Gamma Distribution

Let ‘X’ be a +ve continuous random variable with interval (0,∞) is said to be a gamma distribution having its p.d.f:

$$f(x) = \frac{1}{\Gamma a\, b^a} x^{a-1} e^{-x/b} \qquad (0 \le x \le \infty)$$

And gamma function is

$$\Gamma a\, b^a = \int_0^\infty x^{a-1} e^{-x/b} dx \qquad 0 \le x \le \infty$$

Known as gamma function with parameter a & b.
Where a=notation/location parameter
b=scale parameter
If a=1 then it becomes exponential distribution:

$$f(x) = \frac{1}{b} e^{-x/b} \qquad 0 \le x \le \infty$$

If b=1 then it becomes gamma distribution of single parameter:

$$f(x) = \frac{1}{\Gamma a} x^{a-1} e^{-x} \qquad 0 \le x \le \infty$$

& its function is:

$$\Gamma a = \int_0^\infty x^{a-1} e^{-x} dx \qquad 0 \le x \le \infty$$

If a=b=1 then it becomes standard exponential distribution:

$$f(x) = e^{-x} \qquad 0 \le x \le \infty$$

Properties of gamma distribution

1-Area under the curve is unity.
2-Gamma distribution is a +ve continuous distribution.
3-It has two parameters a & b.
4-The range of the distribution is 0 to ∞.
5-The mean of gamma distribution with single parameter is ‘a’ & its variance is also ‘a’.
6-The mean of the gamma distribution with two parameters is:E(x)= ab & the variance of the gamma distribution with two parameters is:Var(x)= ab^2.
7-The mode of the gamma distribution is:Mode=$b(a-1)$
8- The harmonic mean of the gamma distribution is:H.M=$b(a-1)$.
9-The m.g.f of gamma distribution is: $M_x(t) = (1-bt)^{-a}$

Prove that total area under the curve is unity.

Proof:
Let by definition

Area=$\int f(x)dx$

As x≈ Gamma(a,b)

$$f(x) = \frac{1}{\Gamma a\, b^a} x^{a-1} e^{-x/b}$$

Then

$$Area = \int_0^\infty \frac{1}{\Gamma a\, b^a} x^{a-1} e^{-x/b} dx$$

$$AreA = \frac{1}{\overline{)ab^a}} \int_0^\infty x^{a-1} e^{-x/b} dx \qquad \textbf{(A)}$$

As we know that gamma function is:

$$\overline{)a} b^a = \int_0^\infty x^{a-1} e^{-x/b} dx \qquad \textbf{(B)}$$

Comparing (A) & (B):

a = a & b = b

$$\overline{)a} b^a = \overline{)a} b^a$$

Put in (A)

Area = $\frac{1}{\overline{)a} b^a} \overline{)a} b^a$ = 1 Hence Proved

Find r^th moments about origin. By use it find mean & variance.

Solution:

Let by definition

$$\mu_r' = E(x^r)$$

$$= \int x^r f(x) dx = \frac{1}{\overline{)a} b^a} \int_0^\infty x^r x^{a-1} e^{-x/b} dx$$

$$= \frac{1}{\overline{)a} b^a} \int_0^\infty x^{r+a-1} e^{-x/b} dx \qquad \textbf{(A)}$$

As we know that gamma function is:

$$\overline{)a} b^a = \int_0^\infty x^{a-1} e^{-x/b} dx \qquad \textbf{(B)}$$

Comparing (A) & (B)

a = a+r & b = b

$$\overline{)a} b^a = \overline{)a+r} b^{a+r}$$ **Put in (A)**

$$\mu_r' = \frac{1}{\overline{)a}.b^a} \overline{)a+r}.b^{a+r}$$

$$\mu_r' = \frac{1}{\overline{)a}.b^a} \overline{)a+r}.b^a.b^r = \frac{1}{\overline{)a}} \overline{)a+r}.b^r$$

$$\mu_r' = \frac{b^r \overline{)a+r}}{\overline{)a}} \qquad ©$$

Use rth moments to find mean & variance:

Mean = $\mu_1' = E(x)$

Put r = 1 in eq (C)

$$\mu_1' = \frac{b^1 \overline{)a+1}}{\overline{)a}} = \frac{ba\overline{)a}}{\overline{)a}} = ab$$

Now, put r = 2 in eq.(A)

$$\mu_2' = \frac{b^2\Gamma(a+2)}{\Gamma(a)}$$

$$\mu_2' = \frac{b^2\Gamma(a+1+1)}{\Gamma(a)} = \frac{b^2(a+1)\Gamma(a+1)}{\Gamma(a)} = \frac{b^2(a+1)a\Gamma(a)}{\Gamma(a)} = b^2(a+1)a$$

$$\mu_2' = ab^2(a+1)$$

$$\mathrm{Var(x)} = \mathrm{E(x^2)} - [\mathrm{E(x)}]^2 = ab^2(a+1) - (ab)^2 = a^2b^2 + ab^2 - a^2b^2 = ab^2$$

Question

Derive m.g.f of gamma distribution. Also find mean & variance by using m.g.f.

Solution:

Let by definition

$$M_x(t) = \mu_x(t) = E(e^{tx}) = \int e^{tx} f(x)dx$$

We know that

As x ≈ Gamma(a,b)

$$f(x) = \frac{1}{\Gamma(a)b^a} x^{a-1} e^{-x/b}$$

$$M_x(t) = \frac{1}{\Gamma(a)b^a}\int_0^\infty e^{tx} x^{a-1} e^{-x/b} dx$$

$$M_x(t) = \frac{1}{\Gamma(a)b^a}\int_0^\infty e^{tx - x/b} x^{a-1} dx$$

$$M_x(t) = \frac{1}{\Gamma(a)b^a}\int_0^\infty e^{-x\left(\frac{1}{b}-t\right)} x^{a-1} dx$$

$$M_x(t) = \frac{1}{\Gamma(a)b^a}\int_0^\infty x^{a-1} e^{-x\left(\frac{1-bt}{b}\right)} dx$$

$$M_x(t) = \frac{1}{\Gamma(a)b^a}\int_0^\infty x^{a-1} e^{-\frac{x}{b}(1-bt)} dx$$

$$M_x(t) = \frac{1}{\Gamma(a)b^a}\int_0^\infty x^{a-1} e^{-\frac{x}{b(1-bt)^{-1}}} dx \qquad \text{(A)}$$

As we know that gamma function is:

$$\Gamma(a)b^a = \int_0^\infty x^{a-1} e^{-x/b} dx \qquad \textbf{(B)}$$

Comparing (A) & (B) and we get

a = a   & b = $b(1-bt)^{-1}$

$$\Gamma(a)b^a = \Gamma(a)\{b(1-bt)^{-1}\}^a$$ **Put in (A)**

$$M_x(t) = \frac{1}{\Gamma(a)b^a}\Gamma(a)\{b(1-bt)^{-1}\}^a$$

$$M_x(t) = \frac{1}{b^a}\{b\}^a\{(1-bt)^{-1}\}^a$$

$M_x(t) = (1 - bt)^{-a}$ **Required m.g.f.**

Use it to find mean & variance.

$$E(x) = \mu_1' = \left[\frac{d}{dx} M_x(t)\right]_{t=0} = \left[\frac{d}{dx}(1-bt)^{-a}\right]_{t=0} = \left[-a(1-bt)^{-a-1}(-b)\right]_{t=0} = ab$$

$$E(x^2) = \mu_2' = \left[\frac{d^2}{dt^2} M_x(t)\right]_{t=0} = \frac{d}{dt}\left[\frac{d}{dt} M_x(t)\right]_{t=0}$$

$$E(x^2) = \frac{d}{dt}\left[ab(1-bt)^{-a-1}\right]_{t=0}$$

$$E(x^2) = \left[ab(-a-1)(1-bt)^{-a-2}(-b)\right]_{t=0}$$

$$E(x^2) = ab^2(a+1)$$

$$\text{Var(x)} = \text{E}(\text{x}^2) - [\text{E(x)}]^2 = ab^2(a+1) - (ab)^2 = a^2b^2 + ab^2 - a^2b^2 = ab^2$$

Question

Derive Characteristic function of gamma distribution

Solution:

Let by definition

$\theta_x(t) = M_x(it) = (1 - ibt)^{-a}$ Hence the required result

Question

Derive Cummulent generation function of gamma distribution

Solution:

Let by definition

$$K(t) = \log M_x(t)$$

$$K(t) = \log(1-bt)^{-a} = -\alpha \log(1-bt)$$

$$\log(1-x) = -x - \frac{x^2}{2} - \frac{x^3}{3} - \frac{x^4}{4} - .....$$

$$K(t) = -\alpha\left[-\beta t - \frac{(\beta t)^2}{2} - \frac{(\beta t)^3}{3} - \frac{(\beta t)^4}{4} - .....\right]$$

$$K(t) = \left[\alpha\beta t + \alpha\beta^2 \frac{t^2}{2} + \alpha\beta^3 \frac{t^3}{3} + \alpha\beta \frac{t^4}{4} + ...\right]$$

$$K(t) = \alpha\beta t + \alpha\beta^2 \frac{t^2}{2!} + 2\alpha\beta^3 \frac{t^3}{3!} + 6\alpha\beta \frac{t^4}{4!} + ... \quad \textbf{(A)}$$

General expression of rth cummulent

$$K(t) = K_1 t + K_2 \frac{t^2}{2!} + K_3 \frac{t^3}{3!} + K_4 \frac{t^4}{4!} + ... \quad \textbf{(B)}$$

Comparing (A) And (B) we get

$$K_1 = \mu_1' = Mean = \alpha\beta$$

$$K_2 = \mu_2 = \alpha\beta^2$$

$$K_3 = \mu_3 = 2\alpha\beta^3$$

$$K_4 = \mu_4' = 6\alpha\beta^4$$

$$\mu_4 = K_4 + 3K_2^2$$

$$\mu_4 = 6\alpha\beta^4 + 3(\alpha\beta^2)^2 = 6\alpha\beta^4 + 3\alpha^2\beta^4$$

$$\mu_4 = 3(2+\alpha)\alpha\beta^4$$

Question

Find mode of gamma distribution.

Solution:

If the two conditions are satisfied then mode exists.

$f(x')=0$ or $\frac{d}{dx}\log f(x)=0$

$f(x')<0$ or $\frac{d^2}{dx^2}\log f(x)<0$

As x ≈ Gamma(a,b)

f(x) = $\frac{1}{\overline{)ab^a}} x^{a-1} e^{-x/b}$

Taking log on both sides

$$\log f(x)=\log\left[\frac{1}{\overline{)ab^a}} x^{a-1} e^{-x/b}\right]$$

$$\log f(x)=\left[\log\left(\frac{1}{\overline{)ab^a}}\right)+(a-1)\log x-\frac{x}{b}\log e\right]$$ **:- $\log_e = 1$**

$$\log f(x)=\left[\log\left(\frac{1}{\overline{)ab^a}}\right)+(a-1)\log x-\frac{x}{b}\right]$$

Differentiate w.r.t to 'x':

$$\frac{d\log f(x)}{dx}=\frac{d}{dx}\left[\log\left(\frac{1}{\overline{)ab^a}}\right)+(a-1)\log x-\frac{x}{b}\right]$$

$$\frac{d\log f(x)}{dx}=0+\frac{(a-1)}{x}-\frac{1}{b}$$

$$\frac{d\log f(x)}{dx}=(a-1)x^{-1}-\frac{1}{b}$$ **------------------(i)**

Put $\frac{d\log f(x)}{dx}=0$

$$0=0+\frac{(a-1)}{x}-\frac{1}{b}$$

$$\frac{1}{b}=\frac{(a-1)}{x}$$

$$x=b(a-1)$$

Again diff. eq (i) w.r.t to "x"

$$\frac{d^2\log f(x)}{dx^2}=-1(a-1)x^{-2}$$

$$\frac{d^2\log f(x)}{dx^2}=\frac{-(a-1)}{x^2}$$

$$\frac{d^2\log f(x)}{dx^2}=\frac{-(a-1)}{b^2(a-1)^2}$$

$$\frac{d^2 \log f(x)}{dx^2} = \frac{-1}{b^2(a-1)}$$

$$\frac{d^2 \log f(x)}{dx^2} = -ve < 0 \quad at \quad x = b(a-1)$$

Both conditions are satisfied then mode is existing.

Mode = $\hat{x} = b(a-1)$

Question

Find Harmonic mean of gamma distribution

Solution:

Let by definition

$$H.M = \frac{1}{E\left(\frac{1}{x}\right)} \qquad \textbf{(A)}$$

$$E\left(\frac{1}{x}\right) = \int \frac{1}{x} f(x)dx = \frac{1}{\overline{)ab^a}} \int_0^\infty \frac{1}{x} x^{a-1} e^{-x/b} dx$$

$$= \frac{1}{\overline{)ab^a}} \int_0^\infty x^{(a-1)-1} e^{-x/b} dx \qquad \textbf{(B)}$$

As we know that gamma function is

$$\overline{)a}b^a = \int_0^\infty x^{a-1} e^{-x/b} dx \qquad \textbf{(C)}$$

Comparing (B) & (C):

a = a-1 & b = b

$\overline{)a}b^a = \overline{)a-1}.b^{a-1}$ Put in (B)

$$E\left(\frac{1}{x}\right) = \frac{1}{\overline{)ab^a}} \overline{)a-1} b^{a-1} = \frac{\overline{)a-1} b^a b^{-1}}{\overline{)(a-1)+1} b^a} = \frac{\overline{)a-1} b^{-1}}{(a-1)\overline{)(a-1)}} = \frac{b^{-1}}{(a-1)} = \frac{1}{b(a-1)}$$

Put in eq (A)

H.M = $\frac{1}{E\left(\frac{1}{x}\right)} = \frac{1}{\left(\frac{1}{b(a-1)}\right)} = b(a-1)$ **Required Result.**

Question

Prove that $\beta(a,b) = \frac{\overline{)a}\overline{)b}}{\overline{)a+b}}$

Solution:

We know that

$$\overline{)a} = \int_0^\infty x^{a-1} e^{-x} dx$$

$$\overline{)b} = \int_0^\infty y^{b-1} e^{-y} dy$$

$$\overline{)a}\overline{)b} = \int_0^\infty \int_0^\infty y^{b-1} x^{a-1} e^{-(x+y)} dxdy \qquad \textbf{(A)}$$

Put $z = \frac{x}{x+y}$, $s = x + y$

$z = \frac{x}{s}$, $s - x = y$

$sz = x$, $s(1-z) = y$

Partially differentiate old variables w.r.t to new variables

$\frac{\partial y}{\partial s} = (1-z)$ $\frac{\partial x}{\partial z} = s$

$\frac{\partial y}{\partial z} = -s$ $\frac{\partial x}{\partial s} = z$

$$J = \begin{vmatrix} \frac{\partial x}{\partial s} & \frac{\partial y}{\partial s} \\ \frac{\partial x}{\partial z} & \frac{\partial y}{\partial z} \end{vmatrix} = \begin{vmatrix} z & 1-z \\ s & -s \end{vmatrix} = -zs - s(1-z) = -zs - s + zs = -s$$

$dx.dy = |J| dsdz = sdsdz$

Limits

x & y → 0 then s→ 0

x & y → ∞ then s→ ∞

x & y → 0 then z→ 0

x & y → ∞ then z→ 1

Put in (i)

$$\overline{)a}\,\overline{)b} = \int_0^\infty \int_0^1 (sz)^{a-1} e^{-s} (s(1-z))^{b-1} sdzds$$

$$\overline{)a}\,\overline{)b} = \int_0^\infty \int_0^1 s^{a-1} (z)^{a-1} s^{b-1} (1-z)^{b-1} e^{-s} sdzds$$

$$\overline{)a}\,\overline{)b} = \int_0^\infty \int_0^1 s^{a+b-2+1} (z)^{a-1} (1-z)^{b-1} e^{-s} dzds$$

$$\overline{)a}\,\overline{)b} = \int_0^\infty s^{a+b-1} e^{-s} ds \int_0^1 (z)^{a-1} (1-z)^{b-1} dz \qquad \textbf{(B)}$$

In right hand side 1st function is gamma function & 2nd function is beta function of kind 1st.

As we know that gamma function is:

$$\overline{)a}b^a = \int_0^\infty x^{a-1} e^{-x/b} dx \qquad \textbf{(i)}$$

As we know that beta function is:

$$\beta(a,b) = \int_0^1 x^{a-1} (1-x)^{b-1} dx \qquad \textbf{(ii)}$$

Comparing (B) with (i) & (ii) :

$$\overline{)a}\,\overline{)b} = \overline{)a+b}.\beta(a,b)$$

$$\frac{\overline{)a}\,\overline{)b}}{\overline{)a+b}} = \beta(a,b) \qquad \beta(a,b) = \frac{\overline{)a}\,\overline{)b}}{\overline{)a+b}}$$

Hence proved.

Question

State and prove reproductive property of gamma distribution

Solution**:**

Statement: If $X_i (i = 1,2,3,...,,n)$ are "n" independent random variables from the gamma distribution with parameter (α, β) respectively. Then show that $\sum_{i=1}^{n} X_i$ also follow gamma distribution with parameter $(\sum_{i=1}^{n} \alpha, \beta) = (n\alpha, \beta)$

Proof: It is given that

$X_i (i = 1,2,3,...,,n) \rightarrow Gamma(\alpha_i, \beta)$ and moment generation function

$M_x(t) = (1-\beta t)^{-\alpha_i}$ $\quad\quad\quad\quad\quad i = 1,2,3,...,n$

Let $Z = \sum_{i=1}^{n} X_i$

Let by definition of m,g,f

$M_Z(t) = E(e^{tz})$

$M_Z(t) = E(e^{t\sum_{i=1}^{n} X_i}) = E(e^{tX_1 + tX_2 + tX_3 + ... + tX_n}) = E(e^{tX_1} e^{tX_2} e^{tX_3} .... e^{tX_n})$

As X's are independent then we get

$M_Z(t) = E(e^{tX_1})E(e^{tX_2})E(e^{tX_3}) .... E(e^{tX_n})$

$M_Z(t) = (1-\beta t)^{-\alpha_1}(1-\beta t)^{-\alpha_2}(1-\beta t)^{-\alpha_3} .... (1-\beta t)^{-\alpha_n}$

$M_Z(t) = (1-\beta t)^{-\alpha_1 - \alpha_2 - \alpha_3 - ... \alpha_n}$

$M_Z(t) = (1-\beta t)^{-\sum_{i=1}^{n}\alpha_i}$

$M_Z(t) = (1-\beta t)^{-n\alpha}$

Hence proved $\sum_{i=1}^{n} X_i \rightarrow Gamma(\sum_{i=1}^{n} \alpha_i, \beta)$

Question

If $X_1 \rightarrow Gamma(\alpha_1, 1)$ and $X_2 \rightarrow Gamma(\alpha_2, 1)$ then show that $X_1 + X_2$ also follows gamma distribution with parameters $(\alpha_1 + \alpha_2)$

Proof:

It is given that

$X_1 \rightarrow Gamma(\alpha_1, 1)$ with m.g.f $M_{x_1}(t) = (1-t)^{-\alpha_1}$

Similarly

$X_2 \rightarrow Gamma(\alpha_2, 1)$ with m.g.f $M_{x_2}(t) = (1-t)^{-\alpha_2}$

Let

$Z = X_1 + X_2$

Let by definition of m.g.f

$M_Z(t) = E(e^{tz})$

$M_Z(t) = E(e^{t(X_1 + X_2)})$

$M_Z(t) = E(e^{tX_1 + tX_2})$

$M_Z(t) = E(e^{tX_1} e^{tX_2})$

As X's are independent then we get

$M_Z(t) = E(e^{tX_1})E(e^{tX_2})$

$M_Z(t) = (1-t)^{-\alpha_1}(1-t)^{-\alpha_2} = (1-t)^{-(\alpha_1 + \alpha_2)}$

$Z = X_1 + X_2 \rightarrow Gamma(\alpha_1 + \alpha_2, 1)$ $\quad\quad\quad\quad\quad$ Hence proved

Gamma distribution with single parameter

f(x) = $\frac{1}{\Gamma a} x^{a-1} e^{-x}$ (0 ≤ x ≤ ∞)

And gamma function is

$$\Gamma a = \int_0^\infty x^{a-1} e^{-x} dx$$ 0 ≤ x ≤ ∞

Question

Find r^th moments about origin. By use it finds mean & variance.

Solution:

Let by definition

$$\mu_r' = E\left(x^r\right)$$

$$\mu_r' = \int x^r f(x) dx$$

$$\mu_r' = \frac{1}{\Gamma a} \int_0^\infty x^r x^{a-1} e^{-x} dx$$

$$\mu_r' = \frac{1}{\Gamma a} \int_0^\infty x^{a+r-1} e^{-x} dx$$ **(A)**

As we know that gamma function is:

$$\Gamma a = \int_0^\infty x^{a-1} e^{-x} dx$$ **(B)**

Comparing (A) & (B)

a = a+r

$$\Gamma a = \Gamma a+r$$ **Put in (A)**

$$\mu_r' = \frac{1}{\Gamma a.} \Gamma a+r.$$

Put r=1

$$\mu_1' = \frac{1}{\Gamma a.} \Gamma a+1$$

$$\mu_1' = \frac{1}{\Gamma a.} a \Gamma a$$

$$\mu_1' = a = Mean$$

Now, put r = 2 in eq.(A)

$$\mu_2' = \frac{\Gamma a+2}{\Gamma a}$$

$$\mu_2' = \frac{(a+1)\Gamma a+1}{\Gamma a}$$

$$\mu_2' = \frac{(a+1)a\Gamma a}{\Gamma a}$$

$$\mu_2' = (a+1)a = a^2 + a$$

$$\text{Var(x)} = \text{E}(\text{x}^2) - [\text{E(x)}]^2 = a(a+1) - (a)^2 = a^2 + a - a^2 = a$$

Similarly obtain the others results of single parameter gamma distribution

Mode

The mode of the gamma distribution is: Mode= $(a-1)$

Harmonic mean

The harmonic mean of the gamma distribution is: H.M= $(a-1)$.

Moment generating function

The m.g.f of gamma distribution is: $M_x(t) = (1-t)^{-a}$

Question

Show that $\Gamma(n+1) = n\Gamma(n)$

Solution: As we know that gamma function with single parameter

$$\Gamma(\alpha) = \int_0^\infty x^{a-1} e^{-x} dx \qquad \text{Put } \alpha = \Gamma(n+1)$$

$$\Gamma(n+1) = \int_0^\infty x^{n+1-1} e^{-x} dx$$

$$\Gamma(n+1) = \int_0^\infty x^n e^{-x} dx$$

Integrating by parts

(i) (ii)

$$\Gamma(n+1) = x^n \int_0^\infty e^{-x} dx - \int_0^\infty e^{-x} dx \frac{dx^n}{dx}$$

$$\Gamma(n+1) = x^n \int_0^\infty e^{-x} dx - \int_0^\infty e^{-x} dx \frac{dx^n}{dx}$$

$$\Gamma(n+1) = x^n \left. \frac{e^{-x}}{-1} \right|_0^\infty - \int_0^\infty \frac{e^{-x}}{-1} n x^{n-1} dx$$

$$\Gamma(n+1) = x^n \left. \frac{e^{-x}}{-1} \right|_0^\infty + \int_0^\infty e^{-x} n x^{n-1} dx$$

$$\Gamma(n+1) = 0 + n \int_0^\infty x^{n-1} e^{-x} dx$$

$$\Gamma(n+1) = n \int_0^\infty x^{n-1} e^{-x} dx$$

By comparing gamma function and we get

$\Gamma(n+1) = n\Gamma(n)$ **Hence proved**