Exponential Distribution:-

The exponential distribution is used to model poission process. Poision process is one of the most widely used Counting process It is used in Scenarious where we are counting the occurrence of certain units that appear to happen to certain rate but completely at random.

For Example:-

Suppose that from historical data we know that earthquake occur in certain area with the rate of two per month, other then this information. The timing of earth quakes seen to be completely random. There are some situations in which an object initially in State A can change to state B with constant probability per unit time λ. The time at which the State actually changes is described by exponential random variable with parameter λ.

Properties:-

$$f(x) = \lambda e^{-\lambda x} \; ; \; 0 \le x \le \infty$$

i) The mean of exponential distribution is $1/\lambda$

ii) The varience of exponential distribution is $1/\lambda^2$

iii) The total area with under the curve for exponential distribution is 1.

iv) The median of exponential distribution is $\frac{\ln(2)}{\lambda}$.

v) The m.g.f of exponential distribution function is $M_0(t) = \frac{\lambda}{\lambda - t}$

vi) The skewness or kurtosis of exponential distribution is 2 (positive skewed) 6 (lepto kurtic) respectively.

Total area under the curve for exponential distribution is unity.

$$Area = \int_{-\infty}^{\infty} f(x)dx$$

In area, case of exponential distribution

$$Area = \int_0^{\infty} \lambda e^{-\lambda x} dx$$

$$= \lambda \int_0^{\infty} e^{-\lambda x} dx$$

$$= \lambda \left[\frac{e^{-\lambda x}}{-\lambda}\right]_0^{\infty}$$

$$= -1 \left[e^{-\infty x} - e^{-0 \lambda}\right] \qquad \because e^{-\infty} = 0, \; e^{0} = 1$$

$$= -1 \left[0 - 1\right]$$

$$= -1[-1] = 1 \text{ proved}$$

Mean of Exponential Distribution:-

$$E(x) = \int_{-\infty}^{\infty} x f(x)\,dx$$

$$E(x) = \int_0^{\infty} x \lambda e^{-\lambda x}\,dx$$

$$\int f(x)\,g(x)\,dx = f(x)\int g(x)\,dx - \int \left[\int g(x)\,dx\right] f'(x)\,dx$$

Now $f(x) = x$, $g(x) = \lambda e^{-\lambda x}$

$$= \lambda \int_0^{\infty} x e^{-\lambda x}\,dx$$

$$= \lambda \left[x \frac{e^{-\lambda x}}{-\lambda} \Big|_0^{\infty} \right] - \int_0^{\infty} \frac{e^{-\lambda x}}{-\lambda}\,dx \Big]$$

$$= \frac{\lambda}{-\lambda}\left[x e^{-\lambda x} \Big|_0^{\infty} + \frac{\lambda}{\lambda}\int_0^{\infty} e^{-\lambda x}\,dx \right]$$

$$= -\left[\infty e^{-\infty} - 0 e^{0}\right] + \frac{e^{-\lambda x}}{-\lambda}\Big|_0^{\infty}$$

$$= -[0-0] - \frac{e^{-\lambda x}}{\lambda}\Big|^{\infty}$$

$$= -\frac{1}{\lambda}\left[e^{-\infty} - e^{0}\right]$$

$$= -\frac{1}{\lambda}[0-1] \Rightarrow \frac{1}{\lambda}$$

$E(x) = \frac{1}{\lambda}$ Hence proved

Variance of Exponential Distribution:-

$$Var(x) = E(x^2) - [E(x)]^2 \quad \text{(1)}$$

$$E(x^2) = \int_{-\infty}^{\infty} x^2 f(x)dx$$

$$= \int_0^{\infty} x^2 \lambda e^{-\lambda x} dx$$

$$E(x^2) = \lambda \int_0^{\infty} x^2 e^{-\lambda x} dx$$

$$= \lambda \left[x^2 \frac{e^{-\lambda x}}{-\lambda} \Big|_0^{\infty} - \int_0^{\infty} 2x \frac{e^{-\lambda x}}{-\lambda} dx \right]$$

$$= \lambda \left[-\frac{1}{\lambda} \left[x^2 e^{-\lambda x} \right]_0^{\infty} + \frac{2}{\lambda} \int_0^{\infty} x e^{-\lambda x} dx \right]$$

$$= \lambda \left[-\frac{1}{\lambda} \left[\infty e^{-\infty} - 0 e^{0} \right] + \frac{2}{\lambda} \left[x \frac{e^{-\lambda x}}{-\lambda} \Big|_0^{\infty} - \int \frac{e^{-\lambda x}}{-\lambda} dx \right] \right.$$

$$E(x^2) = \lambda \left[0 + \frac{2}{\lambda} \left[-\frac{1}{\lambda} \left[\infty e^{-\infty} - 0 e^{-0} \right] + \frac{1}{\lambda} \frac{e^{-\lambda x}}{-\lambda} \Big|_0^{\infty} \right] \right]$$

$$= \lambda \left[-\frac{2}{\lambda^2} (0) + \frac{2}{\lambda^3} \left(\frac{e^{-\infty} - e^{0}}{-\lambda} \right) \right]$$

$$= \lambda \left[0 - \frac{2}{\lambda^3} (0 - 1) \right]$$

$$E(x^2) = \lambda \left[\frac{2}{\lambda^3} \right] \Rightarrow \frac{2}{\lambda^2}$$

put in eq (i)

$$Var(x) = \frac{2}{\lambda^2} - \frac{1}{\lambda^2}$$

$$Var(x) = \frac{2-1}{\lambda^2} = \frac{1}{\lambda^2}$$

$$Var(x) = \theta^2 = \frac{1}{\lambda^2}$$

M.g.f of Exponential Distribution.

$$M_x(t) = E(e^{tx})$$

$$= \int_0^\infty e^{tx} f(x) dx$$

$$= \int_0^\infty e^{tx} \cdot \lambda e^{-\lambda x} dx$$

$$= \lambda \int_0^\infty e^{tx - \lambda x} dx$$

$$= \lambda \int_0^\infty e^{(t-\lambda)x} dx$$

$$= \lambda \left[\frac{e^{(t-\lambda)x}}{(t-\lambda)} \right]_0^\infty$$

$$= \frac{\lambda}{t-\lambda} \left[e^{\infty} - e^{0} \right]$$

$$M_x(t) = \frac{\lambda}{t-\lambda} [0-1]$$

$$M_x(t) = \frac{\lambda}{t-\lambda} (-1)$$

$$M_x(t) = \frac{-\lambda}{t-\lambda} \text{ proved}$$

Median of Exponential Distribution.

$$\text{Median} = \int_0^\infty f(x) dx = \frac{1}{2}$$

In case of exponential distribution

$$\int_0^m \lambda e^{-\lambda x}\,dx = \frac{1}{2}$$

$$\lambda\left[\frac{e^{-\lambda x}}{-\lambda}\Big|_0^m\right] = \frac{1}{2}$$

$$-e^{-\lambda x}\Big|_0^m = \frac{1}{2}$$

$$\left[-(e^{-m\lambda}) + 1\right] = \frac{1}{2}$$

$$+e^{-m\lambda} = 1 - \frac{1}{2}$$

$$e^{-m\lambda} = \frac{1}{2}$$

Take natural Log on B. Sides

$$\ln\left(\frac{1}{2}\right) = \ln\left[e^{-m\lambda}\right]$$

$$\ln 1 - \ln 2 = -m\lambda \ln e$$

$$0 - \ln 2 = -m\lambda \ln e \qquad \because \ln(e) = 1$$

$$-\ln 2 = -m\lambda(1)$$

$$\frac{\ln 2}{\lambda} = m$$

$$\boxed{m = \frac{\ln 2}{\lambda}}$$ (Hence proved)

Moments about origion For Exponential Distribution:

$$u_1' = \int_0^\infty x\, f(x)\,dx$$

$$= \int_0^\infty x \lambda e^{-\lambda x}\,dx$$

$$= \lambda \int_0^\infty x e^{-\lambda x}\,dx$$

$$= \lambda \left\{ \frac{x e^{-\lambda x}}{-\lambda} - \int_0^\infty \frac{e^{-\lambda x}}{-\lambda} dx \right\}$$

$$= \lambda \left\{ \left(\frac{\infty e^{\infty}}{-\lambda} - \frac{0 e^{0}}{-\lambda} \right) + \frac{1}{\lambda} \int_0^\infty e^{-\lambda x} dx \right\}$$

$$= \lambda \left\{ (0-0) + \frac{1}{\lambda} \cdot \frac{e^{-\lambda x}}{-\lambda} \Big|_0^\infty \right\}$$

$$= \cancel{\lambda} \left\{ -\frac{1}{\lambda \cancel{^2}} \left[e^{-\infty} - e^{-0} \right] \right.$$

$$= -\frac{1}{\lambda} \{ 0 - 1 \}$$

$$\boxed{\mu_1' = +\frac{1}{\lambda}}$$

$$\mu_2' = \int_0^\infty x^2 f(x) dx$$

$$\mu_2' = \int_0^\infty x^2 \lambda e^{-\lambda x} dx$$

$$\mu_2' = \lambda \int_0^\infty x^2 e^{-\lambda x} dx$$

$$\mu_2' = \lambda \left[x^2 \frac{e^{-\lambda x}}{-\lambda} \Big|_0^\infty - \int_0^\infty 2x \frac{e^{-\lambda x}}{-\lambda} dx \right]$$

$$\mu_2' = \lambda \left[\frac{x^2 e^{-\lambda x}}{-\lambda} \Big|_0^\infty + \frac{2}{\lambda} \left[x \frac{e^{-\lambda x}}{-\lambda} \Big|_0^\infty - \int_0^\infty \frac{e^{-\lambda x}}{-\lambda} dx \right] \right.$$

$$\mu_2' = \lambda \left[- \left[\frac{\infty e^{-\infty}}{\lambda} - \frac{0 e^{0}}{+\lambda} \right] + \frac{2}{\lambda} \left[-\frac{1}{\lambda} \left[\infty e^{-\infty} - 0 e^{0} \right] + \int_0^\infty \frac{e^{-\lambda x}}{\lambda} dx \right] \right.$$

$$\mu_2' = +\frac{2}{\lambda^2} \left[0 + \frac{e^{-\lambda x}}{-\lambda} \Big|_0^\infty \right]$$

$$= -\frac{2}{\lambda} \cdot \frac{\lambda}{\lambda^2} \left[e^{-\infty} - e^{0} \right] \Rightarrow -\frac{2}{\lambda^2} [0 - 1]$$

$$\boxed{\mu_2' = 2/\lambda^2}$$

$$\mu_1' = 1/\lambda$$

$$\mu_2' = 2/\lambda \cdot 1/\lambda = \frac{2}{\lambda^2}$$

$$\mu_3' = 3/\lambda \cdot 2/\lambda^2 = 6/\lambda^3$$

$$\mu_4' = 4/\lambda \cdot 6/\lambda^3 = 24/\lambda^4$$

Moments about mean:-

we find moment about mean by using this formula

$$\mu_1 = \int_0^\infty (x-\mu)\, f(x)\,dx$$

$$\mu_2 = \int_0^\infty (x-\mu)^2 f(x)\,dx$$

$$\mu_3 = \int_0^\infty (x-\mu)^3 f(x)\,dx$$

and so on

when $\mu = 1/\lambda$

But another method to find Moment about mean is

$$\mu_1 = \mu_1' - \mu_1' = 0 \qquad \because \mu_1' = 1/\lambda$$

$$\mu_2 = \theta^2 = \mu_2' - (\mu_1')^2 \qquad \therefore \mu_2' = 2/\lambda^2$$

$$\mu_2 = 2/\lambda^2 - 1/\lambda^2 \qquad \therefore \mu_3' = 6/\lambda^3$$

$$\mu_2 = 1/\lambda^2$$

$$\mu_3 = \mu_3' - 3(\mu_1')(\mu_2') + 2(\mu_1')^3$$

$$\mu_3 = \frac{6}{\lambda^3} - 3\left(\frac{1}{\lambda}\right)\left(\frac{2}{\lambda^2}\right) + 2\left(\frac{1}{\lambda}\right)^3$$

$$= \cancel{\frac{6}{\lambda^3}} - \cancel{\frac{6}{\lambda^3}} + \frac{2}{\lambda^3} = \frac{2}{\lambda^3}$$

$$\mu_3 = \frac{2}{\lambda^3}$$

$$\mu_4 = \mu_4' + 6(\mu_1')^2\mu_2' - 4\mu_1'\mu_3' - 3(\mu_1')^4$$

$$= \frac{24}{\lambda^4} + 6\left(\frac{1}{\lambda^2}\right)\left(\frac{2}{\lambda^2}\right) - 4\left(\frac{1}{\lambda}\right)\left(\frac{6}{\lambda^3}\right) - 3\left(\frac{1}{\lambda}\right)^4$$

$$= \cancel{\frac{24}{\lambda^4}} + \frac{12}{\lambda^4} - \cancel{\frac{24}{\lambda^4}} - \frac{3}{\lambda^4}$$

$$= \frac{12-3}{\lambda^4} = \frac{9}{\lambda^4}$$

$$\mu_4 = \frac{9}{\lambda^4}$$

$\mu_1 = 0$

$\mu_2 = \frac{1}{\lambda^2}$

$\mu_3 = \frac{2}{\lambda^3}$

$\mu_2 = \frac{9}{\lambda^4}$

Skewness :-

$$B_1 = \frac{(\mu_3)^2}{(\mu_2)^3}$$

$$B_1 = \frac{(2/\alpha^3)^2}{(1/\alpha^2)^3}$$

$$= \frac{4}{\cancel{\alpha^6}} \cdot \frac{\cancel{\alpha^6}}{1}$$

$\boxed{B_1 = 4}$

So it is positive skewed

Kurtosis :-

$$B_2 = \frac{U_4}{(U_2)^2}$$

$$B_2 = \frac{9/\alpha^4}{(1/\alpha^2)^2}$$

$$= \frac{9}{\cancel{\alpha^4}} \cdot \frac{\cancel{\alpha^4}}{1}$$

$\boxed{B_2 = 9}$ lepto kurtic