...categories, either procedure can be used.

17.5.2. Pearson's Chi-Square Test For Goodness-of-Fit. A χ^2 statistic can also be applied even when the cell probabilities are not known and they depend upon the unknown parameters of a specified distribution such as the binomial distribution, the Poisson distribution the normal distribution, etc. This test is based on the property that "*the χ^2-test is applicable when the cell probabilities depend upon unknown parameters, provided that the unknown parameters are replaced with their estimates and provided that one degree of freedom is deducted for each parameter estimated.*" When there are k classes/categories and the class probabilities are known, the number of degrees of freedom is $(k-1)$. When the probabilities depend upon m parameters, the degrees of freedom would be $k-1-m$, *i.e.* *d.f.* = number of classes−1−number of parameters estimated from the sample. For example, in a normal distribution, the cell probabilities depend upon the two parameters μ and σ, therefore the degree of freedom is $(k-1-2)$, *i.e.* $(k-3)$.

A *goodness-of-fit* test is a hypothesis test that is concerned with the determination whether results of a sample conform to a hypothesized distribution which may be the uniform, binomial, Poisson, Normal or any other distribution. This is a kind of hypothesis test for problems where we do not know the probability distribution of the random variable under consideration, say X, and we wish to test the hypothesis that X follows a particular distribution. In the test procedure, the range of all possible values of the random variable assumed to follow a particular distribution is divided into k mutually exclusive classes and the probabilities p_i's are calculated for each of the classes, using the estimates of the parameters of the probability distribution specified in H_0. The $n\hat{p}_i$ represents the expected number of observations that fall in the *ith* class and n_i represents the observed number of observations in that class. The differences between observed and expected number of observations can arise from sampling error or from H_0 being false. Small differences are generally attributed to sampling error, large differences which are considered to arise from H_0 being false, are unlikely if the hypothesized distribution gives a satisfactory fit to the sample data (H_0 true). To see whether there is evidence of small or large differences, the test-statistic to use is

$$\chi^2 = \sum_{i=1}^{k} \frac{(n_i - n\hat{p}_i)^2}{n\hat{p}_i} = \sum \frac{(o_i - e_i)^2}{e_i},$$

which, when H_0 is true, has an approximate chi-square distribution with $d.f. = k - 1 -$ number of parameters estimated by sample statistics. The symbols o_i and e_i represent the observed and expected values of n_i respectively.

When the observed values are equal to the expected values, the $\chi^2 = 0$. The larger the differences between observed and expected values, the larger will be the value of χ^2. A small computed value of χ^2 indicates a good fit and it leads to the acceptance of the null hypothesis. A large computed value of χ^2 indicates a poor fit and it leads to the rejection of the null hypothesis. Hence the *rejection region in a goodness-of-fit test* (and all tests that compare frequencies) *will fall in the right tail of the chi-square distribution.*

The procedure for a goodness-of-fit test is as follows:

(i) Formulate the null and alternative hypotheses as

H_0: The population has a specified probability distribution, and

H_1: The population does not have the specified distribution.

(ii) Choose the level of significance α. The commonly used value is at $\alpha = 0.05$.

(iii) The test-statistic to use is

$$\chi^2 = \sum_{i=1}^{k} \frac{(o_i - e_i)^2}{e_i}$$

which, if H_0 is true, has an approximate chi-square distribution with $d.f. = k-1-$number of estimated parameters.

(iv) Determine the critical region, which depends upon α and the degrees of freedom.

(v) Compute the expected values and the value of χ^2.

(vi) Decide as below:

Reject H_0, if the calculated value of χ^2 exceeds the χ^2_α value against the appropriate degrees of freedom from the χ^2-table. Accept H_0, otherwise.

Example 17.10. Five pennies were tossed 1,000 times and the number of heads were observed as given below:

Number of heads	0	1	2	3	4	5
Frequencies	38	144	342	287	164	25

Test whether a binomial distribution gives a satisfactory fit to these data.

(P.U., B.A./B.Sc. 1980)

(i) We state our hypotheses as

H_0: The population distribution is a binomial with $n=5$, but with parameter p unspecified, and

H_1: The population distribution is not a binomial with $n=5$.

(ii) We choose the significance level at $\alpha = 0.05$.

(iii) We use the test-statistic

$$\chi^2 = \sum \frac{(o_i - e_i)^2}{e_i},$$

which, if H_0 is true, has an approximate χ^2-distribution with degrees of freedom $= k-1-$number of estimated parameters. A binomial distribution has two parameters n and p but n is specified ($n=5$). We have to estimate the value of one parameter

p from the sample data. Therefore the degrees of freedom = 6−1−1 = 4 ($\because$ k = 6 categories).

(iv) The critical region is $\chi^2 \geq \chi^2_{0.05,(4)} = 9.49$.

(v) **Computations.** To estimate the value of p, we first compute the mean number of heads, $\bar{x}$. Thus

$$\bar{x} = \frac{\sum fx}{\sum f} = \frac{2470}{1000} = 2.47.$$

Theoretically, $\bar{x} = np$, so that $\hat{p} = \frac{\bar{x}}{n} = \frac{2.47}{5} = 0.494$.

Hence the expected (fitted) frequencies are the terms in the binomial expansion of $1{,}000(0.506+0.494)^5$, which are given in the column headed e_i. Next, we calculate the value of χ^2 as follows:

Number of heads	Observed f, (o_i)	Expected f, (e_i)	$o_i - e_i$	$(o_i - e_i)^2$	$(o_i - e_i)^2/e_i$
0	38	33.2	4.8	23.04	0.69
1	144	161.9	−17.9	320.41	1.98
2	342	316.2	25.8	665.64	2.15
3	287	308.7	−21.7	470.89	1.53
4	164	150.7	13.3	176.89	1.17
5	25	29.4	−4.4	19.36	0.66
Total	1,000	1,000.1	---	---	$\chi^2 = 8.18$

(vi) **Conclusion.** Since the calculated value of $\chi^2 = 8.18$ does not fall in the critical region, we therefore are unable to reject our null hypothesis. We may accept the hypothesis that the distribution of the number of heads is a binomial distribution and conclude that the fit of data is good.