) = \frac{F_0}{b}\left(1 - e^{-b/m\,t}\right) + V_0 e^{-b/m\,t}}$$

Is it easier???

Transfer function & impulse Response

A general T/f.

$$H(s) = \frac{b_1 s^m + b_2 s^{m-1} + \cdots + b_{m+1}}{s^n + a_1 s^{n-1} + \cdots a_0}$$

For physical systems

degree of numerator polynomial is

degree " denominator "

i.e $m \leq n$

such systems are called
"PROPER"

$H(s) = \frac{1}{s}$ is a pure integrator strictly (Proper).

$H(s) = s$ " " " differentiator (improper)

(so a pure differentiator is not physically possible) we can approximate it.

→ Now we are at a pt where we know how to represent physical systems mathematically (e.g. EOMs) then we can find natural resp, impulse response and H/f etc

→ But now let us search ways and tools to ~~find th~~ find (Predict) the response from the EOMs and/or H/f immediately without doing all tedious stuff of solving and predicting.

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Example

$$H(s) = \frac{3s^2 + 16s + 19}{s^2 + 5s + 6}$$

$h(t) = ?$

① Bring it to a strictly proper form

Long division

$$\begin{array}{r|l} & 3 \\ s^2+5s+6 & 3s^2+16s+19 \\ & -\,(3s^2+15s+18) \\ \hline & s+1 \end{array}$$

So $H(s) = 3 + \dfrac{s+1}{s^2+5s+6}$

$$= 3 + \frac{s+1}{(s+2)(s+3)}$$

$\Downarrow$ by partial fractions.

② $H(s) = 3 + \dfrac{2}{s+3} - \dfrac{1}{s+2}$

$$\boxed{h(t) = 3\delta(t) + 2e^{-3t} - e^{-2t}}$$

important ⟹ Notice that when $H(s)$ is proper we will have the i/p appeared directly in the o/p (direct feedthrough)

Now once again look at the t/f

$$H(s) = \frac{Num}{(s+2)(s+3)}$$

poles at -2 and -3,

and when we look at the $h(t)$ we see e^{-2t} and e^{-3t} which means the poles made up the resp: of the sys

⟹ This means that the poles of the t/f are the basic building blocks of the response of the system.

Poles and zeros

$$H(s) = k\,\frac{(s-z_1)(s-z_2)\cdots}{(s-P_1)(s-P_2)\cdots}$$

poles $s = P_1$, $s = P_2$ ...

zeros $s = z_1$, $s = z_2$ ...

poles of the TF are the basic building blocks of the resp: of the system. They are also called "modes" of the systems.

The zeros give the residues for each mode telling how much it contributes in the total resp.

[Poles dictate the quality of resp: while zeros dictate the quantity.]

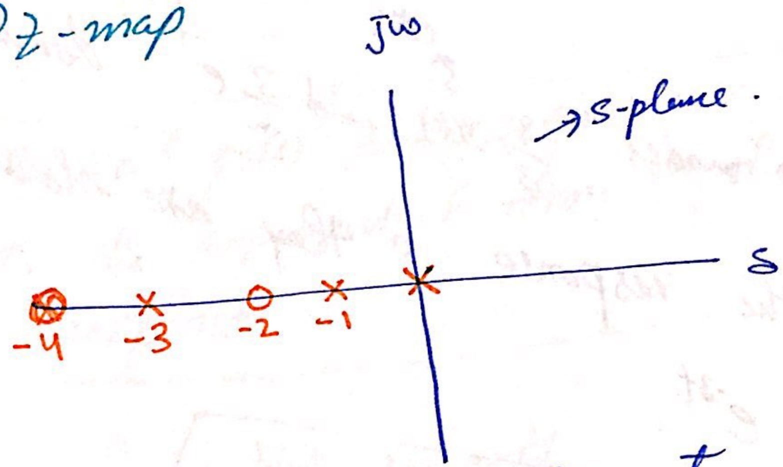
Case 1 :→ Non-Repeated Real Roots.

Example $H(s) = \dfrac{(s+2)(s+4)}{s(s+1)(s+3)}$ → 3^{rd} order, Proper, linear,

↓

T/f of system.

PZ-map



So we want know about the system resp.?

$$H(s) = \frac{A}{s} + \frac{B}{s+1} + \frac{C}{s+3}$$

$$A = \left.\frac{(s+2)(s+4)}{(s+1)(s+3)}\right|_{s=0} = 8/3$$

$$B = -3/2 \,,\quad C = -1/6$$

$$H(s) = \underbrace{\frac{8/3}{s} - \frac{3/2}{s+1} - \frac{1/6}{s+3}}_{\text{Modes or poles of the sys.}}$$

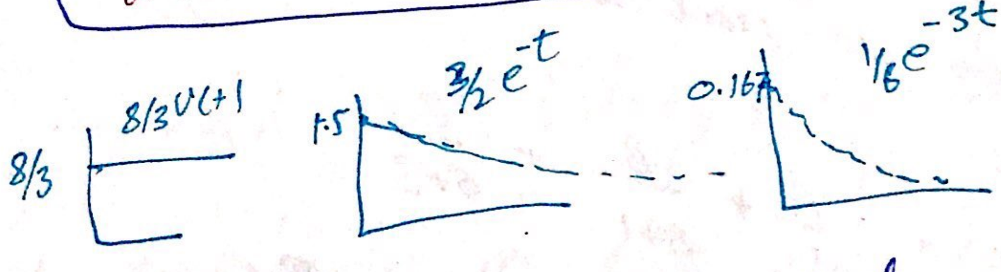
Taking $\mathcal{L}^{-1}$

$$h(t) = \frac{8}{3}U(t) - \frac{3}{2}e^{-t} - \frac{1}{6}e^{-3t}$$

So the impulse response is the linear combination of the modes.

$\Rightarrow$ The modes $\frac{8}{3}U(t)$ and $\frac{3}{2}e^{-t}$ dominates the response $\because$ they are slower than e^{-3t}.

$\boxed{e^{-t} \text{ and } e^{-6t} \text{ which one will decay quickly?}}$



$\rightarrow$ i.e Not all the modes contribute equally in the response.

Notice

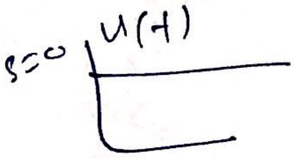
$\rightarrow$ the poles are on the LHP

what if they are on the RHP

(2-9)

→ Look at the response.

The response due to $s=-1$ and $s=-3$ (stable responses) decays exponentially ↑ But the response due to $s=0$ retains itself that is why we do not call an integrator, a stable one



$$\boxed{\text{The real poles always give rise to the exponential terms in the response}}$$

Case 2: Complex Roots

Example: $H(s) = \dfrac{s+1}{s^2+2s+10}$

$$H(s) = \frac{s+1}{(s+1-3j)(s+1+3j)} = \frac{C_1}{s+1-3j} + \frac{C_1^*}{s+1+3j}$$

↑ Complex Conjugate

generally: $\xrightarrow{a+jb}$ $H(s) = \dfrac{C_1}{s+\delta-j\omega} + \dfrac{C_1^*}{s+\delta+j\omega}$

$$= \frac{a+jb}{s+\delta-j\omega} + \frac{a-jb}{s+\delta+j\omega}$$

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$$= \frac{(a+jb)(s+\delta+j\omega) + (a-jb)(s+\delta-j\omega)}{(s+\delta-j\omega)(s+\delta+j\omega)}$$

$$= \frac{2as + 2a\delta - 2b\omega}{(s+\delta)^2 + \omega^2}$$

$$= \frac{2a(s+\delta)}{(s+\delta)^2+\omega^2} - \frac{2b\omega}{(s+\delta)^2+\omega^2}$$

Taking $\mathcal{L}^{-1}$ we get

$$h(t) = 2a\, e^{-\delta t} \cos\omega t - 2b\, e^{-\delta t} \sin\omega t$$

$$= 2e^{-\delta t}\left[a\cos\omega t - b\sin\omega t\right]$$

Let us write

$$a\cos\omega t \mp b\sin\omega t = \gamma\cos(\omega t + \phi)$$

$$= \gamma\left[\cos\omega t\cos\phi - \sin\omega t\sin\phi\right]$$

which means

$$a = \gamma\cos\phi \ , \quad b = \gamma\sin\phi$$

$$a^2 + b^2 = \gamma^2 \Rightarrow \gamma = \sqrt{a^2+b^2}$$

and $\phi = \tan^{-1}\left(\frac{b}{a}\right)$

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So $h(t) = 2e^{-\delta t}\, r \cos(\omega t + \phi)$

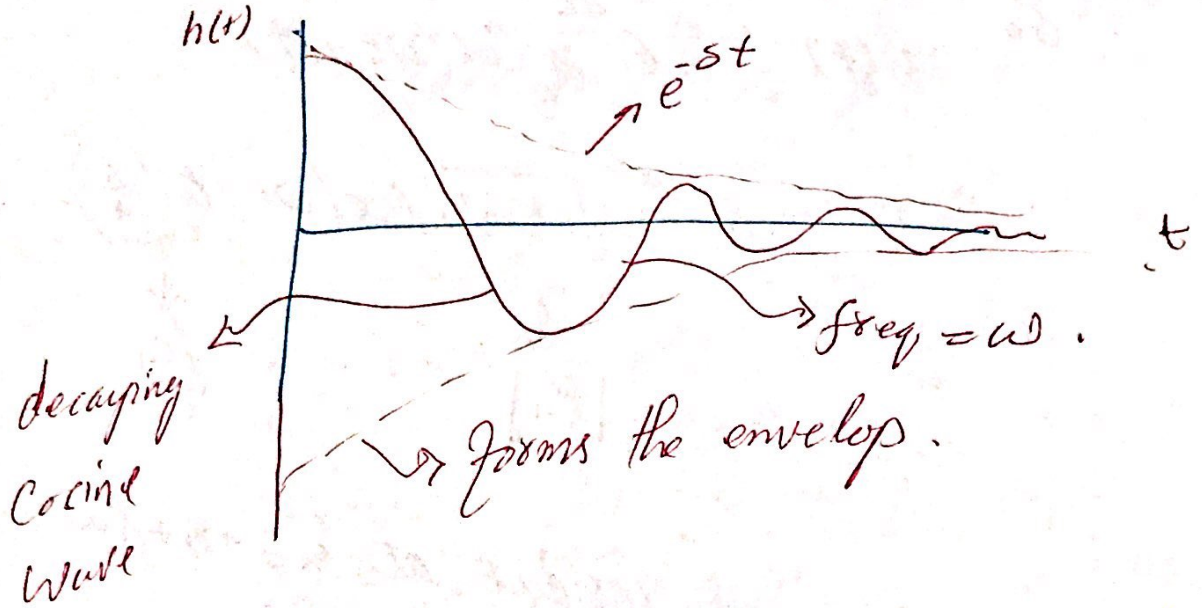
$$= 2e^{-\delta t} \underbrace{\sqrt{a^2+b^2}}_{|C_1|} \cos(\omega t + \underbrace{\tan^{-1}\tfrac{b}{a}}_{\angle C_1})$$

$\rightarrow C_1$ is the residue at $s = -\delta + j\omega$

why complex poles are always complex conjugates???

$\therefore$ The co-efficients of the T/F must always be real as they represent physical quantities like m, b, K, R, C, L etc. that is why the complex roots form a conjugate pair

2-12 Plotting h(t)



$e^{?} \cdot (Cos$

⇒ In case of complex conjugate poles we'll have a sinusoids times exponentials.

$s = -\delta + j\omega$

The real part of the complex pole defines the exponential & hence stability. while the Imaginary part defines the frequency of cosine wave contained within the envelop.

⇒ So by just look at the poles we can tell what type of response do we expect from the system when hit by an impulse.

<u>Example</u>:→ $H(s) = \frac{5s^2 + 16s + 62}{(s+2)(s^2 + 2s + 10)}$

↳ $s = -1 \pm 3j$

what can we say about the impulse response of this sys by just looking at the TF???

$h(t) = Ae^{-2t} + Be^{-t}\cos(3t + \phi)$

↓

This is the info. which this TF gives

Now A, B and ϕ can be found but at least we know that sys is stable and will decay exponentially.

Case 3   Repeated Real Roots

Generally :→ $H(s) = \dfrac{N(s)}{(s-p)^3} = \dfrac{C_1}{s-p} + \dfrac{C_2}{(s-p)^2} + \dfrac{C_3}{(s-p)^3}$

$$C_3 = (s-p)^3 H(s)\Big|_{s=p}$$

$$C_2 = \frac{d}{ds}\left[(s-p)^3 H(s)\right]\Big|_{s=p}$$

$$C_1 = \frac{d^2}{ds^2}\left[(s-p)^3 H(s)\right]\Big|_{s=p}$$

$$h(t) = C_1 e^{pt} + C_2 t e^{pt} + C_3 \frac{t^2}{2!} e^{pt}$$

Final Value Theorem :→

Stability ??

Stability ⇒ All transients in the impulse response die out as $t \to \infty$ ⇒ All poles of the ~~the~~ T/f must have -ve real parts.

⇒ FVT states that

$$\lim_{t \to \infty} y(t) = \lim_{s \to 0} s\,Y(s)$$

iff the sys is stable.

→ if a sys is stable, the ICs always die out to zero. So the FVT is really about the forced steady state response.

freq: domain

$$U(s) \longrightarrow \boxed{H(s)} \longrightarrow Y(s) = H(s)\,U(s)$$

t-domain

$$U(t) \longrightarrow \boxed{h(t)} \longrightarrow y(t) = h(t) * U(t)$$

↳ Cumbersome procedure.

→ Applicable to stable systems only ∵ the final value of an unstable sys even for an impulsive i/p, approaches ∞.

Imp is Notice the power of the tff
→ We could easily tell how the response is going to be
→ What are the basic building blocks of that response (mode)
→ and how much each mode is contributing to the total response.

Why is this imp??
↳ You know the xtics of the sys. the prop of the system. how ~~part~~ its going to respond? will it oscillate? will it dampen out quickly?
→ In short You can see if you like the response of the system or not. and if you don't like the resp. then you can figure out to change it using feedback control.

→ it is also the power of the t/f that without actually doing the cumbersome time domain solution we tell about the final value using FVT.

DC Gain of a system

It is the gain (o/p to i/p ratio) for a cst i/p in steady state.

Sys

$$U(s) \longrightarrow \boxed{H(s)} \longrightarrow Y(s) = H(s)U(s)$$

assume $u(t) \to$ step then $U(s) = \frac{1}{s}$

then $Y(s) = \frac{H(s)}{s}$ $\qquad \frac{Y(s)}{U(s)} = H(s)$ $\qquad Y(s) = H(s)U(s)$

Gain?? $\frac{Y(s)}{U(s)} \to 1$ in steady state

So for gain we see the final value of $Y(s)$ in the steady state

$$y(t)\Big|_{t\to\infty} = s\frac{H(s)}{s}\Big|_{s=0} = \boxed{H(0)}$$

DC gain of the system provided the sys is stable.

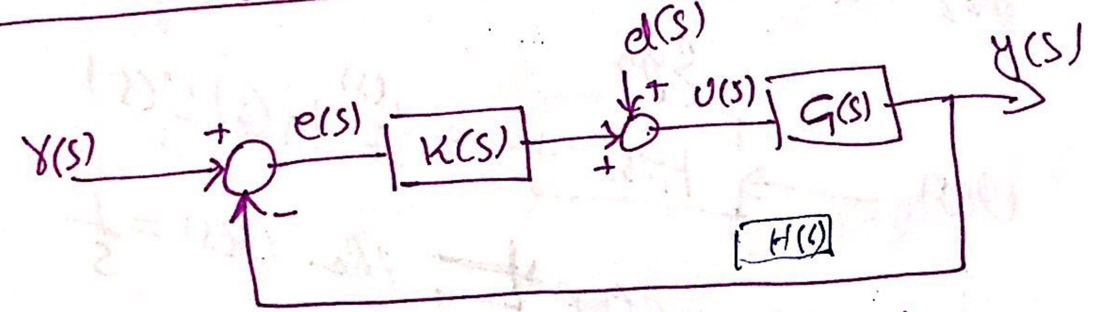
Actually

$$\text{DC gain} = \frac{y(t)|_{t\to\infty}}{\underbrace{U(t)_{t=\infty}}_{\to 1}} = y(t)\Big|_{t=\infty}$$

$\Rightarrow$ DC gain is undefined for unstable systems

A Recap

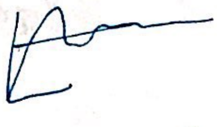
Block Diagram Algebra



$\frac{y(s)}{r(s)} = ?$, $\frac{e(s)}{r(s)} = ?$ $\frac{Y(s)}{d(s)} = ?$, $\frac{U(s)}{r(s)} = ?$

how much the o/p will be affected by d

$$\frac{y}{R} = \frac{GK}{1+GK} \Rightarrow R \to \boxed{GK/1+GK} \to Y$$



$\frac{u(s)}{r(s)}$ $\to$ Tells how big is the control sys or actuator requirement for following r. (Can an actuator sustain for r)

↳ P

Note that the denominator of all the t/f is the same

why ??

∴ The poles or behavior or modes can't change for the same system.

Notice that the numerator in each t/f is diff: which means the contribution of each mode can be different.

$$\frac{\text{num}}{1 + G(s)\,C(s)}$$

Idea of Feedback

Suppose ~~a 1 sys~~ an open-loop system

$$G(s) = \frac{a}{s+a}$$

its impulse response will be

$$a e^{-at}$$

Now assume $a = 0.0001$

then

Now the response is very slow and we are not happy with it

So let us put a proportional controller.

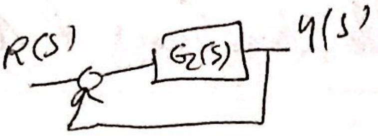
$\boxed{\text{As plant is some physical phenomenon which we can't change.}}$

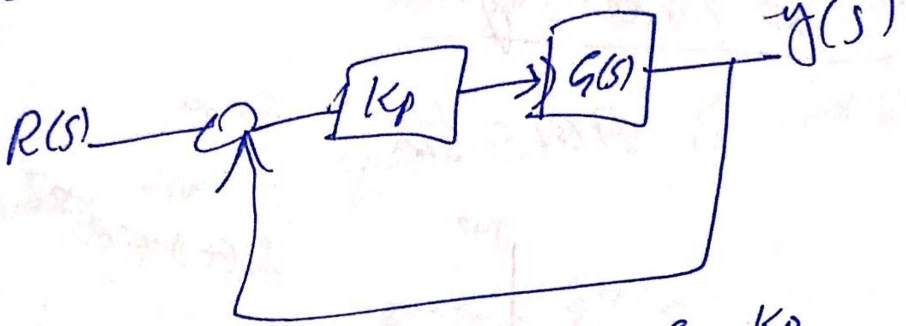
$$\longrightarrow \boxed{K_p} \longrightarrow \boxed{G(s)} \longrightarrow \qquad K_p a e^{-at}$$

$$T/f = \frac{K_p a}{s+a} \Rightarrow K_p a e^{-at}$$

Still the response is very slow even for a large K_p.

Now let us use feedback





$$\frac{Y(s)}{R(s)} = \frac{GK}{1+GK} = \frac{\frac{a}{s+a}K_p}{1+\frac{a}{s+a}K_p}$$

$$= \frac{aK_p}{s+a+aK_p} = \frac{aK_p}{s+a(1+K_p)}$$

Now the pole is at

$$-a(1+K_p)$$

and if $K_p = 1000$ (assume)

then $-0.0001(1001)$ is the pole which is much faster than the O/L pole

→ Observe the advantage of feedback

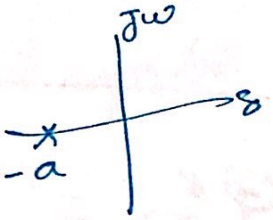
→ it can change the dynamics of the plant and hence the behavior of the resp. (Can alter the poles)

Pole locations & Resp:

(1) First Order Sys

$H(s) = \frac{1}{s+a}$ $\quad s = -a$

~~$h(t) = e^{-at}$~~



impulse Resp: $h(t) = e^{-at}$

Step Resp: $Y(s) = \frac{1}{s}\,\frac{1}{s+a}$

$y(t) = \frac{1}{a}(1-e^{-at})$

$t_r = \frac{2.2}{a}$, $t_s = \frac{4.6}{a}$, $M_p = 0$ (no overshoot)

2nd Order Sys

$$H(s) = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

Stable complex Conjugate poles :→ $s = -\sigma + j\omega_d$

$= -\sigma - j\omega_d$

Denominator polynomial is (Cartesian rep)

$(s+\sigma+j\omega_d)(s+\sigma-j\omega_d)$

$(s+\sigma)^2 + \omega_d^2 \Rightarrow s^2 + \sigma^2 + 2\sigma s + \omega_d^2$

$\Rightarrow s^2 + 2\sigma s + (\sigma^2 + \omega_d^2) \rightarrow (1)$

$s^2 + 2\xi\omega_n s + \omega_n^2 \rightarrow$ ② Standard form ~~for~~ for den poly of a 2^{nd} order Sys.

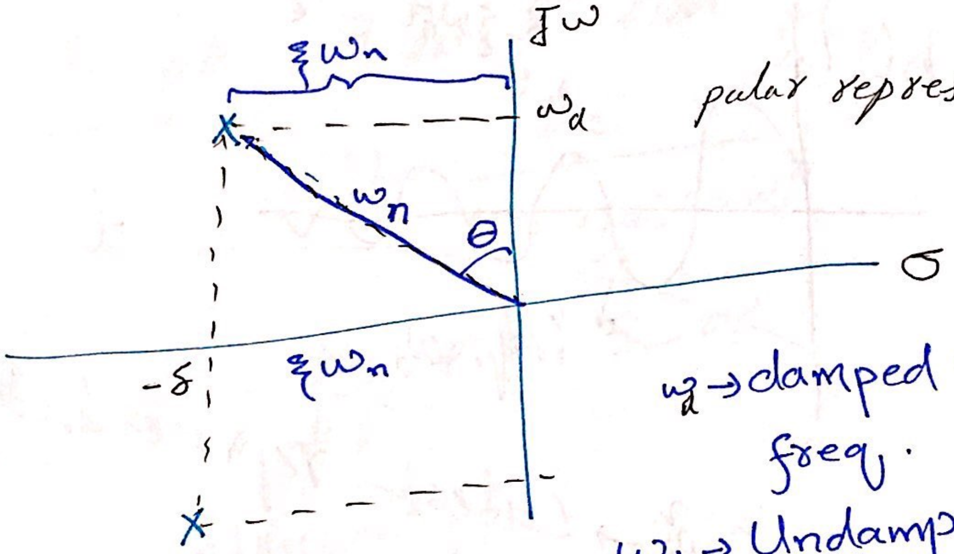
Compare the co-efficients of ① and ②

$2\xi\omega_n = 2\delta \Rightarrow \delta = \xi\omega_n$

$s = \delta + j\omega_d$

and $\omega_n^2 = \delta^2 + \omega_d^2$

Now let see graphically.



polar representation.

$\omega_d \rightarrow$ damped natural freq.

$\omega_n \rightarrow$ Undamped natural freq. (dist. of pole from the origin.

① $\sin\theta = \frac{\xi\omega_n}{\omega_n} = \xi$

② $\boxed{\theta = \sin^{-1}\xi}$

$0 \le \xi \le 1$
$\hookrightarrow$ damping Ratio

③ $\omega_n^2 = \delta^2 + \omega_d^2 = \omega_n^2\xi^2 + \omega_d^2$

$\omega_n^2(1-\xi^2) = \omega_d^2$

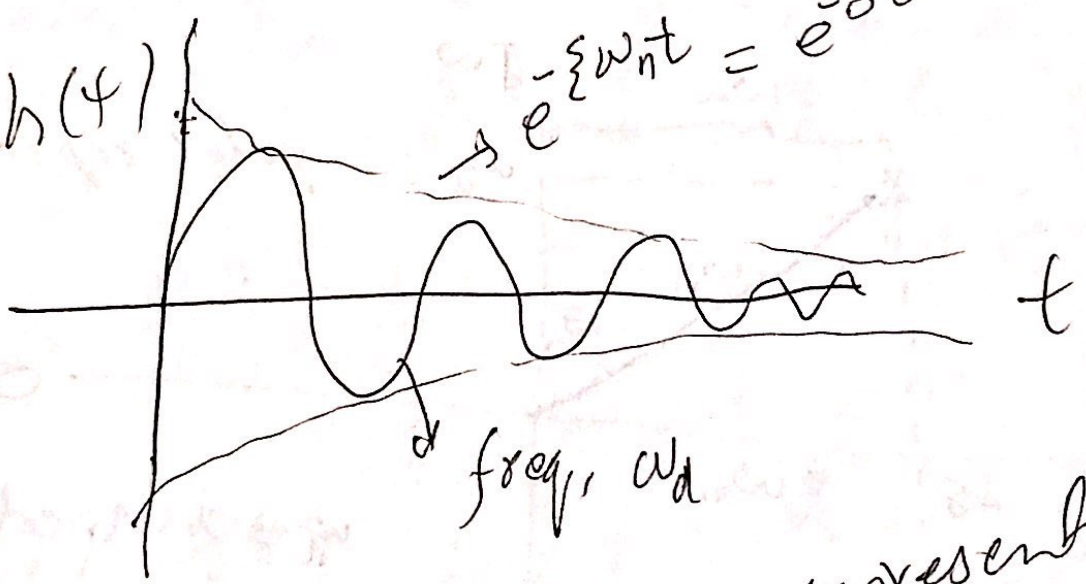
For good damping we need

$$0.5 \le \xi \le 0.8$$

$\theta = 45^\circ$, $\xi = 0.707$

Now $H(s) = \dfrac{\omega_n^2}{s^2 + 2\xi\omega_n s + \omega_n^2}$

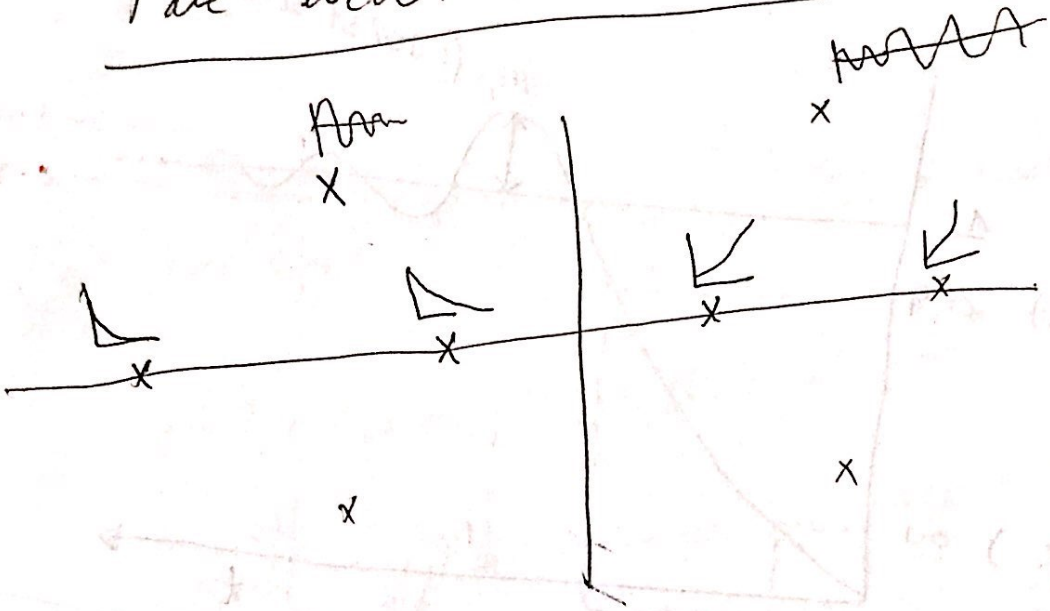
$$\mathcal{L}^{-1}[H(s)] = h(t) = \frac{\omega_n}{\sqrt{1-\xi^2}}\, e^{-\xi\omega_n t} \sin \omega_d t$$



$s^2 + 2\delta s + \delta^2 + \omega_d^2 \rightarrow$ Cartesian representation

$s^2 + 2\xi\omega_n s + \omega_n^2 \rightarrow$ Polar "

Pole location vs Time Response



Example :→ Standard 2^{nd} order System.

$$H(s) = \frac{\omega_n^2}{s^2 + 2\xi\omega_n s + \omega_n^2}$$

$$h(t) = \frac{\omega_n}{\sqrt{1-\xi^2}} e^{-\xi\omega_n t} \sin \omega_d t$$

Step Response ??

$$Y(s) = H(s)\,U(s) \qquad U(s) = \frac{1}{s} \text{ (Step)}$$

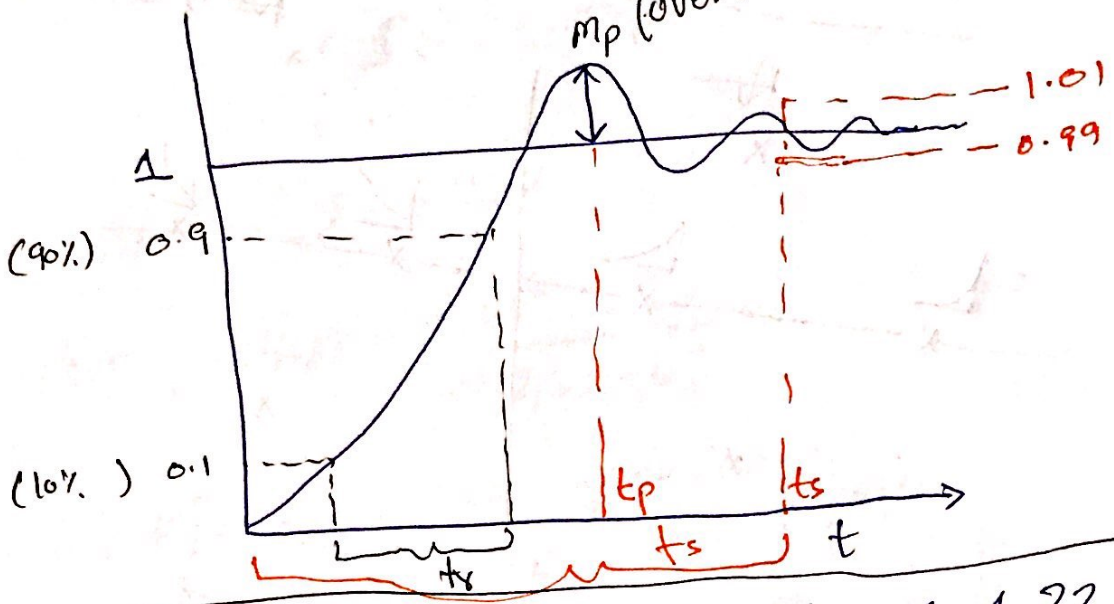
$$Y(s) = \frac{1}{s} \frac{\omega_n^2}{s^2 + 2\xi\omega_n s + \omega_n^2}$$

$$L^{-1}(Y(s)) = y(t) = 1 - e^{-\xi\omega_n t}\left[\cos\omega_d t + \frac{\xi}{\sqrt{1-\xi^2}} \sin \omega_d t\right]$$

(3-2)

$y(t)$ is the step resp. of the standard 2nd order t/f

why does this settle down to 1?
DC gain $H(0) = \frac{\omega_n^2}{\omega_n^2} = 1$
$\downarrow$
if $H(s)$ is stable.



why does this system settles down to 1??

$\rightarrow$ it is because of the DC gain.

$$Y(s) = H(s)\frac{1}{s}$$

$$y(t)\Big|_{t\to\infty} = \lim_{s\to 0} s H(s)\frac{1}{s} = H(0) = 1$$ if $H(s)$ is stable

① $M_p = e^{-\pi \xi/\sqrt{1-\xi^2}} = e^{-\pi \sigma/\omega_d}$ if $0 \leq \xi \leq 1$

② Rise Time (t_r) :→ Time to go from 10% of final value to 90% of final value. $t_r \simeq \frac{1.8}{\omega_n}$

Exact formula

$$t_r = \frac{\pi - \theta}{\omega_d} = \frac{\pi - \tan^{-1}\left(\frac{\sqrt{1-\xi^2}}{\xi}\right)}{\omega_d}$$

③ Settling Time (t_s) :→ Time when $y(t)$ comes within $\pm 1\%$ of the final value and stays there

$$t_s = \frac{4.6}{\xi\omega_n} = \frac{4.6}{\sigma}$$

$e^{-\xi\omega_n t_s} = 0.01$

$-\xi\omega_n t_s = \ln(0.01)$

$t_s = \frac{4.6}{\xi\omega_n}$

④ Peak time (t_p) :→

The time taken by the system to produce the maximum peak.

$$\boxed{t_p = \frac{\pi}{\omega_d}}$$

All these formulas/relations are valid only for standard 2nd order T/F $H(s) = \omega_n^2 / s^2 + 2\xi\omega_n s + \omega_n^2$

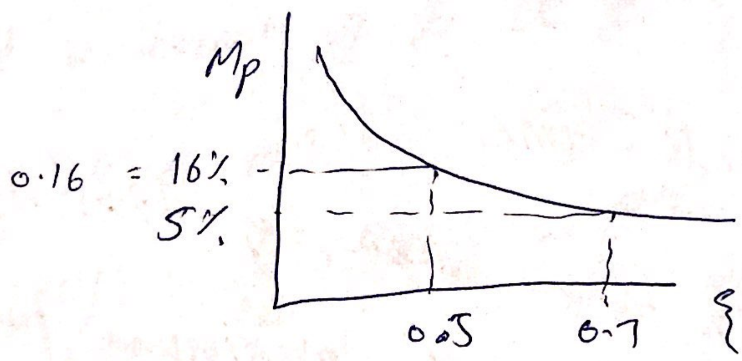
(3-4)

Conclusions

(1) ~~For t_r we need ω_n~~
For a faster rise time ($t_r \downarrow$)
(2) ~~For t_s~~ we need bigger ω_n

(2) For faster ~~set~~ t_s we need bigger $\xi\omega_n$ & hence bigger δ

(3) For $M_p = e^{-\pi \xi/\sqrt{1-\xi^2}}$



increase ξ to decrease overshoot.

All these (M_p, t_p, t_r, t_s) defines the transient Response of the system.