Type – III :

when the R.H.S of the given difference equation is a polynomial i.e.

$f(x) = 1+x^2$, $f(x) = 2+3x+4x^2+9x^3$ etc.

then in order to find P.I. put

$y_x = a_0 + a_1 x + a_2 x^2 + \dots$ upto the highest power of x defined in the difference equation and then evaluate the values of $a_0, a_1, a_2, \dots$

Example. Solve $y_{x+2} - 5y_{x+1} + 6y_x = x^2 - 1$ —①

Sol. Consider the homogeneous difference equation corresponding to the given equation ①

$$y_{x+2} - 5y_{x+1} + 6y_x = 0 \quad \text{—②}$$

Let $y_x = Ab^x$ be the non-trivial sol. to equ ② so it must satisfy ② i.e.

$$Ab^{x+2} - 5Ab^{x+1} + 6Ab^x = 0.$$

$$Ab^x\left[b^2 - 5b + 6\right] = 0$$

$\because Ab^x \neq 0$,

$$b^2 - 5b + 6 = 0.$$

$$(b-2)(b-3) = 0$$

$$b = 2, \quad b = 3$$

$$\text{C.F} = y_x = c_1 2^x + c_2 3^x$$

To find P.I, substitute $y_x = a_0 + a_1 x + a_2 x^2$ in equation ①, ① ⇒

$$a_0 + a_1(x+2) + a_2(x+2)^2 - 5[a_0 + a_1(x+1) + a_2(x+1)^2] +$$
$$+ 6[a_0 + a_1 x + a_2 x^2] = x^2 - 1$$

$\Rightarrow \quad a_0 = 2, \quad a_1 = \frac{3}{2}, \quad a_2 = \frac{1}{2}$

So,

$P.I = y_x^* = 2 + \frac{3}{2}x + \frac{1}{2}x^2$ and

$y_x = c_1 2^x + c_2 3^x + (2 + \frac{3}{2}x + \frac{1}{2}x^2)$

Ans.

Questions:

Q1. Solve $y_{k+2} + 5y_{k+1} - 6y_k = k^2 + k + 1$. —— ①

Sol:- Consider the homogeneous difference equ corresponding to equ. ① i.e.

$$y_{k+2} + 5y_{k+1} - 6y_k = 0 \quad \text{—— ②}$$

Let $y_k = Ab^k$ be the non-trivial solution to ②. ⇒

$$Ab^{k+2} + 5Ab^{k+1} - 6Ab^k = 0.$$

$$\Rightarrow Ab^k[b^2 + 5b - 6] = 0.$$

$\because Ab^k \neq 0$

$$b^2 + 5b - 6 = 0$$

$$\Rightarrow b = \frac{-5 \pm \sqrt{49}}{2}$$

$$b = \frac{-5+7}{2}, \quad b = \frac{-5-7}{2}$$

$$b = 1, \quad b = -6.$$

So C.F is

$$y_k = c_1(1)^k + c_2(-6)^k.$$

To find P.I substitute

$y_k = k(a_0 + a_1 k + a_2 k^2)$ in equ ①

① ⇒

$(k+2)[a_0 + a_1(k+2) + a_2(k+2)^2] + 5(k+1)[a_0 + a_1(k+1) + a_2(k+1)^2] - 6k[a_0 + a_1 k + a_2 k^2] = k^2 + k + 1$

By combining like terms, we get

$a_2 = 1/21$

$a_1 = -1/49$

$a_0 = 83/129$

So P.I is

$y_k^* = k\left[\frac{83}{129} - \frac{1}{49}k + \frac{1}{21}k^2\right]$

and

$G.S = Y_k = c_1(1)^k + c_2(-6)^k + k\left[\frac{83}{129} - \frac{k}{49} + \frac{k^2}{21}\right]$

Ans.

Q2. $y_{k+2} + y_{k+1} + y_k = k^2 + k + 1$ — ①

Sol. corresponding homogeneous equ is

$y_{k+2} + y_{k+1} + y_k = 0$

let $y_k = Ab^k$ be the non-trivial sol....

⇒ $Ab^k[b^2 + b + 1] = 0$

$Ab^k \neq 0$,

$b^2 + b + 1 = 0$

⇒ $b = \frac{-1 + \sqrt{3}i}{2}$, $b = \frac{-1 - \sqrt{3}i}{2}$

$b_1 = \frac{-1+\sqrt{3}i}{2}$, $b_2 = \frac{-1-\sqrt{3}i}{2}$

$R = 1$, $R = 1$

$\theta = \tan^{-1}\left(\frac{\sqrt{3}}{-1}\right) = \frac{-\pi}{3}$, $\theta_2 \tan^{-1}(\sqrt{3}) = \frac{\pi}{3}$

So C.F is

$$y_k = (1)^k \left[A\cos\left(-\frac{k\pi}{3}\right) + B\sin\left(-\frac{k\pi}{3}\right)\right] +$$
$$(1)^k \left[C\cos\left(\frac{k\pi}{3}\right) + D\sin\left(\frac{k\pi}{3}\right)\right].$$

$$y_k = (1)^k \left[(A+C)\cos\frac{k\pi}{3} - (B+D)\sin\left(\frac{k\pi}{3}\right)\right]$$

To find P.I. Substitute

$y_k = a_0 + ka_1 + k^2 a_2$ in ① we obtain,

$a_0 = 11/9$, $a_1 = -1/3$, $a_2 = 1/3$

So,

$$P.I = y_k^* = \frac{11}{9} - \frac{k}{3} + \frac{k^2}{3}$$

and

G.S

$$\Rightarrow Y_k = (1)^k\left[(A+C)\cos\frac{k\pi}{3} - (B+D)\sin\frac{k\pi}{3}\right] + \left(\frac{11}{9} - \frac{k}{3} + \frac{k^2}{3}\right)$$

Ans.

Question: Solve

$$4y_{k+2} + 4y_{k+1} + y_k = k+1$$

Ans: $Y_k = (C_1 + C_2 k)\left(\frac{-1}{2}\right)^k - \frac{1}{27} + \frac{1}{9}k$

Ans.