PART C

Fourier Analysis. Partial Differential Equations

Fourier analysis concerns **periodic phenomena,** as they occur quite frequently in engineering and elsewhere—think of rotating parts of machines, alternating electric currents, or the motion of planets. Related periodic functions may be complicated. This situation poses the important practical task of representing these complicated functions in terms of simple periodic functions, namely, cosines and sines. These representations will be infinite series, called **Fourier series.**[1]

The creation of these series was one of the most path-breaking events in applied mathematics, and we mention that it also had considerable influence on mathematics as a whole, on the concept of a function, on integration theory, on convergence theory for series, and so on (see Ref. [GR7] in App. 1).

Chapter 11 is concerned mainly with Fourier series. However, the underlying ideas can also be extended to *nonperiodic* phenomena. This leads to *Fourier integrals* and *transforms.* A common name for the whole area is **Fourier analysis.**

Chapter 12 deals with the most important **partial differential equations (PDEs)** of physics and engineering. This is the area in which Fourier analysis has its most basic applications, related to boundary and initial value problems of mechanics, heat flow, electrostatics, and other fields.

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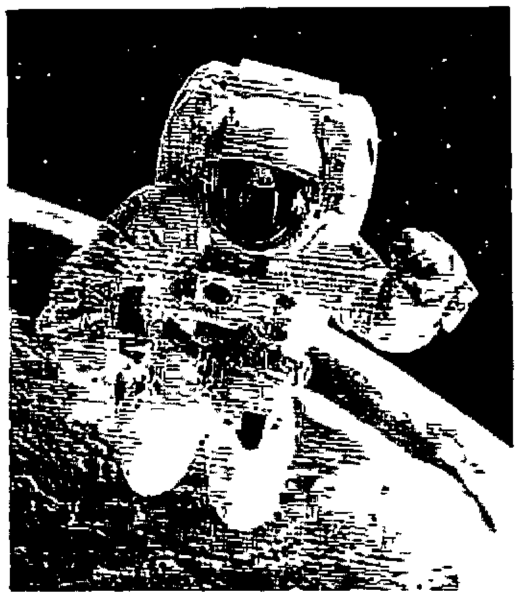
[1]JEAN-BAPTISTE JOSEPH FOURIER (1768–1830), French physicist and mathematician, lived and taught in Paris, accompanied Napoléon in the Egyptian War, and was later made prefect of Grenoble. The beginnings on Fourier series can be found in works by Euler and by Daniel Bernoulli, but it was Fourier who employed them in a systematic and general manner in his main work, *Théorie analytique de la chaleur* (*Analytic Theory of Heat,* Paris, 1822), in which he developed the theory of heat conduction (heat equation; see Sec. 12.5), making these series a most important tool in applied mathematics.



CHAPTER 11

Fourier Series, Integrals, and Transforms

Fourier series (Sec. 11.1) are infinite series designed to represent general periodic functions in terms of simple ones, namely, cosines and sines. They constitute a very important tool, in particular in solving problems that involve ODEs and PDEs.

In this chapter we discuss Fourier series and their engineering use from a practical point of view, in connection with ODEs and with the approximation of periodic functions. Application to PDEs follows in Chap. 12.

The *theory* of Fourier series is complicated, but we shall see that the *application* of these series is rather simple. Fourier series are in a certain sense more universal than the familiar Taylor series in calculus because many *discontinuous* periodic functions of practical interest can be developed in Fourier series but, of course, do not have Taylor series representations.

In the last sections (11.7–11.9) we consider **Fourier integrals** and **Fourier transforms,** which extend the ideas and techniques of Fourier series to nonperiodic functions and have basic applications to PDEs (to be shown in the next chapter).

Prerequisite: Elementary integral calculus (needed for Fourier coefficients)
Sections that may be omitted in a shorter course: 11.4–11.9
References and Answers to Problems: App. 1 Part C, App. 2.

11.1 Fourier Series

Fourier series are the basic tool for representing periodic functions, which play an important role in applications. A function $f(x)$ is called a **periodic function** if $f(x)$ is defined for all real x (perhaps except at some points, such as $x = \pm\pi/2, \pm 3\pi/2, \cdots$ for $\tan x$) and if there is some positive number p, called a **period** of $f(x)$, such that

$$f(x + p) = f(x) \qquad \text{for all } x. \tag{1}$$

The graph of such a function is obtained by periodic repetition of its graph in any interval of length p (Fig. 255).

Familiar periodic functions are the cosine and sine functions. Examples of functions that are not periodic are $x, x^2, x^3, e^x, \cosh x$, and $\ln x$, to mention just a few.

If $f(x)$ has period p, it also has the period $2p$ because (1) implies
$f(x + 2p) = f([x + p] + p) = f(x + p) = f(x)$, etc.; thus for any integer $n = 1, 2, 3, \cdots$,

$$f(x + np) = f(x) \qquad \text{for all } x. \tag{2}$$

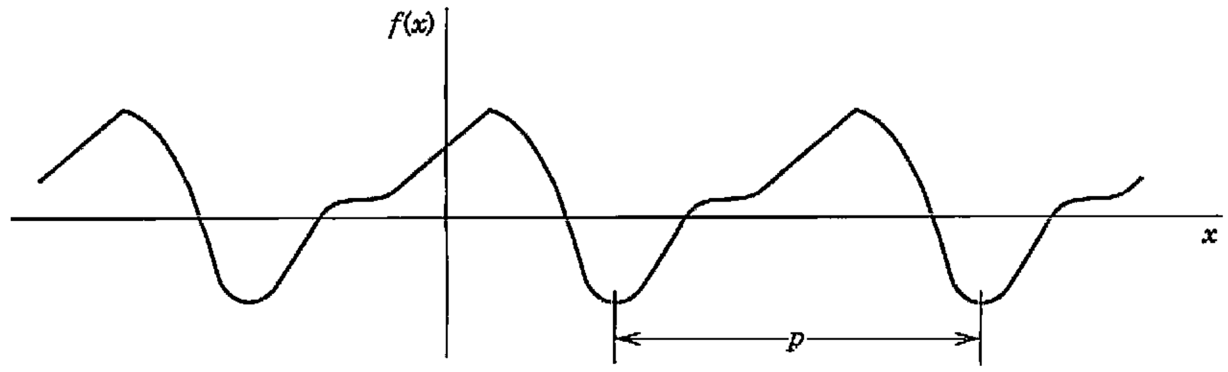


Fig. 255. Periodic function

Furthermore if $f(x)$ and $g(x)$ have period p, then $af(x) + bg(x)$ with any constants a and b also has the period p.

Our problem in the first few sections of this chapter will be the representation of various *functions* $f(x)$ *of period* 2π in terms of the simple functions

$$(3)\quad 1, \qquad \cos x, \quad \sin x, \qquad \cos 2x, \quad \sin 2x, \cdots, \qquad \cos nx, \quad \sin nx, \cdots .$$

All these functions have the period 2π. They form the so-called **trigonometric system.** Figure 256 shows the first few of them (except for the constant 1, which is periodic with any period).

The series to be obtained will be a **trigonometric series,** that is, a series of the form

$$(4)\quad \begin{aligned} &a_0 + a_1 \cos x + b_1 \sin x + a_2 \cos 2x + b_2 \sin 2x + \cdots \\ &= a_0 + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx). \end{aligned}$$

$a_0, a_1, b_1, a_2, b_2, \cdots$ are constants, called the **coefficients** of the series. We see that each term has the period 2π. Hence *if the coefficients are such that the series converges, its sum will be a function of period* 2π.

It can be shown that if the series on the left side of (4) converges, then inserting parentheses on the right gives a series that converges and has the same sum as the series on the left. This justifies the equality in (4).

Now suppose that $f(x)$ is a given function of period 2π and is such that it can be **represented** by a series (4), that is, (4) converges and, moreover, has the sum $f(x)$. Then, using the equality sign, we write

$$(5)\quad f(x) = a_0 + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)$$

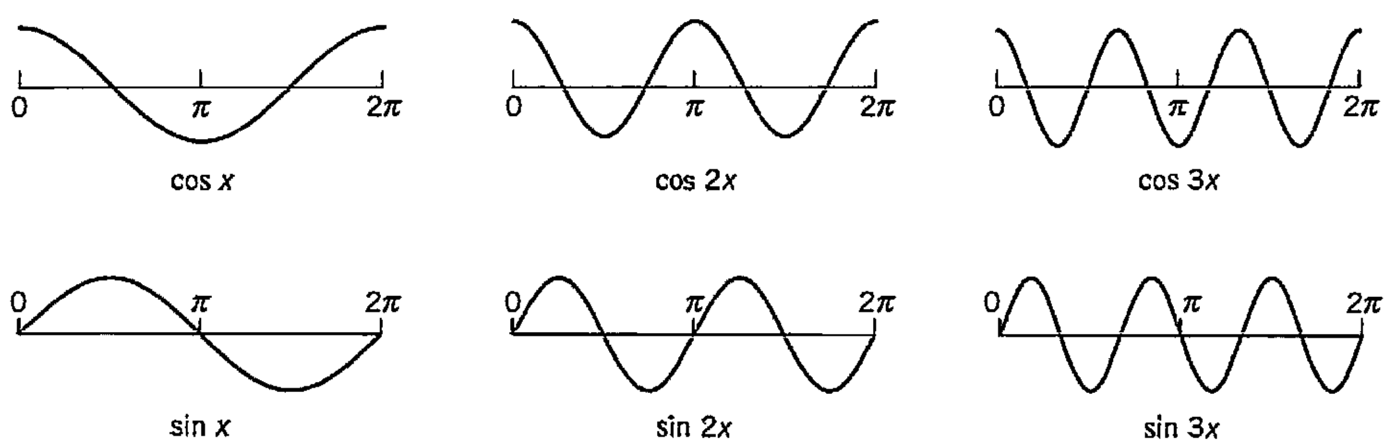


Fig. 256. Cosine and sine functions having the period 2π

and call (5) the **Fourier series** of $f(x)$. We shall prove that in this case the coefficients of (5) are the so-called **Fourier coefficients** of $f(x)$, given by the **Euler formulas**

$$\text{(6)}\quad \begin{aligned} &\text{(a)} && a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\,dx \\ &\text{(b)} && a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx && n = 1, 2, \cdots \\ &\text{(c)} && b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx && n = 1, 2, \cdots. \end{aligned}$$

The name "Fourier series" is sometimes also used in the exceptional case that (5) with coefficients (6) does not converge or does not have the sum $f(x)$—this may happen but is merely of theoretical interest. (For Euler see footnote 4 in Sec. 2.5.)

A Basic Example

Before we derive the Euler formulas (6), let us become familiar with the application of (5) and (6) in the case of an important example. Since your work for other functions will be quite similar, try to fully understand every detail of the integrations, which because of the n involved differ somewhat from what you have practiced in calculus. Do not just routinely use your software, but make observations: How are continuous functions (cosines and sines) able to represent a given discontinuous function? How does the quality of the approximation increase if you take more and more terms of the series? Why are the approximating functions, called the **partial sums** of the series, always zero at 0 and π? Why is the factor $1/n$ (obtained in the integration) important?

EXAMPLE 1 Periodic Rectangular Wave (Fig. 257a)

Find the Fourier coefficients of the periodic function $f(x)$ in Fig. 257a. The formula is

$$\text{(7)}\qquad f(x) = \begin{cases} -k & \text{if} \quad -\pi < x < 0 \\ k & \text{if} \quad 0 < x < \pi \end{cases} \qquad \text{and} \qquad f(x + 2\pi) = f(x).$$

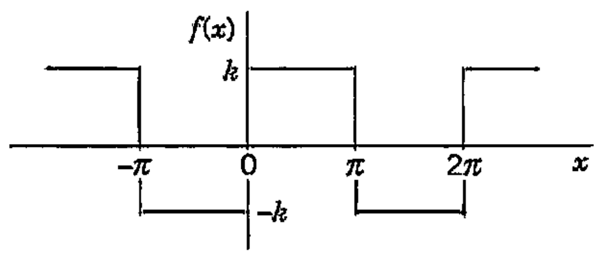
Functions of this kind occur as external forces acting on mechanical systems, electromotive forces in electric circuits, etc. (The value of $f(x)$ at a single point does not affect the integral; hence we can leave $f(x)$ undefined at $x = 0$ and $x = \pm\pi$.)

Solution. From (6a) we obtain $a_0 = 0$. This can also be seen without integration, since the area under the curve of $f(x)$ between $-\pi$ and π is zero. From (6b),

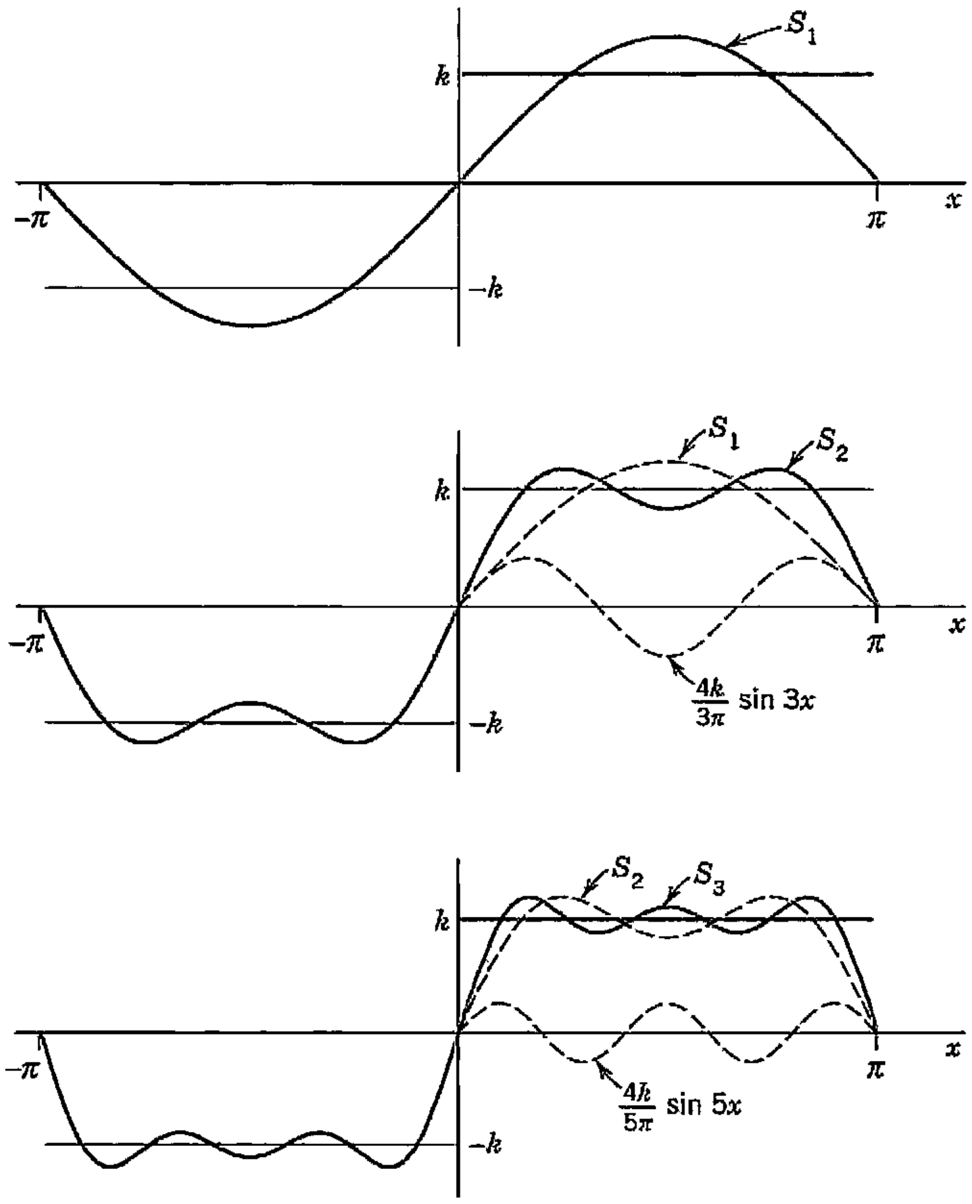
$$\begin{aligned} a_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx = \frac{1}{\pi}\left[\int_{-\pi}^{0} (-k)\cos nx\,dx + \int_{0}^{\pi} k\cos nx\,dx\right] \\ &= \frac{1}{\pi}\left[-k\,\frac{\sin nx}{n}\bigg|_{-\pi}^{0} + k\,\frac{\sin nx}{n}\bigg|_{0}^{\pi}\right] = 0 \end{aligned}$$

because $\sin nx = 0$ at $-\pi$, 0, and π for all $n = 1, 2, \cdots$. Similarly, from (6c) we obtain

$$\begin{aligned} b_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx = \frac{1}{\pi}\left[\int_{-\pi}^{0} (-k)\sin nx\,dx + \int_{0}^{\pi} k\sin nx\,dx\right] \\ &= \frac{1}{\pi}\left[k\,\frac{\cos nx}{n}\bigg|_{-\pi}^{0} - k\,\frac{\cos nx}{n}\bigg|_{0}^{\pi}\right]. \end{aligned}$$



(a) The given function $f(x)$ (Periodic rectangular wave)



(b) The first three partial sums of the corresponding Fourier series

Fig. 257. Eample 1

Since $\cos(-\alpha) = \cos\alpha$ and $\cos 0 = 1$, this yields

$$b_n = \frac{k}{n\pi}\left[\cos 0 - \cos(-n\pi) - \cos n\pi + \cos 0\right] = \frac{2k}{n\pi}(1 - \cos n\pi).$$

Now, $\cos\pi = -1$, $\cos 2\pi = 1$, $\cos 3\pi = -1$, etc.; in general,

$$\cos n\pi = \begin{cases} -1 & \text{for odd } n, \\ 1 & \text{for even } n, \end{cases} \qquad \text{and thus} \qquad 1 - \cos n\pi = \begin{cases} 2 & \text{for odd } n, \\ 0 & \text{for even } n. \end{cases}$$

Hence the Fourier coefficients b_n of our function are

$$b_1 = \frac{4k}{\pi}, \qquad b_2 = 0, \qquad b_3 = \frac{4k}{3\pi}, \qquad b_4 = 0, \qquad b_5 = \frac{4k}{5\pi}, \cdots.$$

Since the a_n are zero, the Fourier series of $f(x)$ is

$$\frac{4k}{\pi}\left(\sin x + \frac{1}{3}\sin 3x + \frac{1}{5}\sin 5x + \cdots\right). \tag{8}$$

The partial sums are

$$S_1 = \frac{4k}{\pi}\sin x, \qquad S_2 = \frac{4k}{\pi}\left(\sin x + \frac{1}{3}\sin 3x\right), \qquad \text{etc.,}$$

Their graphs in Fig. 257 seem to indicate that the series is convergent and has the sum $f(x)$, the given function. We notice that at $x = 0$ and $x = \pi$, the points of discontinuity of $f(x)$, all partial sums have the value zero, the arithmetic mean of the limits $-k$ and k of our function, at these points.

Furthermore, assuming that $f(x)$ is the sum of the series and setting $x = \pi/2$, we have

$$f\left(\frac{\pi}{2}\right) = k = \frac{4k}{\pi}\left(1 - \frac{1}{3} + \frac{1}{5} - + \cdots\right).$$

thus

$$1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + - \cdots = \frac{\pi}{4}.$$

This is a famous result obtained by Leibniz in 1673 from geometric considerations. It illustrates that the values of various series with constant terms can be obtained by evaluating Fourier series at specific points. ■

Derivation of the Euler Formulas (6)

The key to the Euler formulas (6) is the **orthogonality** of (3), a concept of basic importance, as follows.

THEOREM 1

Orthogonality of the Trigonometric System (3)

The trigonometric system (3) *is orthogonal on the interval* $-\pi \leqq x \leqq \pi$ (hence also on $0 \leqq x \leqq 2\pi$ or any other interval of length 2π because of periodicity); *that is, the integral of the product of any two functions in* (3) *over that interval is* 0, *so that for any integers n and m,*

$$\text{(9)} \quad \text{(a)} \int_{-\pi}^{\pi} \cos nx \cos mx \, dx = 0 \qquad (n \neq m)$$

$$\text{(b)} \int_{-\pi}^{\pi} \sin nx \sin mx \, dx = 0 \qquad (n \neq m)$$

$$\text{(c)} \int_{-\pi}^{\pi} \sin nx \cos mx \, dx = 0 \qquad (n \neq m \text{ or } n = m).$$

PROOF This follows simply by transforming the integrands trigonometrically from products into sums. In (9a) and (9b), by (11) in App. A3.1,

$$\int_{-\pi}^{\pi} \cos nx \cos mx \, dx = \frac{1}{2}\int_{-\pi}^{\pi} \cos (n+m)x \, dx + \frac{1}{2}\int_{-\pi}^{\pi} \cos (n-m)x \, dx$$

$$\int_{-\pi}^{\pi} \sin nx \sin mx \, dx = \frac{1}{2}\int_{-\pi}^{\pi} \cos (n-m)x \, dx - \frac{1}{2}\int_{-\pi}^{\pi} \cos (n+m)x \, dx.$$

Since $m \neq n$ (integer!), the integrals on the right are all 0. Similarly, in (9c), for all integer m and n (without exception; do you see why?)

$$\int_{-\pi}^{\pi} \sin nx \cos mx \, dx = \frac{1}{2}\int_{-\pi}^{\pi} \sin (n+m)x \, dx + \frac{1}{2}\int_{-\pi}^{\pi} \sin (n-m)x \, dx = 0 + 0. \quad \blacksquare$$

Application of Theorem 1 to the Fourier Series (5)

We prove (6a). Integrating on both sides of (5) from $-\pi$ to π, we get

$$\int_{-\pi}^{\pi} f(x) \, dx = \int_{-\pi}^{\pi} \left[a_0 + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx) \right] dx.$$

We now assume that termwise integration is allowed. (We shall say in the proof of Theorem 2 when this is true.) Then we obtain

$$\int_{-\pi}^{\pi} f(x) \, dx = a_0 \int_{-\pi}^{\pi} dx + \sum_{n=1}^{\infty} \left(a_n \int_{-\pi}^{\pi} \cos nx \, dx + b_n \int_{-\pi}^{\pi} \sin nx \, dx \right).$$

The first term on the right equals $2\pi a_0$. Integration shows that all the other integrals are 0. Hence division by 2π gives (6a).

We prove (6b). Multiplying (5) on both sides by $\cos mx$ with any *fixed* positive integer m and integrating from $-\pi$ to π, we have

$$(10) \qquad \int_{-\pi}^{\pi} f(x) \cos mx \, dx = \int_{-\pi}^{\pi} \left[a_0 + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx) \right] \cos mx \, dx.$$

We now integrate term by term. Then on the right we obtain an integral of $a_0 \cos mx$, which is 0; an integral of $a_n \cos nx \cos mx$, which is $a_m \pi$ for $n = m$ and 0 for $n \neq m$ by (9a); and an integral of $b_n \sin nx \cos mx$, which is 0 for all n and m by (9c). Hence the right side of (10) equals $a_m \pi$. Division by π gives (6b) (with m instead of n).

We finally prove (6c). Multiplying (5) on both sides by $\sin mx$ with any *fixed* positive integer m and integrating from $-\pi$ to π, we get

$$(11) \qquad \int_{-\pi}^{\pi} f(x) \sin mx \, dx = \int_{-\pi}^{\pi} \left[a_0 + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx) \right] \sin mx \, dx.$$

Integrating term by term, we obtain on the right an integral of $a_0 \sin mx$, which is 0; an integral of $a_n \cos nx \sin mx$, which is 0 by (9c); and an integral of $b_n \sin nx \sin mx$, which is $b_m \pi$ if $n = m$ and 0 if $n \neq m$, by (9b). This implies (6c) (with n denoted by m). This completes the proof of the Euler formulas (6) for the Fourier coefficients. $\blacksquare$

Convergence and Sum of a Fourier Series

The class of functions that can be represented by Fourier series is surprisingly large and general. Sufficient conditions valid in most applications are as follows.

THEOREM 2

Representation by a Fourier Series

Let $f(x)$ *be periodic with period* 2π *and piecewise continuous* (see Sec. 6.1) *in the interval* $-\pi \leqq x \leqq \pi$. *Furthermore, let* $f(x)$ *have a left-hand derivative and a right-hand derivative at each point of that interval. Then the Fourier series* (5) *of* $f(x)$ [with coefficients (6)] *converges. Its sum is* $f(x)$, *except at points* x_0 *where* $f(x)$ *is discontinuous. There the sum of the series is the average of the left- and right-hand limits*[2] *of* $f(x)$ *at* x_0.

PROOF We prove convergence in Theorem 2. We prove convergence for a continuous function $f(x)$ having continuous first and second derivatives. Integrating (6b) by parts, we obtain.

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx = \frac{f(x)\sin nx}{n\pi}\Bigg|_{-\pi}^{\pi} - \frac{1}{n\pi}\int_{-\pi}^{\pi} f'(x)\sin nx\,dx.$$

The first term on the right is zero. Another integration by parts gives

$$a_n = \frac{f'(x)\cos nx}{n^2\pi}\Bigg|_{-\pi}^{\pi} - \frac{1}{n^2\pi}\int_{-\pi}^{\pi} f''(x)\cos nx\,dx.$$

The first term on the right is zero because of the periodicity and continuity of $f'(x)$. Since f'' is continuous in the interval of integration, we have

$$|f''(x)| < M$$

for an appropriate constant M. Furthermore, $|\cos nx| \leqq 1$. It follows that

$$|a_n| = \frac{1}{n^2\pi}\left|\int_{-\pi}^{\pi} f''(x)\cos nx\,dx\right| < \frac{1}{n^2\pi}\int_{-\pi}^{\pi} M\,dx = \frac{2M}{n^2}.$$

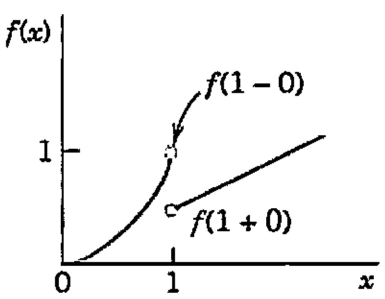


Fig. 258. Left- and right-hand limits

$f(1-0) = 1,$

$f(1+0) = \frac{1}{2}$

of the function

$$f(x) = \begin{cases} x^2 & \text{if } x < 1 \\ x/2 \end{cases}$$

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[2]The **left-hand limit** of $f(x)$ at x_0 is defined as the limit of $f(x)$ as x approaches x_0 from the left and is commonly denoted by $f(x_0 - 0)$. Thus

$$f(x_0 - 0) = \lim_{h\to 0} f(x_0 - h) \text{ as } h \to 0 \text{ through positive values.}$$

The **right-hand limit** is denoted by $f(x_0 + 0)$ and

$$f(x_0 + 0) = \lim_{h\to 0} f(x_0 + h) \text{ as } h \to 0 \text{ through positive values.}$$

The **left-** and **right-hand derivatives** of $f(x)$ at x_0 are defined as the limits of

$$\frac{f(x_0 - h) - f(x_0 - 0)}{-h} \quad \text{and} \quad \frac{f(x_0 + h) - f(x_0 + 0)}{h},$$

respectively, as $h \to 0$ through positive values. Of course if $f(x)$ is continuous at x_0, the last term in both numerators is simply $f(x_0)$.

Similarly, $|b_n| < 2\,M/n^2$ for all n. Hence the absolute value of each term of the Fourier series of $f(x)$ is at most equal to the corresponding term of the series

$$|a_0| + 2M\left(1 + 1 + \frac{1}{2^2} + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{3^2} + \cdots\right)$$

which is convergent. Hence that Fourier series converges and the proof is complete. (Readers already familiar with uniform convergence will see that, by the Weierstrass test in Sec. 15.5, under our present assumptions the Fourier series converges uniformly, and our derivation of (6) by integrating term by term is then justified by Theorem 3 of Sec. 15.5.)

The proof of convergence in the case of a piecewise continuous function $f(x)$ and the proof that under the assumptions in the theorem the Fourier series (5) with coefficients (6) represents $f(x)$ are substantially more complicated; see, for instance, Ref. [C12]. ■

EXAMPLE 2 Convergence at a Jump as Indicated in Theorem 2

The rectangular wave in Example 1 has a jump at $x = 0$. Its left-hand limit there is $-k$ and its right-hand limit is k (Fig. 257). Hence the average of these limits is 0. The Fourier series (8) of the wave does indeed converge to this value when $x = 0$ because then all its terms are 0. Similarly for the other jumps. This is in agreement with Theorem 2. ■

Summary. A Fourier series of a given function $f(x)$ of period 2π is a series of the form (5) with coefficients given by the Euler formulas (6). Theorem 2 gives conditions that are sufficient for this series to converge and at each x to have the value $f(x)$, except at discontinuities of $f(x)$, where the series equals the arithmetic mean of the left-hand and right-hand limits of $f(x)$ at that point.

PROBLEM SET 11.1

1. **(Calculus review)** Review integration techniques for integrals as they are likely to arise from the Euler formulas, for instance, definite integrals of $x \cos nx$, $x^2 \sin nx$, $e^{-2x} \cos nx$, etc.

2–3 FUNDAMENTAL PERIOD

The *fundamental period* is the smallest positive period. Find it for

2. $\cos x$, $\sin x$, $\cos 2x$, $\sin 2x$, $\cos \pi x$, $\sin \pi x$, $\cos 2\pi x$, $\sin 2\pi x$

3. $\cos nx$, $\sin nx$, $\cos \dfrac{2\pi x}{k}$, $\sin \dfrac{2\pi x}{k}$, $\cos \dfrac{2\pi n x}{k}$, $\sin \dfrac{2\pi n x}{k}$

4. Show that $f = const$ is periodic with any period but has no fundamental period.

5. If $f(x)$ and $g(x)$ have period p, show that $h(x) = af(x) + bg(x)$ (a, b, constant) has the period p. Thus all functions of period p form a **vector space.**

6. **(Change of scale)** If $f(x)$ has period p, show that $f(ax)$, $a \neq 0$, and $f(x/b)$, $b \neq 0$, are periodic functions of x of periods p/a and bp, respectively. Give examples.

7–12 GRAPHS OF 2π-PERIODIC FUNCTIONS

Sketch or graph $f(x)$, of period 2π, which for $-\pi < x < \pi$ is given as follows.

7. $f(x) = x$

8. $f(x) = e^{-|x|}$

9. $f(x) = \pi - |x|$

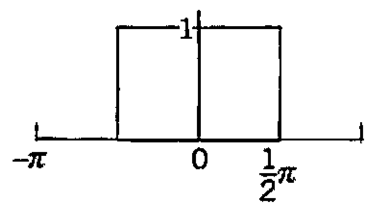
10. $f(x) = |\sin 2x|$

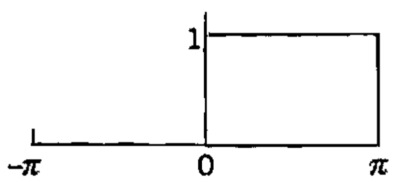
11. $f(x) = \begin{cases} -x^3 & \text{if } -\pi < x < 0 \\ x^3 & \text{if } \ 0 < x < \pi \end{cases}$

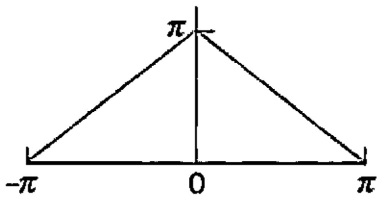
12. $f(x) = \begin{cases} 1 & \text{if } -\pi < x < 0 \\ \cos \frac{1}{2}x & \text{if } \ 0 < x < \pi \end{cases}$

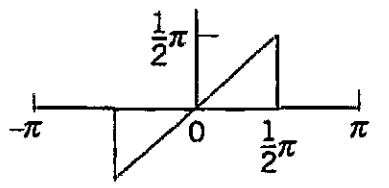
13–24 FOURIER SERIES

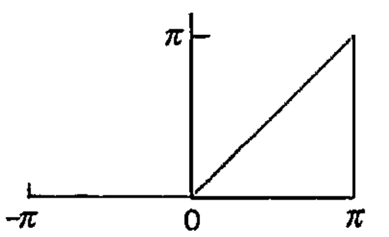
Showing the details of your work, find the Fourier series of the given $f(x)$, which is assumed to have the period 2π. Sketch or graph the partial sums up to that including $\cos 5x$ and $\sin 5x$.

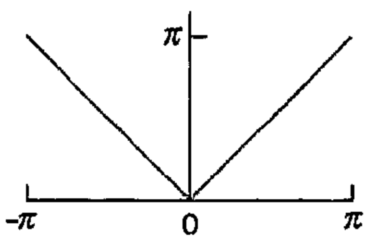
13. 

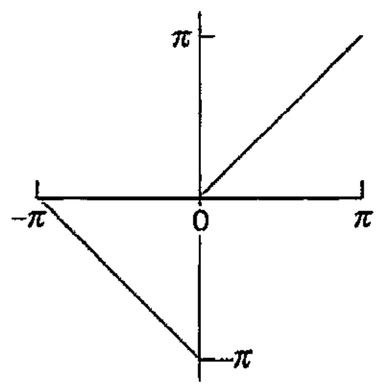
14. 

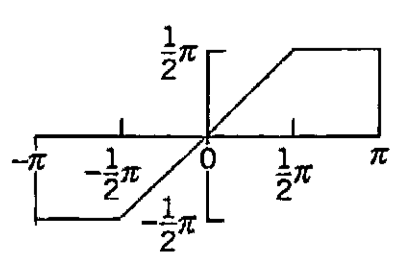
15. 

16. 

17. 

18. 

19. 

20. 

21. $f(x) = x^2 \ (-\pi < x < \pi)$

22. $f(x) = x^2 \ (0 < x < 2\pi)$

23. $f(x) = \begin{cases} x^2 & \text{if } -\frac{1}{2}\pi < x < \frac{1}{2}\pi \\ \frac{1}{4}\pi^2 & \text{if } \frac{1}{2}\pi < x < \frac{3}{2}\pi \end{cases}$

24. $f(x) = \begin{cases} -4x & \text{if } -\pi < x < 0 \\ 4x & \text{if } 0 < x < \pi \end{cases}$

25. **(Discontinuities)** Verify the last statement in Theorem 2 for the discontinuities of $f(x)$ in Prob. 13.

26. **CAS EXPERIMENT. Graphing.** Write a program for graphing partial sums of the following series. Guess from the graph what $f(x)$ the series may represent. Confirm or disprove your guess by using the Euler formulas.

(a) $2(\sin x + \frac{1}{3}\sin 3x + \frac{1}{5}\sin 5x + \cdots)$

$- 2(\frac{1}{2}\sin 2x + \frac{1}{4}\sin 4x + \frac{1}{6}\sin 6x \cdots)$

(b) $\frac{1}{2} + \frac{4}{\pi^2}(\cos x + \frac{1}{9}\cos 3x + \frac{1}{25}\cos 5x + \cdots)$

(c) $\frac{2}{3}\pi^2 + 4(\cos x - \frac{1}{4}\cos 2x + \frac{1}{9}\cos 3x - \frac{1}{16}\cos 4x + - \cdots)$

27. **CAS EXPERIMENT. Order of Fourier Coefficients.** The order seems to be $1/n$ if f is discontinous, and $1/n^2$ if f is continuous but $f' = df/dx$ is discontinuous, $1/n^3$ if f and f' are continuous but f'' is discontinuous, etc. Try to verify this for examples. Try to prove it by integrating the Euler formulas by parts. What is the practical significance of this?

28. **PROJECT. Euler Formulas in Terms of Jumps Without Integration.** Show that for a function whose third derivative is identically zero,

$$a_n = \frac{1}{n\pi}\left[-\sum j_s \sin nx_s - \frac{1}{n}\sum j_s' \cos nx_s + \frac{1}{n^2}\sum j_s'' \sin nx_s\right]$$

$$b_n = \frac{1}{n\pi}\left[\sum j_s \cos nx_s - \frac{1}{n}\sum j_s' \sin nx_s - \frac{1}{n^2}\sum j_s'' \cos nx_s\right]$$

where $n = 1, 2, \cdots$ and we sum over all the jumps j_s, j_s', j_s'' of f, f', f', respectively, located at x_s.

29. Apply the formulas in Project 28 to the function in Prob. 21 and compare the results.

30. **CAS EXPERIMENT. Orthogonality.** Integrate and graph the integral of the product $\cos mx \cos nx$ (with various integer m and n of your choice) from $-a$ to a as a function of a and conclude orthogonality of $\cos mx$ and $\cos nx$ $(m \neq n)$ for $a = \pi$ from the graph. For what m and n will you get orthogonality for $a = \pi/2$, $\pi/3$, $\pi/4$? Other a? Extend the experiment to $\cos mx \sin nx$ and $\sin mx \sin nx$.

11.2 Functions of Any Period $p = 2L$

The functions considered so far had period 2π, for the simplicity of the formulas. Of course, periodic functions in applications will generally have other periods. However, we now show that the transition from period $p = 2\pi$ to a period $2L$ is quite simple. The notation $p = 2L$ is practical because L will be the length of a violin string (Sec. 12.2) or the length of a rod in heat conduction (Sec. 12.5), and so on.

The idea is simply to find and use a ***change of scale*** that gives from a function $g(v)$ of period 2π a function of period $2L$. Now from (5) and (6) in the last section with $g(v)$ instead of $f(x)$ we have the Fourier series

$$g(v) = a_0 + \sum_{n=1}^{\infty} (a_n \cos nv + b_n \sin nv) \tag{1}$$

with coefficients

$$\begin{aligned} a_0 &= \frac{1}{2\pi} \int_{-\pi}^{\pi} g(v)\, dv \\ a_n &= \frac{1}{\pi} \int_{-\pi}^{\pi} g(v) \cos nv\, dv \\ b_n &= \frac{1}{\pi} \int_{-\pi}^{\pi} g(v) \sin nv\, dv. \end{aligned} \tag{2}$$

We can now write the change of scale as $v = kx$ with k such that the old period $v = 2\pi$ gives for the new variable x the new period $x = 2L$. Thus, $2\pi = k2L$. Hence $k = \pi/L$ and

$$v = kx = \pi x/L. \tag{3}$$

This implies $dv = (\pi/L)\, dx$, which upon substitution into (2) cancels $1/2\pi$ and $1/\pi$ and gives instead the factors $1/2L$ and $1/L$. Writing

$$g(v) = f(x), \tag{4}$$

we thus obtain from (1) the **Fourier series** of the function $f(x)$ of period $2L$

$$f(x) = a_0 + \sum_{n=1}^{\infty} \left(a_n \cos \frac{n\pi}{L} x + b_n \sin \frac{n\pi}{L} x \right) \tag{5}$$

with the **Fourier coefficients** of $f(x)$ given by the **Euler formulas**

$$\begin{aligned} &\text{(a)} \quad a_0 = \frac{1}{2L} \int_{-L}^{L} f(x)\, dx \\ &\text{(b)} \quad a_n = \frac{1}{L} \int_{-L}^{L} f(x) \cos \frac{n\pi x}{L}\, dx \qquad n = 1, 2, \cdots \\ &\text{(c)} \quad b_n = \frac{1}{L} \int_{-L}^{L} f(x) \sin \frac{n\pi x}{L}\, dx \qquad n = 1, 2, \cdots . \end{aligned} \tag{6}$$

Just as in Sec. 11.1, we continue to call (5) with any coefficients a **trigonometric series.** And we can integrate from 0 to $2L$ or over any other interval of length $p = 2L$.

EXAMPLE 1 Periodic Rectangular Wave

Find the Fourier series of the function (Fig. 259)

$$f(x) = \begin{cases} 0 & \text{if } -2 < x < -1 \\ k & \text{if } -1 < x < 1 \\ 0 & \text{if } \quad 1 < x < 2 \end{cases} \qquad p = 2L = 4, \quad L = 2.$$

Solution. From (6a) we obtain $a_0 = k/2$ (verify!). From (6b) we obtain

$$a_n = \frac{1}{2}\int_{-2}^{2} f(x)\cos\frac{n\pi x}{2}\,dx = \frac{1}{2}\int_{-1}^{1} k\cos\frac{n\pi x}{2}\,dx = \frac{2k}{n\pi}\sin\frac{n\pi}{2}.$$

Thus $a_n = 0$ if n is even and

$$a_n = 2k/n\pi \quad \text{if} \quad n = 1, 5, 9, \cdots, \qquad a_n = -2k/n\pi \quad \text{if} \quad n = 3, 7, 11, \cdots.$$

From (6c) we find that $b_n = 0$ for $n = 1, 2, \cdots$. Hence the Fourier series is

$$f(x) = \frac{k}{2} + \frac{2k}{\pi}\left(\cos\frac{\pi}{2}x - \frac{1}{3}\cos\frac{3\pi}{2}x + \frac{1}{5}\cos\frac{5\pi}{2}x - + \cdots\right). \qquad \blacksquare$$

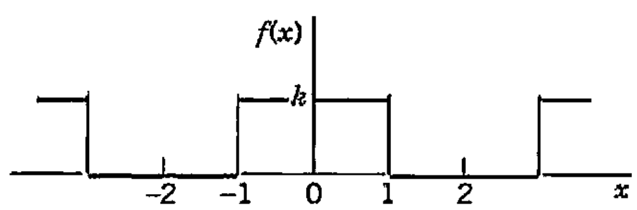


Fig. 259. Example 1

EXAMPLE 2 Periodic Rectangular Wave

Find the Fourier series of the function (Fig. 260)

$$f(x) = \begin{cases} -k & \text{if } -2 < x < 0 \\ \ \ k & \text{if } \quad 0 < x < 2 \end{cases} \qquad p = 2L = 4, \qquad L = 2.$$

Solution. $a_0 = 0$ from (6a). From (6b), with $1/L = 1/2$,

$$\begin{aligned} a_n &= \frac{1}{2}\left[\int_{-2}^{0}(-k)\cos\frac{n\pi x}{2}\,dx + \int_{0}^{2} k\cos\frac{n\pi x}{2}\,dx\right] \\ &= \frac{1}{2}\left[-\frac{2k}{n\pi}\sin\frac{n\pi x}{2}\bigg|_{-2}^{0} + \frac{2k}{n\pi}\sin\frac{n\pi x}{2}\bigg|_{0}^{2}\right] = 0, \end{aligned}$$

so that the Fourier series has no cosine terms. From (6c),

$$\begin{aligned} b_n &= \frac{1}{2}\left[\frac{2k}{n\pi}\cos\frac{n\pi x}{2}\bigg|_{-2}^{0} - \frac{2k}{n\pi}\cos\frac{n\pi x}{2}\bigg|_{0}^{2}\right] \\ &= \frac{k}{n\pi}(1 - \cos n\pi - \cos n\pi + 1) = \begin{cases} 4k/n\pi & \text{if } n = 1, 3, \cdots \\ 0 & \text{if } n = 2, 4, \cdots. \end{cases} \end{aligned}$$

Hence the Fourier series of $f(x)$ is

$$f(x) = \frac{4k}{\pi}\left(\sin\frac{\pi}{2}x + \frac{1}{3}\sin\frac{3\pi}{2}x + \frac{1}{5}\sin\frac{5\pi}{2}x + \cdots\right).$$

It is interesting that we could have derived this from (8) in Sec. 11.1, namely, by the scale change (3). Indeed, writing v instead of x, we have in (8), Sec. 11.1,

$$\frac{4k}{\pi}\left(\sin v + \frac{1}{3}\sin 3v + \frac{1}{5}\sin 5v + \cdots\right).$$

Since the period 2π in v corresponds to $2L = 4$, we have $k = \pi/L = \pi/2$ and $v = kx = \pi x/2$ in (3); hence we obtain the Fourier series of $f(x)$, as before. ■

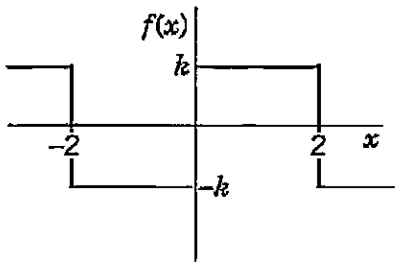


Fig. 260. Example 2

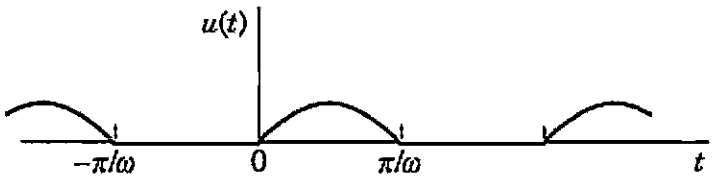


Fig. 261. Half-wave rectifier

EXAMPLE 3 Half-Wave Rectifier

A sinusoidal voltage $E \sin \omega t$, where t is time, is passed through a half-wave rectifier that clips the negative portion of the wave (Fig. 261). Find the Fourier series of the resulting periodic function

$$u(t) = \begin{cases} 0 & \text{if } -L < t < 0, \\ E\sin\omega t & \text{if } \quad 0 < t < L \end{cases} \qquad p = 2L = \frac{2\pi}{\omega}, \qquad L = \frac{\pi}{\omega}.$$

Solution. Since $u = 0$ when $-L < t < 0$, we obtain from (6a), with t instead of x,

$$a_0 = \frac{\omega}{2\pi}\int_0^{\pi/\omega} E\sin\omega t\, dt = \frac{E}{\pi}$$

and from (6b), by using formula (11) in App. A3.1 with $x = \omega t$ and $y = n\omega t$,

$$a_n = \frac{\omega}{\pi}\int_0^{\pi/\omega} E\sin\omega t\cos n\omega t\, dt = \frac{\omega E}{2\pi}\int_0^{\pi/\omega}[\sin(1+n)\omega t + \sin(1-n)\omega t]\, dt.$$

If $n = 1$, the integral on the right is zero, and if $n = 2, 3, \cdots$, we readily obtain

$$a_n = \frac{\omega E}{2\pi}\left[-\frac{\cos(1+n)\omega t}{(1+n)\omega} - \frac{\cos(1-n)\omega t}{(1-n)\omega}\right]_0^{\pi/\omega}$$

$$= \frac{E}{2\pi}\left(\frac{-\cos(1+n)\pi + 1}{1+n} + \frac{-\cos(1-n)\pi + 1}{1-n}\right).$$

If n is odd, this is equal to zero, and for even n we have

$$a_n = \frac{E}{2\pi}\left(\frac{2}{1+n} + \frac{2}{1-n}\right) = -\frac{2E}{(n-1)(n+1)\pi} \qquad (n = 2, 4, \cdots).$$

In a similar fashion we find from (6c) that $b_1 = E/2$ and $b_n = 0$ for $n = 2, 3, \cdots$. Consequently,

$$u(t) = \frac{E}{\pi} + \frac{E}{2}\sin\omega t - \frac{2E}{\pi}\left(\frac{1}{1\cdot 3}\cos 2\omega t + \frac{1}{3\cdot 5}\cos 4\omega t + \cdots\right).$$ ■

PROBLEM SET 11.2

1–11 **FOURIER SERIES FOR PERIOD $p = 2L$**

Find the Fourier series of the function $f(x)$, of period $p = 2L$, and sketch or graph the first three partial sums. (Show the details of your work.)

1. $f(x) = -1\ (-2 < x < 0),\ f(x) = 1\ (0 < x < 2),\ p = 4$
2. $f(x) = 0\ (-2 < x < 0),\ f(x) = 4\ (0 < x < 2),\ p = 4$
3. $f(x) = x^2 \quad (-1 < x < 1), \quad p = 2$
4. $f(x) = \pi x^3/2 \quad (-1 < x < 1), \quad p = 2$
5. $f(x) = \sin \pi x \quad (0 < x < 1), \quad p = 1$
6. $f(x) = \cos \pi x \quad (-\frac{1}{2} < x < \frac{1}{2}), \quad p = 1$
7. $f(x) = |x| \quad (-1 < x < 1), \quad p = 2$
8. $f(x) = \begin{cases} 1 + x \text{ if } -1 < x < 0 \\ 1 - x \text{ if } \quad 0 < x < 1, \end{cases} \quad p = 2$
9. $f(x) = 1 - x^2 \quad (-1 < x < 1), \quad p = 2$
10. $f(x) = 0\ (-2 < x < 0),\ f(x) = x\ (0 < x < 2),\ p = 4$
11. $f(x) = -x \quad (-1 < x < 0), \quad f(x) = x \quad (0 < x < 1),$ $f(x) = 1 \quad (1 < x < 3), \quad p = 4$
12. **(Rectifier)** Find the Fourier series of the function obtained by passing the voltage $v(t) = V_0 \cos 100\pi t$ through a half-wave rectifier.
13. Show that the familiar identities $\cos^3 x = \frac{3}{4} \cos x + \frac{1}{4} \cos 3x$ and $\sin^3 x = \frac{3}{4} \sin x - \frac{1}{4} \sin 3x$ can be interpreted as Fourier series expansions. Develop $\cos^4 x$.
14. Obtain the series in Prob. 7 from that in Prob. 8.
15. Obtain the series in Prob. 6 from that in Prob. 5.
16. Obtain the series in Prob. 3 from that in Prob. 21 of Problem Set 11.1.
17. Using Prob. 3, show that $1 - \frac{1}{4} + \frac{1}{9} - \frac{1}{16} + - \cdots = \frac{1}{12}\pi^2.$
18. Show that $1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \cdots = \frac{1}{6}\pi^2.$
19. **CAS PROJECT. Fourier Series of $2L$-Periodic Functions.** **(a)** Write a program for obtaining partial sums of a Fourier series (1).

    (b) Apply the program to Probs. 2–5, graphing the first few partial sums of each of the four series on common axes. Choose the first five or more partial sums until they approximate the given function reasonably well. Compare and comment.
20. **CAS EXPERIMENT. Gibbs Phenomenon.** The partial sums $s_n(x)$ of a Fourier series show oscillations near a discontinuity point. These oscillations do not disappear as n increases but instead become sharp "spikes." They were explained mathematically by J. W. Gibbs[3]. Graph $s_n(x)$ in Prob. 10. When $n = 50$, say, you will see those oscillations quite distinctly. Consider other Fourier series of your choice in a similar way. Compare.

11.3 Even and Odd Functions. Half-Range Expansions

The function in Example 1, Sec. 11.2, is *even,* and its Fourier series has only *cosine* terms. The function in Example 2, Sec. 11.2, is *odd,* and its Fourier series has only *sine* terms.

Recall that g is **even** if $g(-x) = g(x)$, so that its graph is symmetric with respect to the vertical axis (Fig. 262). A function h is **odd** if $h(-x) = -h(x)$ (Fig. 263).

Now the cosine terms in the Fourier series (5), Sec. 11.2, are even and the sine terms are odd. So it should not be a surprise that an even function is given by a series of cosine terms and an odd function by a series of sine terms. Indeed, the following holds.

[3]JOSIAH WILLARD GIBBS (1839–1903), American mathematician, professor of mathematical physics at Yale from 1871 on, one of the founders of vector calculus [another being O. Heaviside (see Sec. 6.1)], mathematical thermodynamics, and statistical mechanics. His work was of great importance to the development of mathematical physics.

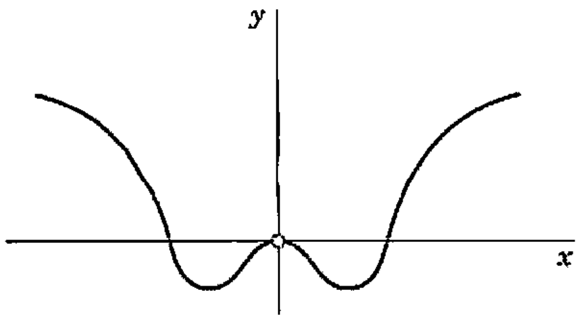


Fig. 262. Even function

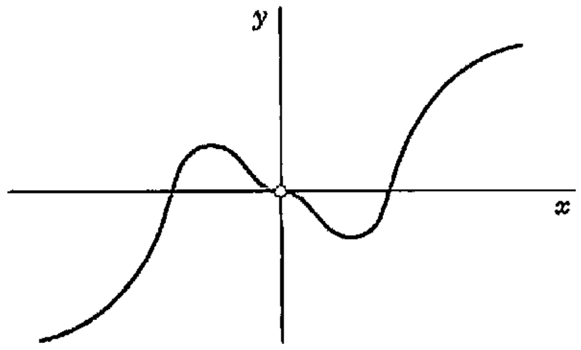


Fig. 263. Odd function

THEOREM 1

Fourier Cosine Series, Fourier Sine Series

The Fourier series of an even function of period 2L is a **"Fourier cosine series"**

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos \frac{n\pi}{L} x \qquad (f \text{ even}) \tag{1}$$

with coefficients (note: integration from 0 to L only!)

$$a_0 = \frac{1}{L} \int_0^L f(x)\, dx, \qquad a_n = \frac{2}{L} \int_0^L f(x) \cos \frac{n\pi x}{L}\, dx, \qquad n = 1, 2, \cdots. \tag{2}$$

The Fourier series of an ***odd*** *function of period 2L is a* **"Fourier sine series"**

$$f(x) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi}{L} x \qquad (f \text{ odd}) \tag{3}$$

with coefficients

$$b_n = \frac{2}{L} \int_0^L f(x) \sin \frac{n\pi x}{L}\, dx. \tag{4}$$

PROOF Since the definite integral of a function gives the area under the curve of the function between the limits of integration, we have

$$\int_{-L}^{L} g(x)\, dx = 2 \int_0^L g(x)\, dx \qquad \text{for even } g$$

$$\int_{-L}^{L} h(x)\, dx = 0 \qquad \text{for odd } h$$

as is obvious from the graphs of g and h. (Give a formal proof.) Now let f be even. Then (6a), Sec. 11.2, gives a_0 in (2). Also, the integrand in (6b), Sec. 11.2, is even (a product of even functions is even), so that (6b) gives a_n in (2). Furthermore, the integrand in (6c), Sec. 11.2, is the even f times the odd sine, so that the integrand (the product) is odd, the integral is zero, and there are no sine terms in (1).

Similarly, if f is odd, the integrals for a_0 and a_n in (6a) and (6b), Sec. 11.2, are zero, f times the sine in (6c) is even, (6c) implies (4), and there are no cosine terms in (3). ■

The Case of Period 2π. If $L = \pi$, then $f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos nx$ (f even) with coefficients

$$(2^*) \qquad a_0 = \frac{1}{\pi}\int_0^{\pi} f(x)\,dx, \qquad a_n = \frac{2}{\pi}\int_0^{\pi} f(x)\cos nx\,dx, \qquad n = 1, 2, \cdots$$

and $f(x) = \sum_{n=1}^{\infty} b_n \sin nx$ (f odd) with coefficients

$$(4^*) \qquad b_n = \frac{2}{\pi}\int_0^{\pi} f(x)\sin nx\,dx, \qquad n = 1, 2, \cdots.$$

For instance, $f(x)$ in Example 1, Sec. 11.1, is odd and is represented by a Fourier sine series.

Further simplifications result from the following property, whose very simple proof is left to the student.

THEOREM 2

Sum and Scalar Multiple

The Fourier coefficients of a sum $f_1 + f_2$ are the sums of the corresponding Fourier coefficients of f_1 and f_2.

The Fourier coefficients of cf are c times the corresponding Fourier coefficients of f.

EXAMPLE 1 Rectangular Pulse

The function $f^*(x)$ in Fig. 264 is the sum of the function $f(x)$ in Example 1 of Sec 11.1 and the constant k. Hence, from that example and Theorem 2 we conclude that

$$f^*(x) = k + \frac{4k}{\pi}\left(\sin x + \frac{1}{3}\sin 3x + \frac{1}{5}\sin 5x + \cdots\right).$$ ■

EXAMPLE 2 Half-Wave Rectifier

The function $u(t)$ in Example 3 of Sec. 11.2 has a Fourier cosine series plus a single term $v(t) = (E/2)\sin \omega t$. We conclude from this and Theorem 2 that $u(t) - v(t)$ must be an even function. Verify this graphically. (See Fig. 265.) ■

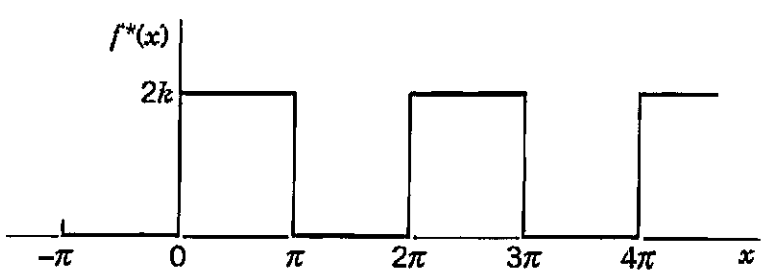


Fig. 264. Example 1

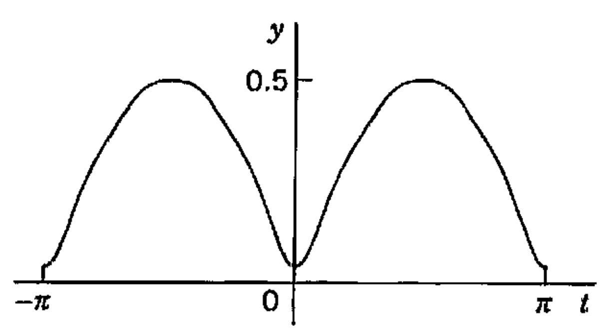
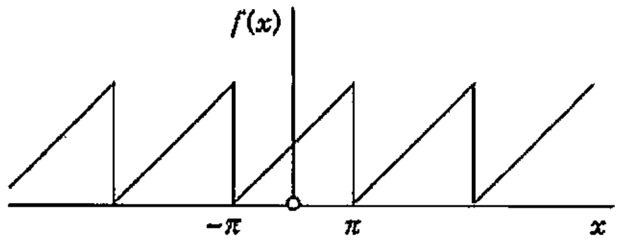


Fig. 265. $u(t) - v(t)$ with $E = 1$, $\omega = 1$

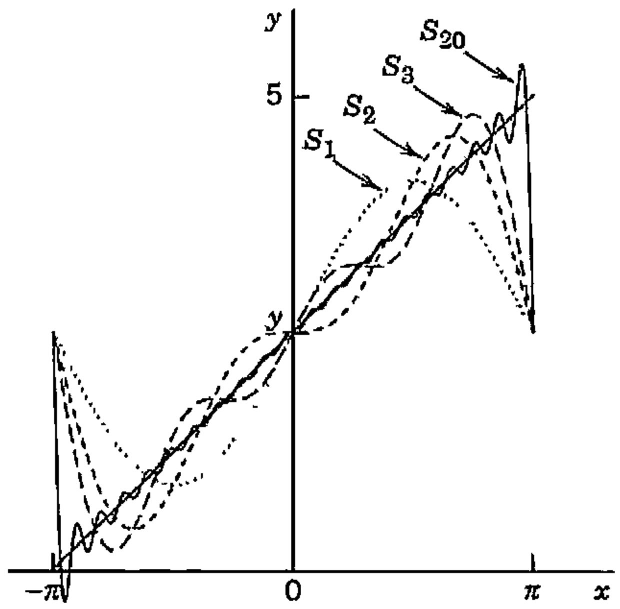
EXAMPLE 3 Sawtooth Wave

Find the Fourier series of the function (Fig. 266)

$$f(x) = x + \pi \quad \text{if} \quad -\pi < x < \pi \qquad \text{and} \qquad f(x + 2\pi) = f(x).$$



(a) The function $f(x)$



(b) Partial sums S_1, S_2, S_3, S_{20}

Fig. 266. Example 3

Solution. We have $f = f_1 + f_2$, where $f_1 = x$ and $f_2 = \pi$. The Fourier coefficients of f_2 are zero, except for the first one (the constant term), which is π. Hence, by Theorem 2, the Fourier coefficients a_n, b_n are those of f_1, except for a_0, which is π. Since f_1 is odd, $a_n = 0$ for $n = 1, 2, \cdots$, and

$$b_n = \frac{2}{\pi}\int_0^{\pi} f_1(x) \sin nx \, dx = \frac{2}{\pi}\int_0^{\pi} x \sin nx \, dx.$$

Integrating by parts, we obtain

$$b_n = \frac{2}{\pi}\left[\frac{-x \cos nx}{n}\Bigg|_0^{\pi} + \frac{1}{n}\int_0^{\pi} \cos nx \, dx\right] = -\frac{2}{n}\cos n\pi.$$

Hence $b_1 = 2$, $b_2 = -2/2$, $b_3 = 2/3$, $b_4 = -2/4, \cdots$, and the Fourier series of $f(x)$ is

$$f(x) = \pi + 2\left(\sin x - \frac{1}{2}\sin 2x + \frac{1}{3}\sin 3x - + \cdots\right).$$ ■

Half-Range Expansions

Half-range expansions are Fourier series. The idea is simple and useful. Figure 267 explains it. We want to represent $f(x)$ in Fig. 267a by a Fourier series, where $f(x)$ may be the shape of a distorted violin string or the temperature in a metal bar of length L, for example. (Corresponding problems will be discussed in Chap. 12.) Now comes the idea.

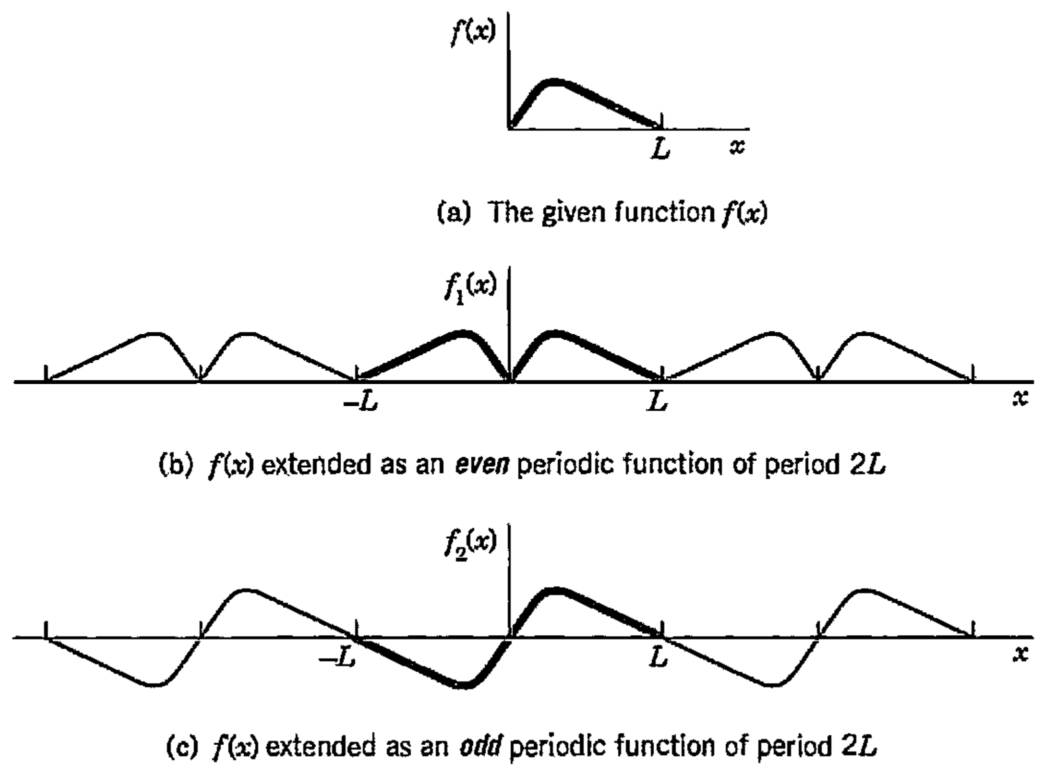


Fig. 267. (a) Function $f(x)$ given on an interval $0 \leqq x \leqq L$

(b) **Even extension** to the full "range" (interval) $-L \leqq x \leqq L$ (heavy curve) and the periodic extension of period $2L$ to the x-axis

(c) **Odd extension** to $-L \leqq x \leqq L$ (heavy curve) and the periodic extension of period $2L$ to the x-axis

We could extend $f(x)$ as a function of period L and develop the extended function into a Fourier series. But this series would in general contain *both* cosine *and* sine terms. We can do better and get simpler series. Indeed, for our given f we can calculate Fourier coefficients from (2) or from (4) in Theorem 1. And we have a choice and can take what seems more practical. If we use (2), we get (1). This is the **even periodic extension** f_1 of f in Fig. 267b. If we choose (4) instead, we get (3), the **odd periodic extension** f_2 of f in Fig. 267c.

Both extensions have period $2L$. This motivates the name **half-range expansions:** f is given (and of physical interest) only on half the range, half the interval of periodicity of length $2L$.

Let us illustrate these ideas with an example that we shall also need in Chap. 12.

EXAMPLE 4 "Triangle" and Its Half-Range Expansions

Find the two half-range expansions of the function (Fig. 268)

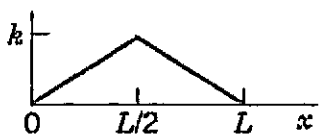


Fig. 268. The given function in Example 4

$$f(x) = \begin{cases} \dfrac{2k}{L}x & \text{if } 0 < x < \dfrac{L}{2} \\[2mm] \dfrac{2k}{L}(L-x) & \text{if } \dfrac{L}{2} < x < L. \end{cases}$$

Solution. *(a) Even periodic extension.* From (2) we obtain

$$a_0 = \frac{1}{L}\left[\frac{2k}{L}\int_0^{L/2} x\,dx + \frac{2k}{L}\int_{L/2}^{L}(L-x)\,dx\right] = \frac{k}{2},$$

$$a_n = \frac{2}{L}\left[\frac{2k}{L}\int_0^{L/2} x\cos\frac{n\pi}{L}x\,dx + \frac{2k}{L}\int_{L/2}^{L}(L-x)\cos\frac{n\pi}{L}x\,dx\right].$$

We consider a_n. For the first integral we obtain by integration by parts

$$\int_0^{L/2} x \cos \frac{n\pi}{L} x \, dx = \frac{Lx}{n\pi} \sin \frac{n\pi}{L} x \Big|_0^{L/2} - \frac{L}{n\pi} \int_0^{L/2} \sin \frac{n\pi}{L} x \, dx$$

$$= \frac{L^2}{2n\pi} \sin \frac{n\pi}{2} + \frac{L^2}{n^2\pi^2} \left(\cos \frac{n\pi}{2} - 1 \right).$$

Similarly, for the second integral we obtain

$$\int_{L/2}^{L} (L - x) \cos \frac{n\pi}{L} x \, dx = \frac{L}{n\pi} (L - x) \sin \frac{n\pi}{L} x \Big|_{L/2}^{L} + \frac{L}{n\pi} \int_{L/2}^{L} \sin \frac{n\pi}{L} x \, dx$$

$$= \left(0 - \frac{L}{n\pi} \left(L - \frac{L}{2} \right) \sin \frac{n\pi}{2} \right) - \frac{L^2}{n^2\pi^2} \left(\cos n\pi - \cos \frac{n\pi}{2} \right).$$

We insert these two results into the formula for a_n. The sine terms cancel and so does a factor L^2. This gives

$$a_n = \frac{4k}{n^2\pi^2} \left(2 \cos \frac{n\pi}{2} - \cos n\pi - 1 \right).$$

Thus,

$$a_2 = -16k/(2^2\pi^2), \qquad a_6 = -16k/(6^2\pi^2), \qquad a_{10} = -16k/(10^2\pi^2), \cdots$$

and $a_n = 0$ if $n \neq 2, 6, 10, 14, \cdots$. Hence the first half-range expansion of $f(x)$ is (Fig. 269a)

$$f(x) = \frac{k}{2} - \frac{16k}{\pi^2} \left(\frac{1}{2^2} \cos \frac{2\pi}{L} x + \frac{1}{6^2} \cos \frac{6\pi}{L} x + \cdots \right).$$

This Fourier cosine series represents the even periodic extension of the given function $f(x)$, of period $2L$.

(b) *Odd periodic extension.* Similarly, from (4) we obtain

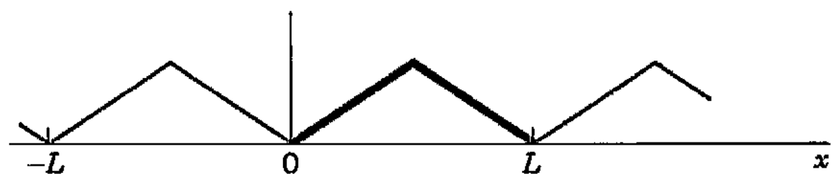
$$b_n = \frac{8k}{n^2\pi^2} \sin \frac{n\pi}{2}. \tag{5}$$

Hence the other half-range expansion of $f(x)$ is (Fig. 269b)

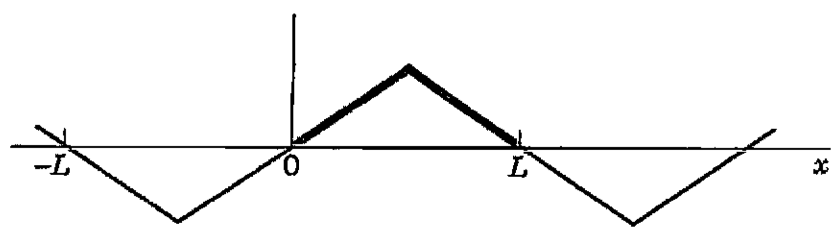
$$f(x) = \frac{8k}{\pi^2} \left(\frac{1}{1^2} \sin \frac{\pi}{L} x - \frac{1}{3^2} \sin \frac{3\pi}{L} x + \frac{1}{5^2} \sin \frac{5\pi}{L} x - + \cdots \right).$$

This series represents the odd periodic extension of $f(x)$, of period $2L$.

Basic applications of these results will be shown in Secs. 12.3 and 12.5. ■



(a) Even extension



(b) Odd extension

Fig. 269. Periodic extensions of $f(x)$ in Example 4

PROBLEM SET 11.3

1–9 EVEN AND ODD FUNCTIONS

Are the following functions even, odd, or neither even nor odd?

1. $|x|$, $x^2 \sin nx$, $x + x^2$, $e^{-|x|}$, $\ln x$, $x \cosh x$
2. $\sin(x^2)$, $\sin^2 x$, $x \sinh x$, $|x^3|$, $e^{\pi x}$, xe^x, $\tan 2x$, $x/(1 + x^2)$

Are the following functions, which are assumed to be periodic of period 2π, even, odd, or neither even nor odd?

3. $f(x) = x^3 \quad (-\pi < x < \pi)$
4. $f(x) = x^2 \quad (-\pi/2 < x < 3\pi/2)$
5. $f(x) = e^{-4x} \quad (-\pi < x < \pi)$
6. $f(x) = x^3 \sin x \quad (-\pi < x < \pi)$
7. $f(x) = x|x| - x^3 \quad (-\pi < x < \pi)$
8. $f(x) = 1 - x + x^3 - x^5 \quad (-\pi < x < \pi)$
9. $f(x) = 1/(1 + x^2)$ if $-\pi < x < 0$, $f(x) = -1/(1 + x^2)$ if $0 < x < \pi$

10. **PROJECT. Even and Odd Functions.** **(a)** Are the following expressions even or odd? Sums and products of even functions and of odd functions. Products of even times odd functions. Absolute values of odd functions. $f(x) + f(-x)$ and $f(x) - f(-x)$ for arbitrary $f(x)$.

    (b) Write e^{kx}, $1/(1 - x)$, $\sin(x + k)$, $\cosh(x + k)$ as sums of an even and an odd function.

    (c) Find all functions that are both even and odd.

    (d) Is $\cos^3 x$ even or odd? $\sin^3 x$? Find the Fourier series of these functions. Do you recognize familiar identities?

11–16 FOURIER SERIES OF EVEN AND ODD FUNCTIONS

Is the given function even or odd? Find its Fourier series. Sketch or graph the function and some partial sums. (Show the details of your work.)

11. $f(x) = \pi - |x| \quad (-\pi < x < \pi)$
12. $f(x) = 2x|x| \quad (-1 < x < 1)$
13. $f(x) = \begin{cases} x & \text{if } -\pi/2 < x < \pi/2 \\ \pi - x & \text{if } \pi/2 < x < 3\pi/2 \end{cases}$
14. $f(x) = \begin{cases} \pi e^{-x} & \text{if } -\pi < x < 0 \\ \pi e^{x} & \text{if } 0 < x < \pi \end{cases}$
15. $f(x) = \begin{cases} 2 & \text{if } -2 < x < 0 \\ 0 & \text{if } 0 < x < 2 \end{cases}$
16. $f(x) = \begin{cases} 1 - \frac{1}{2}|x| & \text{if } -2 < x < 2 \\ 0 & \text{if } 2 < x < 6 \end{cases} \quad (p = 8)$

17–25 HALF-RANGE EXPANSIONS

Find (a) the Fourier cosine series, (b) the Fourier sine series. Sketch $f(x)$ and its two periodic extensions. (Show the details of your work.)

17. $f(x) = 1 \quad (0 < x < 2)$
18. $f(x) = x \quad (0 < x < \frac{1}{2})$
19. $f(x) = 2 - x \quad (0 < x < 2)$
20. $f(x) = \begin{cases} 0 & (0 < x < 2) \\ 1 & (2 < x < 4) \end{cases}$
21. $f(x) = \begin{cases} 1 & (0 < x < 1) \\ 2 & (1 < x < 2) \end{cases}$
22. $f(x) = \begin{cases} x & (0 < x < \pi/2) \\ \pi/2 & (\pi/2 < x < \pi) \end{cases}$
23. $f(x) = x \quad (0 < x < L)$
24. $f(x) = x^2 \quad (0 < x < L)$
25. $f(x) = \pi - x \quad (0 < x < \pi)$
26. Illustrate the formulas in the proof of Theorem 1 with examples. Prove the formulas.

11.4 Complex Fourier Series. *Optional*

In this optional section we show that the Fourier series

$$f(x) = a_0 + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx) \tag{1}$$

can be written in complex form, which sometimes simplifies calculations (see Example 1, on page 498). This complex form can be obtained because in complex, the exponential function e^{it} and $\cos t$ and $\sin t$ are related by the basic **Euler formula** (see (11) in Sec. 2.2)

$$e^{it} = \cos t + i \sin t. \qquad \text{Thus} \qquad e^{-it} = \cos t - i \sin t. \tag{2}$$

Conversely, by adding and subtracting these two formulas, we obtain

$$\text{(3)} \qquad \text{(a)} \quad \cos t = \frac{1}{2}(e^{it} + e^{-it}), \qquad \text{(b)} \quad \sin t = \frac{1}{2i}(e^{it} - e^{-it}).$$

From (3), using $1/i = -i$ in $\sin t$ and setting $t = nx$ in both formulas, we get

$$\begin{aligned} a_n \cos nx + b_n \sin nx &= \frac{1}{2}\, a_n(e^{inx} + e^{-inx}) + \frac{1}{2i}\, b_n(e^{inx} - e^{-inx}) \\ &= \frac{1}{2}\,(a_n - ib_n)e^{inx} + \frac{1}{2}\,(a_n + ib_n)e^{-inx}. \end{aligned}$$

We insert this into (1). Writing $a_0 = c_0$, $\frac{1}{2}(a_n - ib_n) = c_n$, and $\frac{1}{2}(a_n + ib_n) = k_n$, we get from (1)

$$\text{(4)} \qquad f(x) = c_0 + \sum_{n=1}^{\infty} (c_n e^{inx} + k_n e^{-inx}).$$

The coefficients $c_1, c_2, \cdots$, and $k_1, k_2, \cdots$ are obtained from (6b), (6c) in Sec. 11.1 and then (2) above with $t = nx$,

$$\text{(5)} \qquad \begin{aligned} c_n &= \frac{1}{2}\,(a_n - ib_n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)(\cos nx - i \sin nx)\, dx = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)e^{-inx}\, dx \\ k_n &= \frac{1}{2}\,(a_n + ib_n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)(\cos nx + i \sin nx)\, dx = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)e^{inx}\, dx. \end{aligned}$$

Finally, we can combine (5) into a single formula by the trick of writing $k_n = c_{-n}$. Then (4), (5), and $c_0 = a_0$ in (6a) of Sec. 11.1 give (summation from $-\infty$!)

$$\text{(6)} \qquad \begin{aligned} f(x) &= \sum_{n=-\infty}^{\infty} c_n e^{inx}, \\ c_n &= \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)e^{-inx}\, dx, \qquad n = 0, \pm 1, \pm 2, \cdots. \end{aligned}$$

This is the so-called *complex form of the Fourier series* or, more briefly, the **complex Fourier series,** of $f(x)$. The c_n are called the **complex Fourier coefficients** of $f(x)$.

For a function of period $2L$ our reasoning gives the **complex Fourier series**

$$\text{(7)} \qquad \begin{aligned} f(x) &= \sum_{n=-\infty}^{\infty} c_n e^{in\pi x/L}, \\ c_n &= \frac{1}{2L}\int_{-L}^{L} f(x)e^{-in\pi x/L}\, dx, \qquad n = 0, \pm 1, \pm 2, \cdots. \end{aligned}$$

EXAMPLE 1 Complex Fourier Series

Find the complex Fourier series of $f(x) = e^x$ if $-\pi < x < \pi$ and $f(x + 2\pi) = f(x)$ and obtain from it the usual Fourier series.

Solution. Since $\sin n\pi = 0$ for integer n, we have

$$e^{\pm in\pi} = \cos n\pi \pm i \sin n\pi = \cos n\pi = (-1)^n.$$

With this we obtain from (6) by integration

$$c_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^x e^{-inx}\, dx = \frac{1}{2\pi}\frac{1}{1-in} e^{x-inx}\Big|_{x=-\pi}^{\pi} = \frac{1}{2\pi}\frac{1}{1-in}(e^{\pi} - e^{-\pi})(-1)^n.$$

On the right,

$$\frac{1}{1-in} = \frac{1+in}{(1-in)(1+in)} = \frac{1+in}{1+n^2} \qquad \text{and} \qquad e^{\pi} - e^{-\pi} = 2\sinh \pi.$$

Hence the complex Fourier series is

$$e^x = \frac{\sinh \pi}{\pi} \sum_{n=-\infty}^{\infty} (-1)^n \frac{1+in}{1+n^2} e^{inx} \qquad (-\pi < x < \pi). \tag{8}$$

From this let us derive the real Fourier series. Using (2) with $t = nx$ and $i^2 = -1$, we have in (8)

$$(1 + in)e^{inx} = (1 + in)(\cos nx + i \sin nx) = (\cos nx - n \sin nx) + i(n \cos nx + \sin nx).$$

Now (8) also has a corresponding term with $-n$ instead of n. Since $\cos(-nx) = \cos nx$ and $\sin(-nx) = -\sin nx$, we obtain in this term

$$(1 - in)e^{-inx} = (1 - in)(\cos nx - i \sin nx) = (\cos nx - n \sin nx) - i(n \cos nx + \sin nx).$$

If we add these two expressions, the imaginary parts cancel. Hence their sum is

$$2(\cos nx - n \sin nx), \qquad n = 1, 2, \cdots.$$

For $n = 0$ we get 1 (not 2) because there is only one term. Hence the real Fourier series is

$$e^x = \frac{2\sinh \pi}{\pi}\left[\frac{1}{2} - \frac{1}{1+1^2}(\cos x - \sin x) + \frac{1}{1+2^2}(\cos 2x - 2\sin 2x) - + \cdots\right]. \tag{9}$$

In Fig. 270 the poor approximation near the jumps at $\pm\pi$ is a case of the Gibbs phenomenon (see CAS Experiment 20 in Problem Set 11.2). ■

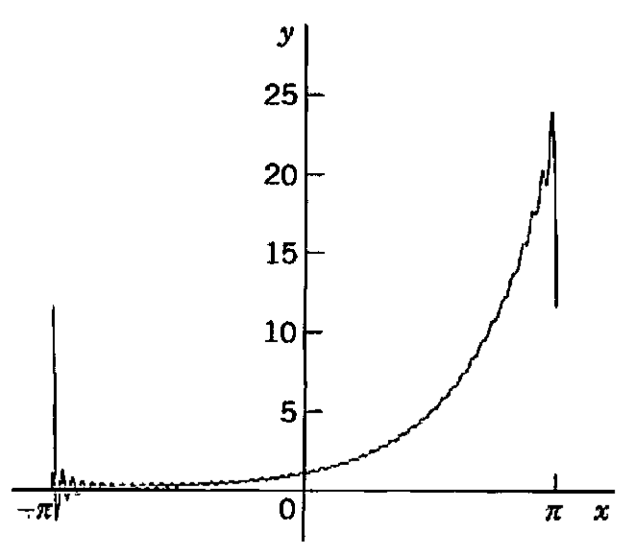


Fig. 270. Partial sum of (9), terms from $n = 0$ to 50

PROBLEM SET 11.4

1. **(Calculus review)** Review complex numbers.
2. **(Even and odd functions)** Show that the complex Fourier coefficients of an even function are real and those of an odd function are pure imaginary.
3. **(Fourier coefficients)** Show that
$a_0 = c_0$, $a_n = c_n + c_{-n}$, $b_n = i(c_n - c_{-n})$.
4. Verify the calculations in Example 1.
5. Find further terms in (9) and graph partial sums with your CAS.
6. Obtain the real series in Example 1 directly from the Euler formulas in Sec. 11.

7–13 COMPLEX FOURIER SERIES

Find the complex Fourier series of the following functions. (Show the details of your work.)

7. $f(x) = -1$ if $-\pi < x < 0$, $f(x) = 1$ if $0 < x < \pi$
8. Convert the series in Prob. 7 to real form.
9. $f(x) = x \quad (-\pi < x < \pi)$
10. Convert the series in Prob. 9 to real form.
11. $f(x) = x^2 \quad (-\pi < x < \pi)$
12. Convert the series in Prob. 11 to real form.
13. $f(x) = x \quad (0 < x < 2\pi)$
14. **PROJECT. Complex Fourier Coefficients.** It is very interesting that the c_n in (6) can be derived directly by a method similar to that for a_n and b_n in Sec. 11.1. For this, multiply the series in (6) by e^{-imx} with fixed integer m, and integrate termwise from $-\pi$ to π on both sides (allowed, for instance, in the case of uniform convergence) to get

$$\int_{-\pi}^{\pi} f(x)e^{-imx}\,dx = \sum_{n=-\infty}^{\infty} c_n \int_{-\pi}^{\pi} e^{i(n-m)x}\,dx.$$

Show that the integral on the right equals 2π when $n = m$ and 0 when $n \neq m$ [use (3b)], so that you get the coefficient formula in (6).

11.5 Forced Oscillations

Fourier series have important applications in connection with ODEs and PDEs. We show this for a basic problem modeled by an ODE. Various applications to PDEs will follow in Chap. 12. This will show the enormous usefulness of Euler's and Fourier's ingenious idea of splitting up periodic functions into the simplest ones possible.

From Sec. 2.8 we know that forced oscillations of a body of mass m on a spring of modulus k are governed by the ODE

$$my'' + cy' + ky = r(t) \qquad (1)$$

where $y = y(t)$ is the displacement from rest, c the damping constant, k the spring constant (spring modulus), and $r(t)$ the external force depending on time t. Figure 271 shows the model and Fig. 272 its electrical analog, an RLC-circuit governed by

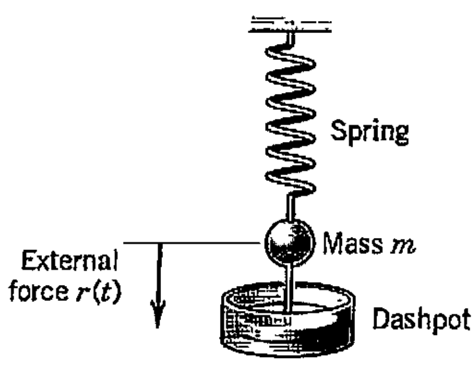


Fig. 271. Vibrating system under consideration

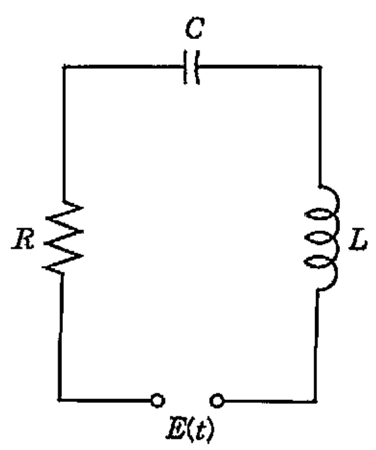


Fig. 272. Electrical analog of the system in Fig. 271 (RLC-circuit)

(1*) $$LI'' + RI' + \frac{1}{C} I = E'(t)$$ (Sec. 2.9).

We consider (1). If $r(t)$ is a sine or cosine function and if there is damping ($c > 0$), then the steady-state solution is a harmonic oscillation with frequency equal to that of $r(t)$. However, if $r(t)$ is not a pure sine or cosine function but is any other periodic function, then the steady-state solution will be a superposition of harmonic oscillations with frequencies equal to that of $r(t)$ and integer multiples of the latter. And if one of these frequencies is close to the (practical) resonant frequency of the vibrating system (see Sec. 2.8), then the corresponding oscillation may be the dominant part of the response of the system to the external force. This is what the use of Fourier series will show us. Of course, this is quite surprising to an observer unfamiliar with Fourier series, which are highly important in the study of vibrating systems and resonance. Let us discuss the entire situation in terms of a typical example.

EXAMPLE 1 Forced Oscillations under a Nonsinusoidal Periodic Driving Force

In (1), let $m = 1$ (gm), $c = 0.05$ (gm/sec), and $k = 25$ (gm/sec^2), so that (1) becomes

(2) $$y'' + 0.05y' + 25y = r(t)$$

where $r(t)$ is measured in gm · cm/sec^2. Let (Fig. 273)

$$r(t) = \begin{cases} t + \frac{\pi}{2} & \text{if } -\pi < t < 0, \\ -t + \frac{\pi}{2} & \text{if } \quad 0 < t < \pi, \end{cases} \qquad r(t + 2\pi) = r(t).$$

Find the steady-state solution $y(t)$.

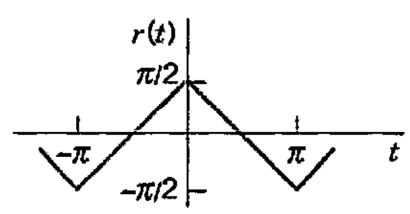


Fig. 273. Force in Example 1

Solution. We represent $r(t)$ by a Fourier series, finding

(3) $$r(t) = \frac{4}{\pi}\left(\cos t + \frac{1}{3^2}\cos 3t + \frac{1}{5^2}\cos 5t + \cdots\right)$$

(take the answer to Prob. 11 in Problem Set 11.3 minus $\frac{1}{2}\pi$ and write t for x). Then we consider the ODE

(4) $$y'' + 0.05y' + 25y = \frac{4}{n^2\pi}\cos nt \qquad (n = 1, 3, \cdots)$$

whose right side is a single term of the series (3). From Sec. 2.8 we know that the steady-state solution $y_n(t)$ of (4) is of the form

(5) $$y_n = A_n \cos nt + B_n \sin nt.$$

By substituting this into (4) we find that

$$(6) \qquad A_n = \frac{4(25 - n^2)}{n^2 \pi D_n}, \qquad B_n = \frac{0.2}{n\pi D_n}, \qquad \text{where} \qquad D_n = (25 - n^2)^2 + (0.05n)^2.$$

Since the ODE (2) is linear, we may expect the steady-state solution to be

$$(7) \qquad y = y_1 + y_3 + y_5 + \cdots$$

where y_n is given by (5) and (6). In fact, this follows readily by substituting (7) into (2) and using the Fourier series of $r(t)$, provided that termwise differentiation of (7) is permissible. (Readers already familiar with the notion of uniform convergence [Sec. 15.5] may prove that (7) may be differentiated term by term.)

From (6) we find that the amplitude of (5) is (a factor $\sqrt{D_n}$ cancels out)

$$C_n = \sqrt{A_n^2 + B_n^2} = \frac{4}{n^2 \pi \sqrt{D_n}}.$$

Numeric values are

$$C_1 = 0.0531$$
$$C_3 = 0.0088$$
$$C_5 = 0.2037$$
$$C_7 = 0.0011$$
$$C_9 = 0.0003.$$

Figure 274 shows the input (multiplied by 0.1) and the output. For $n = 5$ the quantity D_n is very small, the denominator of C_5 is small, and C_5 is so large that y_5 is the dominating term in (7). Hence the output is almost a harmonic oscillation of five times the frequency of the driving force, a little distorted due to the term y_1, whose amplitude is about 25% of that of y_5. You could make the situation still more extreme by decreasing the damping constant c. Try it. ■

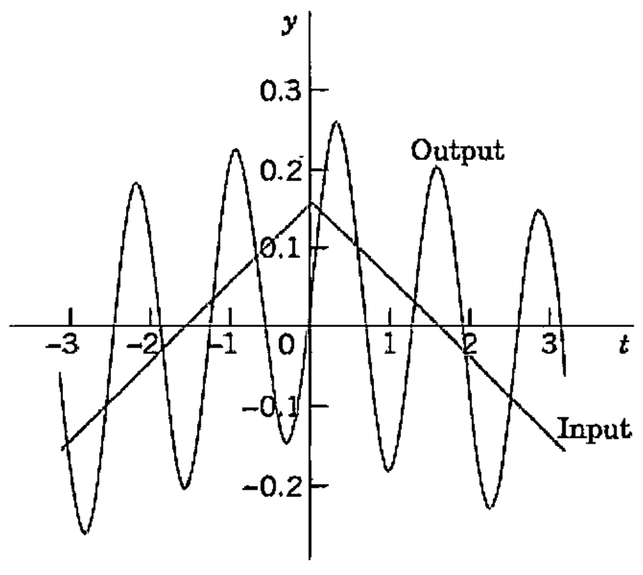


Fig. 274. Input and steady-state output in Example 1

PROBLEM SET 11.5

1. **(Coefficients)** Derive the formula for C_n from A_n and B_n.
2. **(Spring constant)** What would happen to the amplitudes C_n in Example 1 (and thus to the form of the vibration) if we changed the spring constant to the value 9? If we took a stiffer spring with $k = 81$? First guess.
3. **(Damping)** In Example 1 change c to 0.02 and discuss how this changes the output.
4. **(Input)** What would happen in Example 1 if we replaced $r(t)$ with its derivative (the rectangular wave)? What is the ratio of the new C_n to the old ones?

5–11 GENERAL SOLUTION

Find a general solution of the ODE $y'' + \omega^2 y = r(t)$ with $r(t)$ as given. (Show the details of your work.)

5. $r(t) = \cos \omega t,\ \omega = 0.5, 0.8, 1.1, 1.5, 5.0, 10.0$
6. $r(t) = \cos \omega_1 t + \cos \omega_2 t \quad (\omega^2 \neq \omega_1^{\,2}, \omega_2^{\,2})$
7. $r(t) = \sum_{n=1}^{N} a_n \cos nt, \quad |\omega| \neq 1, 2, \cdots, N$
8. $r(t) = \sin t + \frac{1}{3}\sin 3t + \frac{1}{5}\sin 5t + \frac{1}{7}\sin 7t$
9. $r(t) = \begin{cases} t + \pi & \text{if} \quad -\pi < t < 0 \\ -t + \pi & \text{if} \quad 0 < t < \pi \end{cases}$
   and $r(t + 2\pi) = r(t),\ |\omega| \neq 0, 1, 3, \cdots$
10. $r(t) = \begin{cases} t & \text{if} \quad -\pi/2 < t < \pi/2 \\ \pi - t & \text{if} \quad \pi/2 < t < 3\pi/2 \end{cases}$
    and $r(t + 2\pi) = r(t), \quad |\omega| \neq 1, 3, 5, \cdots$
11. $r(t) = \frac{\pi}{4}|\sin t|$ if $-\pi < t < \pi$ and
    $r(t + 2\pi) = r(t), \quad |\omega| \neq 0, 2, 4, \cdots$
12. **(CAS Program)** Write a program for solving the ODE just considered and for jointly graphing input and output of an initial value problem involving that ODE. Apply the program to Probs. 5 and 9 with initial values of your choice.
13. **(Sign of coefficients)** Some A_n in Example 1 are positive and some negative. Is this physically understandable?

14–17 STEADY-STATE DAMPED OSCILLATIONS

Find the steady-state oscillation of $y'' + cy' + y = r(t)$ with $c > 0$ and $r(t)$ as given. (Show the details of your work.)

14. $r(t) = a_n \cos nt$
15. $r(t) = \sin 3t$
16. $r(t) = \begin{cases} \pi t & \text{if} \quad -\pi/2 < t < \pi/2 \\ \pi(\pi - t) & \text{if} \quad \pi/2 < t < 3\pi/2 \end{cases}$
    and $r(t + 2\pi) = r(t)$
17. $r(t) = \sum_{n=1}^{N} b_n \sin nt$
18. **CAS EXPERIMENT. Maximum Output Term.** Graph and discuss outputs of $y'' + cy' + ky = r(t)$ with $r(t)$ as in Example 1 for various c and k with emphasis on the maximum C_n and its ratio to the second largest $|C_n|$.

19–20 *RLC*-CIRCUIT

Find the steady-state current $I(t)$ in the *RLC*-circuit in Fig. 272, where $R = 100\ \Omega$, $L = 10$ H, $C = 10^{-2}$ F and $E(t)$ V as follows and periodic with period 2π. Sketch or graph the first four partial sums. Note that the coefficients of the solution decrease rapidly.

19. $E(t) = 200t(\pi^2 - t^2) \quad (-\pi < t < \pi)$
20. $E(t) = \begin{cases} 100(\pi t + t^2) & \text{if} \quad -\pi < t < 0 \\ 100(\pi t - t^2) & \text{if} \quad 0 < t < \pi \end{cases}$

11.6 Approximation by Trigonometric Polynomials

Fourier series play a prominent role in differential equations. Another field in which they have major applications is **approximation theory,** which concerns the approximation of functions by other (usually simpler) functions. In connection with Fourier series the idea is as follows.

Let $f(x)$ be a function on the interval $-\pi \leqq x \leqq \pi$ that can be represented on this interval by a Fourier series. Then the ***N*th partial sum** of the series

$$f(x) \approx a_0 + \sum_{n=1}^{N} (a_n \cos nx + b_n \sin nx) \tag{1}$$

is an approximation of the given $f(x)$. It is natural to ask whether (1) is the "best" approximation of f by a **trigonometric polynomial of degree *N*,** that is, by a function of the form

$$F(x) = A_0 + \sum_{n=1}^{N} (A_n \cos nx + B_n \sin nx) \qquad (N \text{ fixed}) \tag{2}$$

where "best" means that the "error" of the approximation is as small as possible.

Of course, we must first define what we mean by the **error** E of such an approximation. We could choose the maximum of $|f - F|$. But in connection with Fourier series it is better to choose a definition that measures the goodness of agreement between f and F *on the whole interval* $-\pi \leqq x \leqq \pi$. This seems preferable, in particular if f has jumps: F in Fig. 275 is a good overall approximation of f, but the maximum of $|f - F|$ (more precisely, the *supremum*) is large (it equals at least half the jump of f at x_0). We choose

$$E = \int_{-\pi}^{\pi} (f - F)^2 \, dx. \tag{3}$$

This is called the **square error** of F relative to the function f on the interval $-\pi \leqq x \leqq \pi$. Clearly, $E \geqq 0$.

N being fixed, we want to determine the coefficients in (2) such that E is minimum. Since $(f - F)^2 = f^2 - 2fF + F^2$, we have

$$E = \int_{-\pi}^{\pi} f^2 \, dx - 2 \int_{-\pi}^{\pi} fF \, dx + \int_{-\pi}^{\pi} F^2 \, dx. \tag{4}$$

We square (2), insert it into the last integral in (4), and evaluate the occurring integrals. This gives integrals of $\cos^2 nx$ and $\sin^2 nx$ ($n \geqq 1$), which equal π, and integrals of $\cos nx$, $\sin nx$, and $(\cos nx)(\sin mx)$, which are zero (just as in Sec. 11.1). Thus

$$\int_{-\pi}^{\pi} F^2 \, dx = \int_{-\pi}^{\pi} \left[A_0 + \sum_{n=1}^{N} (A_n \cos nx + B_n \sin nx) \right]^2 dx$$

$$= \pi(2A_0^2 + A_1^2 + \cdots + A_N^2 + B_1^2 + \cdots + B_N^2).$$

We now insert (2) into the integral of fF in (4). This gives integrals of $f \cos nx$ as well as $f \sin nx$, just as in Euler's formulas, Sec. 11.1, for a_n and b_n (each multiplied by A_n or B_n). Hence

$$\int_{-\pi}^{\pi} fF \, dx = \pi(2A_0 a_0 + A_1 a_1 + \cdots + A_N a_N + B_1 b_1 + \cdots + B_N b_N).$$

With these expressions, (4) becomes

$$\begin{aligned} E = \int_{-\pi}^{\pi} f^2 \, dx &- 2\pi \left[2A_0 a_0 + \sum_{n=1}^{N} (A_n a_n + B_n b_n) \right] \\ &+ \pi \left[2A_0^2 + \sum_{n=1}^{N} (A_n^2 + B_n^2) \right]. \end{aligned} \tag{5}$$

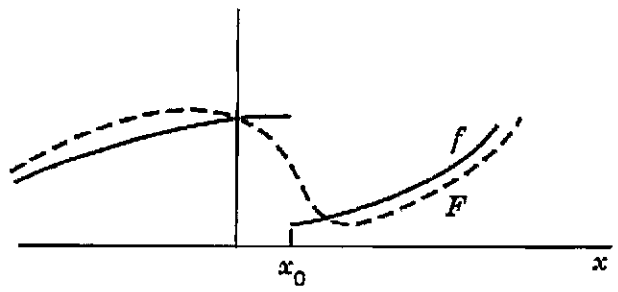


Fig. 275. Error of approximation

We now take $A_n = a_n$ and $B_n = b_n$ in (2). Then in (5) the second line cancels half of the integral-free expression in the first line. Hence for this choice of the coefficients of F the square error, call it E^*, is

$$E^* = \int_{-\pi}^{\pi} f^2\,dx - \pi\left[2a_0^{\,2} + \sum_{n=1}^{N} (a_n^{\,2} + b_n^{\,2})\right]. \tag{6}$$

We finally subtract (6) from (5). Then the integrals drop out and we get terms $A_n^{\,2} - 2A_n a_n + a_n^{\,2} = (A_n - a_n)^2$ and similar terms $(B_n - b_n)^2$:

$$E - E^* = \pi\left\{2(A_0 - a_0)^2 + \sum_{n=1}^{N} [(A_n - a_n)^2 + (B_n - b_n)^2]\right\}.$$

Since the sum of squares of real numbers on the right cannot be negative,

$$E - E^* \geqq 0, \qquad \text{thus} \qquad E \geqq E^*,$$

and $E = E^*$ if and only if $A_0 = a_0, \cdots, B_N = b_N$. This proves the following fundamental minimum property of the partial sums of Fourier series.

THEOREM 1

Minimum Square Error

The square error of F in (2) (*with fixed N*) *relative to f on the interval $-\pi \leqq x \leqq \pi$ is minimum if and only if the coefficients of F in* (2) *are the Fourier coefficients of f. This minimum value E^* is given by* (6).

From (6) we see that E^* cannot increase as N increases, but may decrease. Hence *with increasing N the partial sums of the Fourier series of f yield better and better approximations to f*, considered from the viewpoint of the square error.

Since $E^* \geqq 0$ and (6) holds for every N, we obtain from (6) the important **Bessel's inequality**

$$2a_0^{\,2} + \sum_{n=1}^{\infty} (a_n^{\,2} + b_n^{\,2}) \leqq \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)^2\,dx \tag{7}$$

for the Fourier coefficients of any function f for which integral on the right exists. (For F. W. Bessel see Sec. 5.5.)

It can be shown (see [C12] in App. 1) that for such a function f, **Parseval's theorem** holds; that is, formula (7) holds with the equality sign, so that it becomes **Parseval's identity**[4]

$$2a_0^{\,2} + \sum_{n=1}^{\infty} (a_n^{\,2} + b_n^{\,2}) = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)^2\,dx. \tag{8}$$

[4]MARC ANTOINE PARSEVAL (1755–1836), French mathematician. A physical interpretation of the identity follows in the next section.

EXAMPLE 1 Minimum Square Error for the Sawtooth Wave

Compute the minimum square error E^* of $F(x)$ with $N = 1, 2, \cdots, 10, 20, \cdots, 100$ and 1000 relative to

$$f(x) = x + \pi \qquad (-\pi < x < \pi)$$

on the interval $-\pi \leqq x \leqq \pi$.

Solution. $F(x) = \pi + 2(\sin x - \frac{1}{2}\sin 2x + \frac{1}{3}\sin 3x - + \cdots + \frac{(-1)^{N+1}}{N}\sin Nx)$ by Example 3 in Sec. 11.3. From this and (6),

$$E^* = \int_{-\pi}^{\pi} (x + \pi)^2\, dx - \pi\left(2\pi^2 + 4\sum_{n=1}^{N} \frac{1}{n^2}\right).$$

Numeric values are:

N	E^*	N	E^*	N	E^*	N	E^*
1	8.1045	6	1.9295	20	0.6129	70	0.1782
2	4.9629	7	1.6730	30	0.4120	80	0.1561
3	3.5666	8	1.4767	40	0.3103	90	0.1389
4	2.7812	9	1.3216	50	0.2488	100	0.1250
5	2.2786	10	1.1959	60	0.2077	1000	0.0126

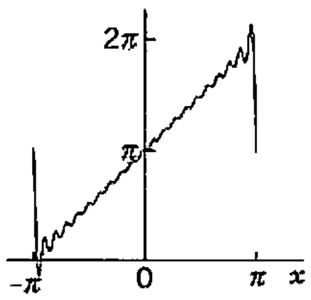


Fig. 276. F with $N = 20$ in Example 1

$F = S_1, S_2, S_3$ are shown in Fig. 266 in Sec. 11.3, and $F = S_{20}$ is shown in Fig. 276. Although $|f(x) - F(x)|$ is large at $\pm\pi$ (how large?), where f is discontinuous, F approximates f quite well on the whole interval, except near $\pm\pi$, where "waves" remain owing to the Gibbs phenomenon (see CAS Experiment 20 in Problem Set 11.2).

Can you think of functions f for which E^* decreases more quickly with increasing N? ■

This is the end of our discussion of Fourier series, which has emphasized the practical aspects of these series, as needed in applications. In the last three sections of this chapter we show how ideas and techniques in Fourier series can be extended to ***nonperiodic*** functions.

PROBLEM SET 11.6

1–9 MINIMUM SQUARE ERROR

Find the trigonometric polynomial $F(x)$ of the form (2) for which the square error with respect to the given $f(x)$ on the interval $-\pi \leqq x \leqq \pi$ is minimum, and compute the minimum value for $N = 1, 2, \cdots, 5$ (or also for larger values if you have a CAS).

1. $f(x) = x \ (-\pi < x < \pi)$
2. $f(x) = x^2 \ (-\pi < x < \pi)$
3. $f(x) = |x| \ (-\pi < x < \pi)$
4. $f(x) = x^3 \ (-\pi < x < \pi)$
5. $f(x) = |\sin x| \ (-\pi < x < \pi)$
6. $f(x) = e^{-|x|} \ (-\pi < x < \pi)$
7. $f(x) = \begin{cases} -1 & \text{if} \quad -\pi < x < 0 \\ 1 & \text{if} \quad 0 < x < \pi \end{cases}$
8. $f(x) = \begin{cases} x & \text{if} \quad -\frac{1}{2}\pi < x < \frac{1}{2}\pi \\ 0 & \text{if} \quad \frac{1}{2}\pi < x < \frac{3}{2}\pi \end{cases}$
9. $f(x) = x(x + \pi)$ if $-\pi < x < 0$, $f(x) = x(-x + \pi)$ if $0 < x < \pi$
10. **CAS EXPERIMENT. Size and Decrease of E^*.** Compare the size of the minimum square error E^* for functions of your choice. Find experimentally the factors on which the decrease of E^* with N depends. For each function considered find the smallest N such that $E^* < 0.1$.
11. **(Monotonicity)** Show that the minimum square error (6) is a monotone decreasing function of N. How can you use this in practice?

12–16 **PARSEVAL'S IDENTITY**

Using Parseval's identity, prove that the series have the indicated sums. Compute the first few partial sums to see that the convergence is rapid.

12. $1 + \frac{1}{3^4} + \frac{1}{5^4} + \frac{1}{7^4} + \cdots = \frac{\pi^4}{96} = 1.01467\,8032$

(Use Prob. 15 in Sec. 11.1.)

13. $1 + \frac{1}{3^2} + \frac{1}{5^2} + \cdots = \frac{\pi^2}{8} = 1.23370\,0550$

(Use Prob. 13 in Sec. 11.1.)

14. $\frac{1}{1^2 \cdot 3^2} + \frac{1}{3^2 \cdot 5^2} + \frac{1}{5^2 \cdot 7^2} + \cdots$

$= \frac{\pi^2}{16} - \frac{1}{2} = 0.11685\,0275$

(Use Prob. 5, this set.)

15. $1 + \frac{1}{2^4} + \frac{1}{3^4} + \cdots = \frac{\pi^4}{90} = 1.08232\,3234$

(Use Prob. 21 in Sec. 11.1.)

16. $1 + \frac{1}{3^6} + \frac{1}{5^6} + \frac{1}{7^6} + \cdots = \frac{\pi^6}{960} = 1.00144\,7078$

(Use Prob. 9, this set.)

11.7 Fourier Integral

Fourier series are powerful tools for problems involving functions that are periodic or are of interest on a finite interval only. Sections 11.3 and 11.5 first illustrated this, and various further applications follow in Chap. 12. Since, of course, many problems involve functions that are **nonperiodic** *and are of interest on the whole x-axis,* we ask what can be done to extend the method of Fourier series to snch functions. This idea will lead to "Fourier integrals."

In Example 1 we start from a special function f_L of period $2L$ and see what happens to its Fourier series if we let $L \to \infty$. Then we do the same for an *arbitrary* function f_L of period $2L$. This will motivate and suggest the main result of this section, which is an integral representation given in Theorem 1 (below).

EXAMPLE 1 **Rectangular Wave**

Consider the periodic rectangular wave $f_L(x)$ of period $2L > 2$ given by

$$f_L(x) = \begin{cases} 0 & \text{if} \quad -L < x < -1 \\ 1 & \text{if} \quad -1 < x < \ 1 \\ 0 & \text{if} \quad \ 1 < x < L. \end{cases}$$

The left part of Fig. 277 shows this function for $2L = 4, 8, 16$ as well as the nonperiodic function $f(x)$, which we obtain from f_L if we let $L \to \infty$,

$$f(x) = \lim_{L\to\infty} f_L(x) = \begin{cases} 1 & \text{if } -1 < x < 1 \\ 0 & \text{otherwise.} \end{cases}$$

We now explore what happens to the Fourier coefficients of f_L as L increases. Since f_L is even, $b_n = 0$ for all n. For a_n the Euler formulas (6), Sec. 11.2, give

$$a_0 = \frac{1}{2L}\int_{-1}^{1} dx = \frac{1}{L}, \qquad a_n = \frac{1}{L}\int_{-1}^{1} \cos\frac{n\pi x}{L}\,dx = \frac{2}{L}\int_{0}^{1} \cos\frac{n\pi x}{L}\,dx = \frac{2}{L}\,\frac{\sin(n\pi/L)}{n\pi/L}.$$

This sequence of Fourier coefficients is called the **amplitude spectrum** of f_L because $|a_n|$ is the maximum amplitude of the wave $a_n \cos(n\pi x/L)$. Figure 277 shows this spectrum for the periods $2L = 4, 8, 16$. We see that for increasing L these amplitudes become more and more dense on the positive w_n-axis, where $w_n = n\pi/L$. Indeed, for $2L = 4, 8, 16$ we have 1, 3, 7 amplitudes per "half-wave" of the function $(2 \sin w_n)/(Lw_n)$ (dashed in the figure). Hence for $2L = 2^k$ we have $2^{k-1} - 1$ amplitudes per half-wave, so that these amplitudes will eventually be everywhere dense on the positive w_n-axis (and will decrease to zero).

The outcome of this example gives an intuitive impression of what about to expect if we turn from our special function to an arbitrary one. as we shall do next. ■

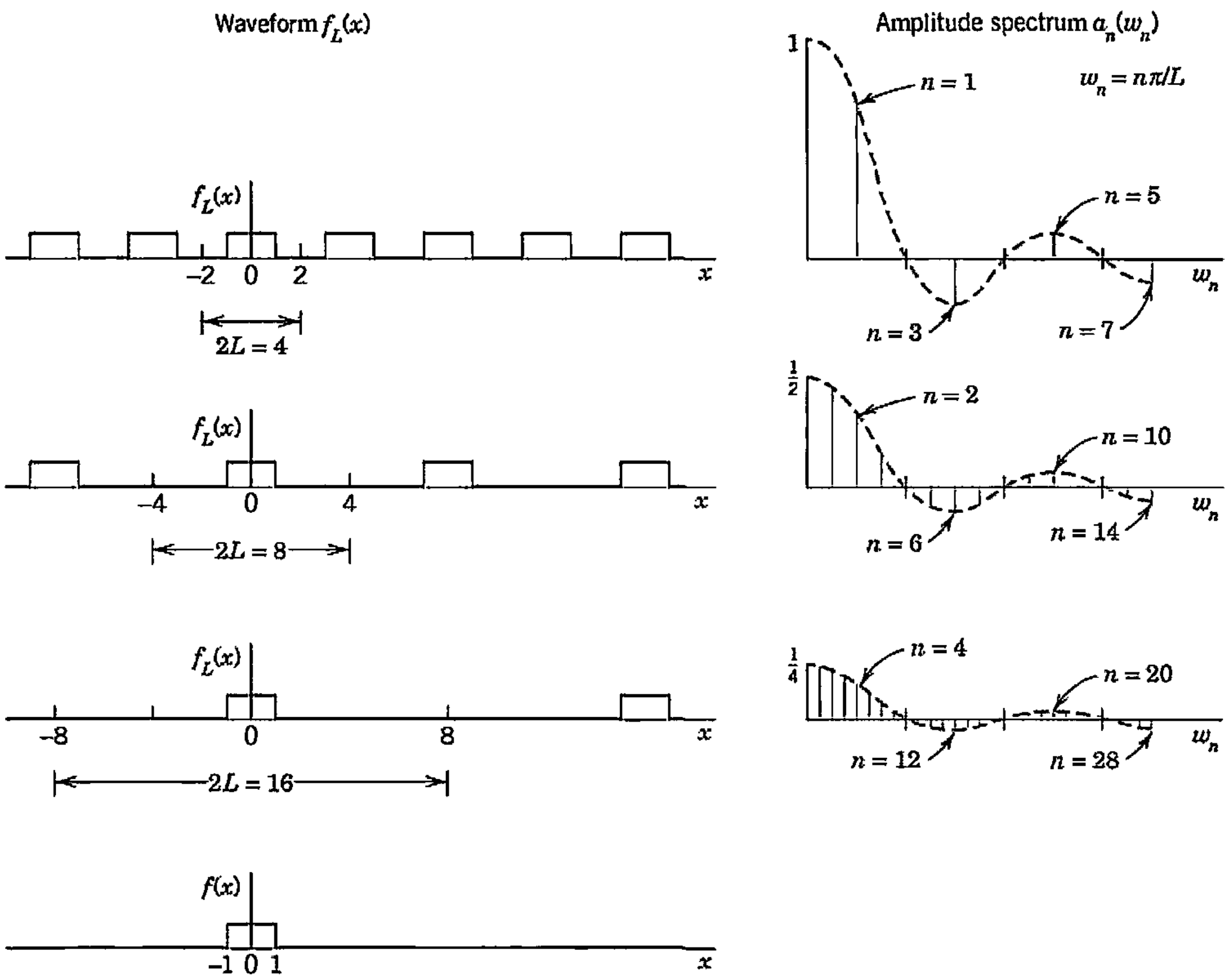


Fig. 277. Waveforms and amplitude spectra in Example 1

From Fourier Series to Fourier Integral

We now consider any periodic function $f_L(x)$ of period $2L$ that can be represented by a Fourier series

$$f_L(x) = a_0 + \sum_{n=1}^{\infty} (a_n \cos w_n x + b_n \sin w_n x), \qquad w_n = \frac{n\pi}{L}$$

and find out what happens if we let $L \to \infty$. Together with Example 1 the present calculation will suggest that we should expect an integral (instead of a series) involving $\cos wx$ and $\sin wx$ with w no longer restricted to integer multiples $w = w_n = n\pi/L$ of π/L but taking *all* values. We shall also see what form such an integral might have.

If we insert a_n and b_n from the Euler formulas (6), Sec. 11.2, and denote the variable of integration by v, the Fourier series of $f_L(x)$ becomes

$$f_L(x) = \frac{1}{2L} \int_{-L}^{L} f_L(v)\, dv + \frac{1}{L} \sum_{n=1}^{\infty} \left[\cos w_n x \int_{-L}^{L} f_L(v) \cos w_n v\, dv + \sin w_n x \int_{-L}^{L} f_L(v) \sin w_n v\, dv \right].$$

We now set

$$\Delta w = w_{n+1} - w_n = \frac{(n+1)\pi}{L} - \frac{n\pi}{L} = \frac{\pi}{L}.$$

Then $1/L = \Delta w/\pi$, and we may write the Fourier series in the form

$$(1)\qquad f_L(x) = \frac{1}{2L}\int_{-L}^{L} f_L(v)\,dv + \frac{1}{\pi}\sum_{n=1}^{\infty}\Bigg[(\cos w_n x)\,\Delta w \int_{-L}^{L} f_L(v)\cos w_n v\,dv + (\sin w_n x)\,\Delta w \int_{-L}^{L} f_L(v)\sin w_n v\,dv\Bigg].$$

This representation is valid for any fixed L, arbitrarily large, but finite.

We now let $L \to \infty$ and assume that the resulting nonperiodic function

$$f(x) = \lim_{L\to\infty} f_L(x)$$

is **absolutely integrable** on the x-axis; that is, the following (finite!) limits exist:

$$(2)\qquad \lim_{a\to-\infty}\int_a^0 |f(x)|\,dx + \lim_{b\to\infty}\int_0^b |f(x)|\,dx \qquad \left(\text{written } \int_{-\infty}^{\infty} |f(x)|\,dx\right).$$

Then $1/L \to 0$, and the value of the first term on the right side of (1) approaches zero. Also $\Delta w = \pi/L \to 0$ and it seems ***plausible*** that the infinite series in (1) becomes an integral from 0 to ∞, which represents $f(x)$, namely,

$$(3)\quad f(x) = \frac{1}{\pi}\int_0^{\infty}\left[\cos wx \int_{-\infty}^{\infty} f(v)\cos wv\,dv + \sin wx \int_{-\infty}^{\infty} f(v)\sin wv\,dv\right] dw.$$

If we introduce the notations

$$(4)\qquad A(w) = \frac{1}{\pi}\int_{-\infty}^{\infty} f(v)\cos wv\,dv, \qquad B(w) = \frac{1}{\pi}\int_{-\infty}^{\infty} f(v)\sin wv\,dv$$

we can write this in the form

$$(5)\qquad f(x) = \int_0^{\infty} [A(w)\cos wx + B(w)\sin wx]\,dw.$$

This is called a representation of $f(x)$ by a **Fourier integral.**

It is clear that our naive approach merely ***suggests*** the representation (5), but by no means establishes it; in fact, the limit of the series in (1) as Δw approaches zero is not the definition of the integral (3). Sufficient conditions for the validity of (5) are as follows.

THEOREM 1

Fourier Integral

If $f(x)$ is piecewise continuous (see Sec. 6.1) *in every finite interval and has a right-hand derivative and a left-hand derivative at every point* (see Sec 11.1) *and if the integral* (2) *exists, then $f(x)$ can be represented by a Fourier integral* (5) *with A and B given by* (4). *At a point where $f(x)$ is discontinuous the value of the Fourier integral equals the average of the left- and right-hand limits of $f(x)$ at that point* (see Sec. 11.1). (Proof in Ref. [C12]; see App. 1.)

Applications of Fourier Integrals

The main application of Fourier integrals is in solving ODEs and PDEs, as we shall see for PDEs in Sec. 12.6. However, we can also use Fourier integrals in integration and in discussing functions defined by integrals, as the next examples (2 and 3) illustrate.

EXAMPLE 2 Single Pulse, Sine Integral

Find the Fourier integral representation of the function

$$f(x) = \begin{cases} 1 & \text{if} \quad |x| < 1 \\ 0 & \text{if} \quad |x| > 1 \end{cases} \qquad \text{(Fig. 278).}$$

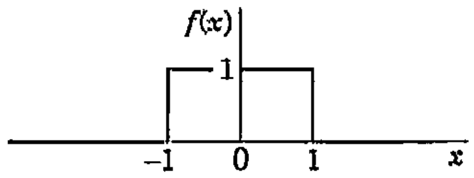


Fig. 278. Example 2

Solution. From (4) we obtain

$$A(w) = \frac{1}{\pi}\int_{-\infty}^{\infty} f(v) \cos wv \, dv = \frac{1}{\pi}\int_{-1}^{1} \cos wv \, dv = \frac{\sin wv}{\pi w}\bigg|_{-1}^{1} = \frac{2 \sin w}{\pi w}$$

$$B(w) = \frac{1}{\pi}\int_{-1}^{1} \sin wv \, dv = 0$$

and (5) gives the answer

$$\text{(6)} \qquad f(x) = \frac{2}{\pi}\int_{0}^{\infty} \frac{\cos wx \sin w}{w} \, dw.$$

The average of the left- and right-hand limits of $f(x)$ at $x = 1$ is equal to $(1 + 0)/2$, that is, $1/2$.

Furthermore, from (6) and Theorem 1 we obtain (multiply by $\pi/2$)

$$\text{(7)} \qquad \int_{0}^{\infty} \frac{\cos wx \sin w}{w} \, dw = \begin{cases} \pi/2 & \text{if} \quad 0 \leqq x < 1, \\ \pi/4 & \text{if} \quad x = 1, \\ 0 & \text{if} \quad x > 1. \end{cases}$$

We mention that this integral is called **Dirichlet's discontinous factor.** (For P. L. Dirichlet see Sec. 10.8.)

The case $x = 0$ is of particular interest. If $x = 0$, then (7) gives

$$\text{(8*)} \qquad \int_{0}^{\infty} \frac{\sin w}{w} \, dw = \frac{\pi}{2}.$$

We see that this integral is the limit of the so-called **sine integral**

$$\text{(8)} \qquad \text{Si}(u) = \int_{0}^{u} \frac{\sin w}{w} \, dw$$

as $u \to \infty$. The graphs of Si(u) and of the integrand are shown in Fig. 279.

In the case of a Fourier series the graphs of the partial sums are approximation curves of the curve of the periodic function represented by the series. Similarly, in the case of the Fourier integral (5), approximations are obtained by replacing ∞ by numbers a. Hence the integral

$$\text{(9)} \qquad \frac{2}{\pi}\int_{0}^{a} \frac{\cos wx \sin w}{w} \, dw$$

approximates the right side in (6) and therefore $f(x)$.

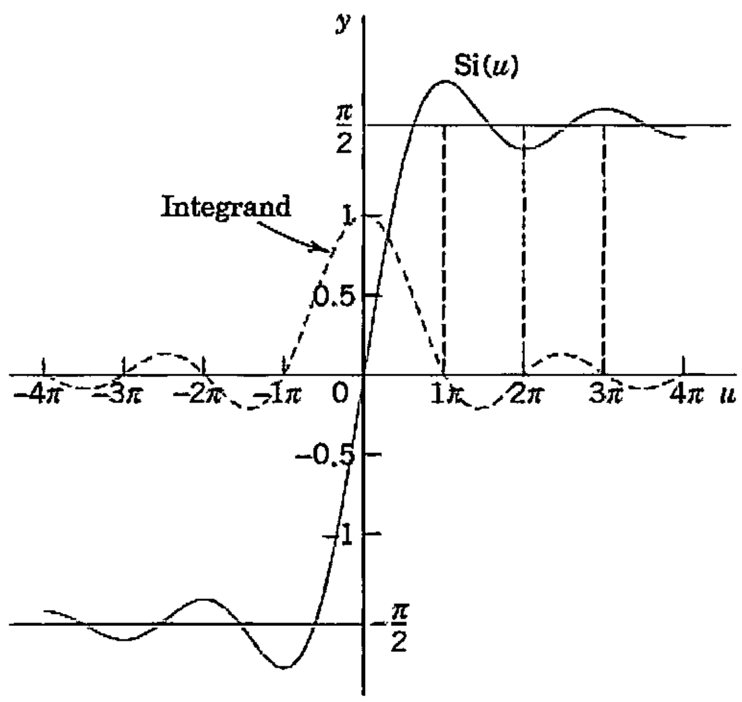


Fig. 279. Sine integral Si(u) and integrand

Figure 280 shows oscillations near the points of discontinuity of $f(x)$. We might expect that these oscillations disappear as a approaches infinity. But this is not true; with increasing a, they are shifted closer to the points $x = \pm 1$. This unexpected behavior, which also occurs in connection with Fourier series, is known as the **Gibbs phenomenon.** (See also Problem Set 11.2.) We can explain it by representing (9) in terms of sine integrals as follows. Using (11) in App. A3.1, we have

$$\frac{2}{\pi}\int_0^a \frac{\cos wx \sin w}{w}\,dw = \frac{1}{\pi}\int_0^a \frac{\sin (w + wx)}{w}\,dw + \frac{1}{\pi}\int_0^a \frac{\sin (w - wx)}{w}\,dw.$$

In the first integral on the right we set $w + wx = t$. Then $dw/w = dt/t$, and $0 \leqq w \leqq a$ corresponds to $0 \leqq t \leqq (x + 1)a$. In the last integral we set $w - wx = -t$. Then $dw/w = dt/t$, and $0 \leqq w \leqq a$ corresponds to $0 \leqq t \leqq (x - 1)a$. Since $\sin(-t) = -\sin t$, we thus obtain

$$\frac{2}{\pi}\int_0^a \frac{\cos wx \sin w}{w}\,dw = \frac{1}{\pi}\int_0^{(x+1)a} \frac{\sin t}{t}\,dt - \frac{1}{\pi}\int_0^{(x-1)a} \frac{\sin t}{t}\,dt.$$

From this and (8) we see that our integral (9) equals

$$\frac{1}{\pi}\,\mathrm{Si}(a[x + 1]) - \frac{1}{\pi}\,\mathrm{Si}(a[x - 1])$$

and the oscillations in Fig. 280 result from those in Fig. 279. The increase of a amounts to a transformation of the scale on the axis and causes the shift of the oscillations (the waves) toward the points of discontinuity -1 and 1. ■

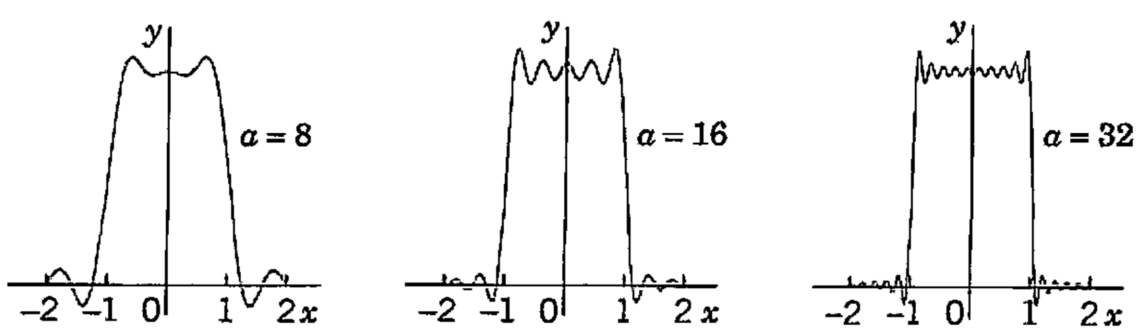


Fig. 280. The integral (9) for a = 8, 16, and 32

Fourier Cosine Integral and Fourier Sine Integral

For an even or odd function the Fourier integral becomes simpler. Just as in the case of Fourier series (Sec. 11.3), this is of practical interest in saving work and avoiding errors. The simplifications follow immediately from the formulas just obtained.

Indeed, if $f(x)$ is an ***even*** function, then $B(w) = 0$ in (4) and

$$A(w) = \frac{2}{\pi}\int_0^\infty f(v)\cos wv\, dv. \tag{10}$$

The Fourier integral (5) then reduces to the **Fourier cosine integral**

$$f(x) = \int_0^\infty A(w)\cos wx\, dw \qquad (f \text{ even}). \tag{11}$$

Similarly, if $f(x)$ is ***odd,*** then in (4) we have $A(w) = 0$ and

$$B(w) = \frac{2}{\pi}\int_0^\infty f(v)\sin wv\, dv. \tag{12}$$

The Fourier integral (5) then reduces to the **Fourier sine integral**

$$f(x) = \int_0^\infty B(w)\sin wx\, dw \qquad (f \text{ odd}). \tag{13}$$

Evaluation of Integrals

Earlier in this section we pointed out that the main application of the Fourier integral is in differential equations but that Fourier integral representations also help in evaluating certain integrals. To see this, we show the method for an important case, the Laplace integrals.

EXAMPLE 3 Laplace Integrals

We shall derive the Fourier cosine and Fourier sine integrals of $f(x) = e^{-kx}$, where $x > 0$ and $k > 0$ (Fig. 281). The result will be used to evaluate the so-called Laplace integrals.

Fig. 281. $f(x)$ in Example 3

Solution. (a) From (10) we have $A(w) = \frac{2}{\pi}\int_0^\infty e^{-kv}\cos wv\, dv$. Now, by integration by parts,

$$\int e^{-kv}\cos wv\, dv = -\frac{k}{k^2+w^2}\, e^{-kv}\left(-\frac{w}{k}\sin wv + \cos wv\right).$$

If $v = 0$, the expression on the right equals $-k/(k^2 + w^2)$. If v approaches infinity, that expression approaches zero because of the exponential factor. Thus

$$A(w) = \frac{2k/\pi}{k^2 + w^2}. \tag{14}$$

By substituting this into (11) we thus obtain the Fourier cosine integral representation

$$f(x) = e^{-kx} = \frac{2k}{\pi}\int_0^\infty \frac{\cos wx}{k^2 + w^2}\, dw \qquad (x > 0,\quad k > 0).$$

From this representation we see that

$$\int_0^\infty \frac{\cos wx}{k^2 + w^2}\, dw = \frac{\pi}{2k} e^{-kx} \qquad (x > 0,\quad k > 0). \tag{15}$$

(b) Similarly, from (12) we have $B(w) = \frac{2}{\pi}\int_0^\infty e^{-kv} \sin wv\, dv$. By integration by parts,

$$\int e^{-kv} \sin wv\, dv = -\frac{w}{k^2 + w^2} e^{-kv}\left(\frac{k}{w}\sin wv + \cos wv\right).$$

This equals $-w/(k^2 + w^2)$ if $v = 0$, and approaches 0 as $v \to \infty$. Thus

$$B(w) = \frac{2w/\pi}{k^2 + w^2}. \tag{16}$$

From (13) we thus obtain the Fourier sine integral representation

$$f(x) = e^{-kx} = \frac{2}{\pi}\int_0^\infty \frac{w \sin wx}{k^2 + w^2}\, dw.$$

From this we see that

$$\int_0^\infty \frac{w \sin wx}{k^2 + w^2}\, dw = \frac{\pi}{2} e^{-kx} \qquad (x > 0,\quad k > 0). \tag{17}$$

The integrals (15) and (17) are called the **Laplace integrals.** ■

PROBLEM SET 11.7

1–6 EVALUATION OF INTEGRALS

Show that the given integral represents the indicated function. *Hint.* Use (5), (11), or (13); the integral tells you which one, and its value tells you what function to consider. (Show the details of your work.)

1. $\displaystyle\int_0^\infty \frac{\cos xw + w \sin xw}{1 + w^2}\, dw = \begin{cases} 0 & \text{if } x < 0 \\ \pi/2 & \text{if } x = 0 \\ \pi e^{-x} & \text{if } x > 0 \end{cases}$

2. $\displaystyle\int_0^\infty \frac{\sin w - w \cos w}{w^2} \sin xw\, dw = \begin{cases} \pi x/2 & \text{if } 0 < x < 1 \\ \pi/4 & \text{if } x = 1 \\ 0 & \text{if } x > 1 \end{cases}$

3. $\displaystyle\int_0^\infty \frac{\cos xw}{1 + w^2}\, dw = \frac{\pi}{2} e^{-x} \text{ if } x > 0$

4. $\displaystyle\int_0^\infty \frac{\sin w}{w} \cos xw\, dw = \begin{cases} \pi/2 & \text{if } 0 \leqq x < 1 \\ \pi/4 & \text{if } x = 1 \\ 0 & \text{if } x > 1 \end{cases}$

5. $\displaystyle\int_0^\infty \frac{\cos(\pi w/2)}{1 - w^2} \cos xw\, dw = \begin{cases} \frac{\pi}{2}\cos x & \text{if } 0 < |x| < \pi/2 \\ 0 & \text{if } |x| \geqq \pi/2 \end{cases}$

6. $\displaystyle\int_0^\infty \frac{\sin \pi w \sin xw}{1 - w^2}\, dw = \begin{cases} \frac{\pi}{2}\sin x & \text{if } 0 \leqq x \leqq \pi \\ 0 & \text{if } x > \pi \end{cases}$

7–12 FOURIER COSINE INTEGRAL REPRESENTATIONS

Represent $f(x)$ as an integral (11).

7. $f(x) = \begin{cases} 1 & \text{if } 0 < x < a \\ 0 & \text{if } x > a \end{cases}$

8. $f(x) = \begin{cases} x^2 & \text{if } 0 < x < a \\ 0 & \text{if } x > a \end{cases}$

9. $f(x) = \begin{cases} x & \text{if } 0 < x < 1 \\ 0 & \text{if } x > 1 \end{cases}$

10. $f(x) = \begin{cases} x/2 & \text{if } 0 < x < 1 \\ 1 - x/2 & \text{if } 1 < x < 2 \\ 0 & \text{if } x > 2 \end{cases}$

11. $f(x) = \begin{cases} \sin x & \text{if } 0 < x < \pi \\ 0 & \text{if } x > \pi \end{cases}$

12. $f(x) = \begin{cases} e^{-x} & \text{if } 0 < x < a \\ 0 & \text{if } x > a \end{cases}$

13. **CAS EXPERIMENT. Approximate Fourier Cosine Integrals.** Graph the integrals in Prob. 7, 9, and 11 as functions of x. Graph approximations obtained by replacing ∞ with finite upper limits of your choice. Compare the quality of the approximations. Write a short report on your empirical results and observations.

14–19 FOURIER SINE INTEGRAL REPRESENTATIONS

Represent $f(x)$ as an integral (13).

14. $f(x) = \begin{cases} 1 & \text{if } 0 < x < a \\ 0 & \text{if } x > a \end{cases}$

15. $f(x) = \begin{cases} \sin x & \text{if } 0 < x < \pi \\ 0 & \text{if } x > \pi \end{cases}$

16. $f(x) = \begin{cases} 1 - x^2 & \text{if } 0 < x < 1 \\ 0 & \text{if } x > 1 \end{cases}$

17. $f(x) = \begin{cases} \pi - x & \text{if } 0 < x < \pi \\ 0 & \text{if } x > \pi \end{cases}$

18. $f(x) = \begin{cases} \cos x & \text{if } 0 < x < \pi \\ 0 & \text{if } x > \pi \end{cases}$

19. $f(x) = \begin{cases} a - x & \text{if } 0 < x < a \\ 0 & \text{if } x > a \end{cases}$

20. **PROJECT. Properties of Fourier Integrals**

(a) Fourier cosine integral. Show that (11) implies

(a1) $$f(ax) = \frac{1}{a}\int_0^\infty A\left(\frac{w}{a}\right)\cos xw \, dw$$
$(a > 0)$ *(Scale change)*

(a2) $$xf(x) = \int_0^\infty B^*(w)\sin xw \, dw,$$
$$B^* = -\frac{dA}{dw}, \qquad A \text{ as in (10)}$$

(a3) $$x^2 f(x) = \int_0^\infty A^*(w)\cos xw \, dw,$$
$$A^* = -\frac{d^2A}{dw^2}.$$

(b) Solve Prob. 8 by applying (a3) to the result of Prob. 7.

(c) Verify (a2) for $f(x) = 1$ if $0 < x < a$ and $f(x) = 0$ if $x > a$.

(d) Fourier sine integral. Find formulas for the Fourier sine integral similar to those in (a).

11.8 Fourier Cosine and Sine Transforms

An **integral transform** is a transformation in the form of an integral that produces from given functions new functions depending on a different variable. These transformations are of interest mainly as tools for solving ODEs, PDEs, and integral equations, and they often also help in handling and applying special functions. The **Laplace transform** (Chap. 6) is of this kind and is by far the most important integral transform in engineering.

The next in order of importance are Fourier transforms. We shall see that these transforms can be obtained from the Fourier integral in Sec. 11.7 in a rather simple fashion. In this section we consider two of them, which are real, and in the next section a third one that is complex.

Fourier Cosine Transform

For an ***even*** function $f(x)$, the Fourier integral is the Fourier cosine integral

$$(1) \quad \text{(a)} \quad f(x) = \int_0^\infty A(w) \cos wx \, dw, \qquad \text{where} \qquad \text{(b)} \quad A(w) = \frac{2}{\pi} \int_0^\infty f(v) \cos wv \, dv$$

[see (10), (11), Sec. 11.7]. We now set $A(w) = \sqrt{2/\pi}\, \hat{f}_c(w)$, where c suggests "cosine." Then from (1b), writing $v = x$, we have

$$(2) \qquad \hat{f}_c(w) = \sqrt{\frac{2}{\pi}} \int_0^\infty f(x) \cos wx \, dx$$

and from (1a),

$$(3) \qquad f(x) = \sqrt{\frac{2}{\pi}} \int_0^\infty \hat{f}_c(w) \cos wx \, dw.$$

ATTENTION! In (2) we integrate with respect to x and in (3) with respect to w. Formula (2) gives from $f(x)$ a new function $\hat{f}_c(w)$, called the **Fourier cosine transform** of $f(x)$. Formula (3) gives us back $f(x)$ from $\hat{f}_c(w)$, and we therefore call $f(x)$ the **inverse Fourier cosine transform** of $\hat{f}_c(w)$.

The process of obtaining the transform $\hat{f}_c$ from a given f is also called the **Fourier cosine transform** or the *Fourier cosine transform method.*

Fourier Sine Transform

Similarly, for an ***odd*** function $f(x)$, the Fourier integral is the Fourier sine integral [see (12), (13), Sec. 11.7]

$$(4) \quad \text{(a)} \quad f(x) = \int_0^\infty B(w) \sin wx \, dw, \qquad \text{where} \qquad \text{(b)} \quad B(w) = \frac{2}{\pi} \int_0^\infty f(v) \sin wv \, dv.$$

We now set $B(w) = \sqrt{2/\pi}\, \hat{f}_s(w)$, where s suggests "sine." Then from (4b), writing $v = x$, we have

$$(5) \qquad \hat{f}_s(w) = \sqrt{\frac{2}{\pi}} \int_0^\infty f(x) \sin wx \, dx.$$

This is called the **Fourier sine transform** of $f(x)$. Similarly, from (4a) we have

$$(6) \qquad f(x) = \sqrt{\frac{2}{\pi}} \int_0^\infty \hat{f}_s(w) \sin wx \, dw.$$

This is called the **inverse Fourier sine transform** of $\hat{f}_s(w)$. The process of obtaining $\hat{f}_s(w)$ from $f(x)$ is also called the **Fourier sine transform** or the *Fourier sine transform method.*

Other notations are

$$\mathcal{F}_c(f) = \hat{f}_c, \qquad \mathcal{F}_s(f) = \hat{f}_s$$

and $\mathcal{F}_c^{-1}$ and $\mathcal{F}_s^{-1}$ for the inverses of $\mathcal{F}_c$ and $\mathcal{F}_s$, respectively.