Markov Chain:

A markove chain is the Stochastic model describing a Sequence of Possible event in which the Probability of each event depends only on the state attained in the Privious event.

In simpler terms it is a process for which Prediction can be made regarding future outcomes based solely on its presented state and most importantly a such prediction are just as good as the ones, That could be made knowing the process's full of history. This is used to simplify Predictions about the future state of a stochastic process.

Background:

Andrei Markov was

a Russian Methimatician who lived between (1856-1922). He was poorly performing student and the only subject he didn't have any difficulties in was mathimatics.

His most famous studies were with Markov Chain. hence the name and his first paper on the subject was published in 1906.

Stochastic:

In Probability there are a stochastic process in mathimatical object usually defined as a family of random variable written as

$$\{X_\beta : \beta \in A\}$$

where A is the index set and β is the index Perameter.

Types of Stochastic Matrics

There are several diffirent types of stochastic matrices described as follows.

Right Stochastic matrix

A Right stochastic matrix is a square matrix of non negative real numbers whose rows add up to one.

Left Stochastic Matrix.

A left stochastic matrix is a square matrix of non negative real numbers whose columns add up to one.

Doubly Stochastic Matrix.

A doubly stochastic matrix is a square matrix of non negative real numbers whose with each row and column adding up to one

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Transition: Probability

The changes of the system are called Transition and Probability accociated with various state changes are called Transition Probability. and it is totally concerned with a matrix called Transition matrix.

or

movement of Probability from one state to another state. Mathimatically it can be writen as

$$P_{ij} = P\{X_{n+1} = j \mid X_n = i\}.$$

Since Markov chain does't depends on past its future depends on Preasent.

Now The Transition Probability matrix is written as

$$P = \begin{bmatrix} P_{11} & P_{12} & P_{13} & \cdots & P_{1j} \\ P_{21} & P_{22} & P_{23} & \cdots & P_{2j} \\ P_{31} & P_{32} & P_{33} & \cdots & P_{3j} \\ \vdots & \vdots & \vdots & & \vdots \\ P_{i1} & P_{i2} & P_{i3} & \cdots & P_{ij} \end{bmatrix}$$

where i number of rows and j number

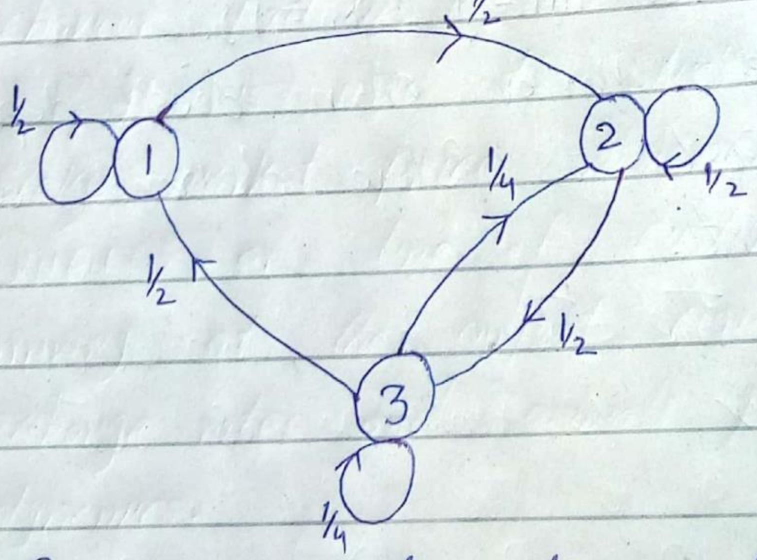
of columes.
Since in Transition matrix sum of each row is unity i.e

$$\sum_{n=1}^{j} P_{in} = 1$$

Also each element in matrix is non negative i.e

$$P_{ij} \geq 0 \; ; \; \forall \; i, j$$

State diagram



By using state diagram the Transition Probability matrix is can be written as;

$$
\begin{array}{cccc}
 & 1 & 2 & 3 \\
\end{array}
$$

$$
P = \begin{array}{c} 1 \\ 2 \\ 3 \end{array}
\begin{bmatrix}
\frac{1}{2} & \frac{1}{2} & 0 \\
0 & \frac{1}{2} & \frac{1}{2} \\
\frac{1}{2} & \frac{1}{4} & \frac{1}{4}
\end{bmatrix}
$$

A zero element in the transition matrix indicate that the transition is not possible.

Defination: A state diagram is a type of diagram which is used to describe the behaviour of the system. State diagram requires that the system described is composed of a finite number of states.

Step to draw a state diagram:

1. Identify the initial state and the final terminating states.
2. Identify the possible states in which object can exist.
3. Label the events which trigger these transitions.

Application and uses of Markov Chain:

Markov Chain have many applications as statistical models of real world processes such as studying cruise control systems in motor vehicals.

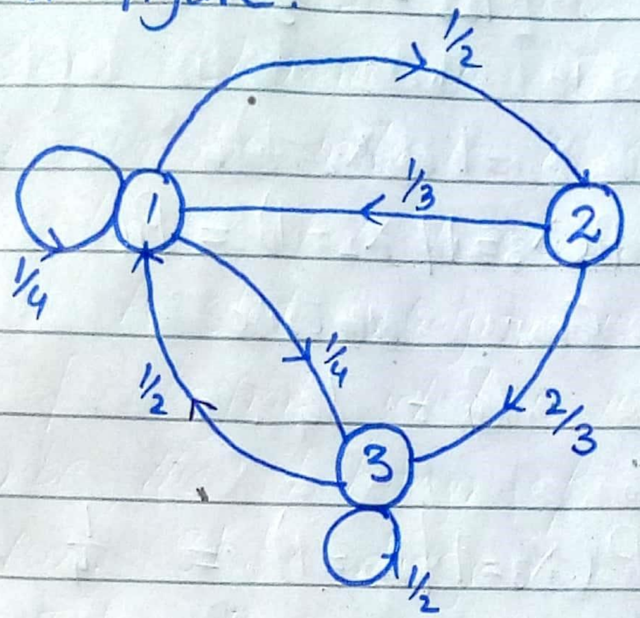
It is used to examine and pridict the behaviour of consumer in the term of this brand of interested loyality and switching behaviour patterns to other brand.

It is used to study the stoke market price movement.

Queues or line of customers arriving at the airport, currency exchange rates and animals populations dynamics.

Example 1:

Consider a Markov chain is Shown in figure:



Find:

a) $P(X_4 = 3 \mid X_3 = 2)$.

First of all we made a Transition probability matrix by using state diagram: i.e

$$P = \begin{array}{c} \\ \\ \\ \end{array}\begin{array}{ccc} 1 & 2 & 3 \\ \end{array}$$

$$P = \left[\begin{array}{ccc} 1/4 & 1/2 & 1/4 \\ 1/3 & 0 & 2/3 \\ 1/2 & 0 & 1/2 \end{array}\right]\begin{array}{c} 1 \\ 2 \\ 3 \end{array}$$

Now we use formula to find the Probability of Transition matrix as

$$P(X_{n+1} = j \mid X_n = i) = P_{ij}.$$

Now $P(X_4=3 \mid X_3=2) = P_{23}$

from matrix

$$\boxed{P_{23} = \frac{2}{3}}$$ Ans

b) $P(X_3=1 \mid X_2=1)$.

$P(X_3=1 \mid X_2=1) = P_{11}$

From matrix

$P_{11} = 1/4$

So

$P(X_3=1 \mid X_2=1) = \frac{1}{4}$. Ans

c) If we know $P(X_0=1) = \frac{1}{3}$

find $P(X_0=1, X_1=2)$.

$P(X_0=1, X_1=2) = P(X_0=1)\, P(X_1=2 \mid X_0=1)$

$= \left(\frac{1}{3}\right)(P_{12})$

$= \left(\frac{1}{3}\right)\left(\frac{1}{2}\right)$

$= \frac{1}{6}$.

d) If we know $P(X_0=1) = 1/3$

find $P(X_0=1, X_1=2, X_2=3)$.

$P(X_0=1, X_1=2, X_2=3)$

$= P(X_0=1)\, P(X_1=2 \mid X_0=1)$

$P(X_2=3 \mid X_1=2)$

$= \left(\frac{1}{3}\right)(P_{12})(P_{23})$

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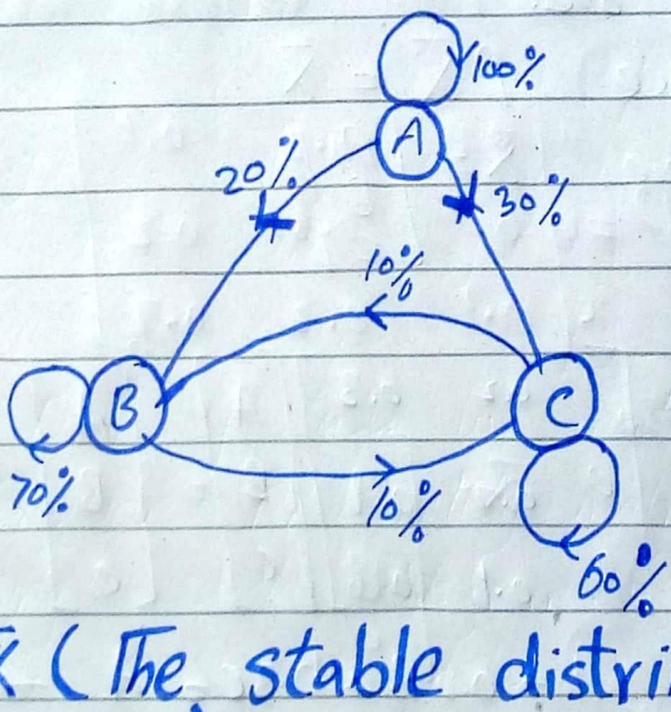
$$= \left(\frac{1}{3}\right)\left(\frac{1}{2}\right)\left(\frac{2}{3}\right) = \frac{1}{9}$$

$$= \frac{1}{9} \quad \text{Ans}$$

Example 2.

Consider a Markov Chain is Shown in figure.



Find $\bar{X}$ (The stable distribution Matrix).

Solution:

By using above Markov chain The Probability Transition matrix can be written as;

$$P = \begin{array}{c} \\ A \\ B \\ C \end{array} \begin{array}{c} \begin{array}{ccc} A & B & C \end{array} \\ \begin{bmatrix} 1 & 0.2 & 0.3 \\ 0 & 0.7 & 0.1 \\ 0 & 0.1 & 0.6 \end{bmatrix} \end{array}$$

Note that it is left stochastic matrix and note That inmatrix There are non-negative real numbers whose column add up to one.

Since $P\bar{X} = \bar{X}$

where $\bar{X} = \begin{bmatrix} A \\ B \\ C \end{bmatrix}$

$$\begin{bmatrix} 1 & 0.2 & 0.3 \\ 0 & 0.7 & 0.1 \\ 0 & 0.1 & 0.6 \end{bmatrix} \begin{bmatrix} A \\ B \\ C \end{bmatrix} = \begin{bmatrix} A \\ B \\ C \end{bmatrix}$$

$\Rightarrow \quad A + 0.2B + 0.3C = A \rightarrow (i)$

$0.7B + 0.1C = B \rightarrow (ii)$

$0.1B + 0.6C = C \rightarrow (iii)$

$ii) \Rightarrow \quad 7B + C = 10B$

$\Rightarrow \quad C = 10B - 7B = 3B \rightarrow (iv)$

$(iii) \Rightarrow \quad B + 6C \cancel{+} = 10C$

$\Rightarrow B = 10C - 6C$

$\Rightarrow B = 4C \rightarrow (v)$

use equation (v) in (iv)

$C = 3(4C) = 12C$

$11C = 0 \Rightarrow C = 0$

(iv) $\Rightarrow 3B = C = 0$

$\Rightarrow B = 0$

since $A + B + C = 1$

$A + 0 + 0 = 1$

$\Rightarrow A = 1$

So The required set $\bar{X}$ is

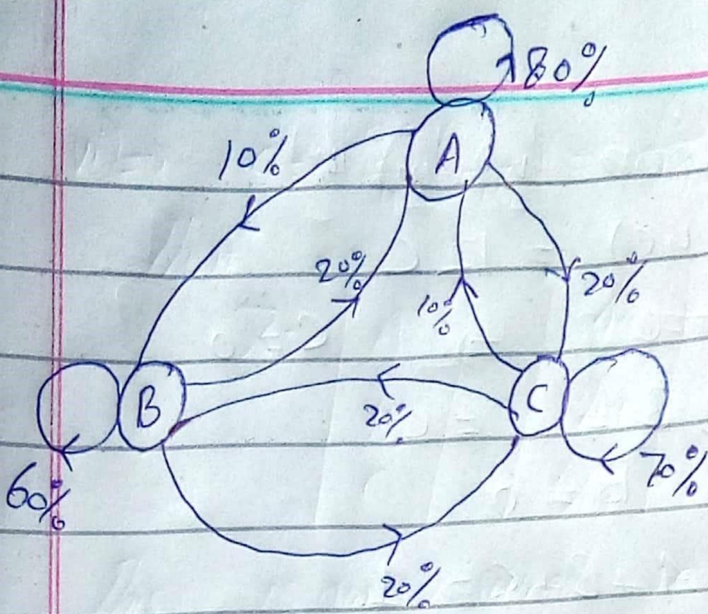
$$\bar{X} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}.$$

Example 3:

10% of A shoppers will switch to B And 10% to C.
20% of B shoppers will switch to A and 20% to C
10% of C shoppers will switch to A and 20% to B.
Find $\bar{X}$ (state distribution Matrix).

Solution.

The given state diagram is



Now By using state diagram the required Transition matrix is

$$P = \begin{array}{c} \\ A \\ B \\ C \end{array}\begin{array}{c} \begin{array}{ccc} A & B & C \end{array} \\ \begin{bmatrix} 0.8 & 0.2 & 0.1 \\ 0.1 & 0.6 & 0.2 \\ 0.1 & 0.2 & 0.7 \end{bmatrix} \end{array}$$

Also

$$\bar{X} = \begin{bmatrix} A \\ B \\ C \end{bmatrix}$$

And $A + B + C = 1 \longrightarrow ①$

Now we find $P\bar{X} = \bar{X}$

$$\begin{bmatrix} 0.8 & 0.2 & 0.1 \\ 0.1 & 0.6 & 0.2 \\ 0.1 & 0.2 & 0.7 \end{bmatrix} \begin{bmatrix} A \\ B \\ C \end{bmatrix} = \begin{bmatrix} A \\ B \\ C \end{bmatrix}$$

$0.8A + 0.2B + 0.1C = A$ —(i

$0.1A + 0.6B + 0.2C = B$ —(ii

$0.1A + 0.2B + 0.7C = C$ —(iii

(i) $\Rightarrow$ $8A + 2B + 1C = 10A$

$\Rightarrow$ $2B + C = 2A$ $\longrightarrow$ (iv

ii) $\Rightarrow$ $A + 6B + 2C = 10B$ x by 10 (ii)

$\Rightarrow$ $2C = 4B - A$ $\longrightarrow$ (v

iv) $-$ $C = 2A - 2B$ put in (v)

(v) $\Rightarrow$ $2(2A - 2B) = 4B - A$

$4A - 4B = 4B - A$

$5A = 8B$ $\longrightarrow$

$\Rightarrow$ $B = \frac{5}{8}A$ $\longrightarrow$ vii)

v) $\Rightarrow$ $C = 2A - 2\left(\frac{5}{8}A\right)$

$= 2A - \frac{5}{4}A$

$C = \frac{3}{4}A$ $\longrightarrow$ (vi)

Put all in (1)

$A + B + C = 1$

$A + \frac{5}{8}A + \frac{3}{4}A = 1$

$\Rightarrow$

$\frac{19}{8}A = 1$

$\Rightarrow$ $A = \frac{8}{19}$

vii) $\Rightarrow$

$B = \frac{5}{8}\left(\frac{8}{19}\right) = \frac{5}{19}$

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vi) $\Rightarrow$ $C = \frac{3}{4}\left(\frac{8}{19}\right) = \frac{6}{19}$

Then

$$\bar{X} = \begin{bmatrix} A \\ B \\ C \end{bmatrix} = \begin{bmatrix} 8/19 \\ 5/19 \\ 6/19 \end{bmatrix}$$

This is our required result Proved.