CONTENT:

→ History.
→ Introduction
→ Literature review.
→ Definition of integral equation.
→ Structure of integral equation.
→ Types.
→ Classification of types
→ Laplace transformation.
→ Stepwise methodology of types.
→ Examples-
→ Conclusion.

HISTORY:

The foundations for the discovery of the integral were first laid by "Cavalieri", an Italian Mathematician, in around 1635. Cavalieri's work centered around the observation that a curve can be considered to be sketched by a moving point and an area to be sketched by a moving line. "J. Fourier" (1768-1830) is the initiator of the theory of integral equations - A term integral equation first suggested by "du Bois-Reymond" in 1885. Du Bois-Reymond define an integral equation is understood an equation in which the unknown function occurs under one or more signs of definite integration.

Late eighteenth and early ninetieth century "Laplace", "Fourier", "Poission", "Lionville" and "Able" studies some special type of integral equation. The pioneering systematic investigations goes back to late nineteenth and early twentieth century work of Voltera, Fredholm and Hilbert. In 1877, Voltera published a series of famous papers in which he singled out the notion of a functional and pioneered in the development of a theory of functions in the theory of linear integral equation of special type.

Fredholm presented the fundamentals of the Fredholm integral equation theory in a paper published in 1903 in the Acta mathematica. This paper beame famous almost overnight and soon took its rightful place among the gems of modern mathematics. Hilbert followed Fredholm's famous paper with a series of papers in the Nachrichten of the Guttingen Academy.

In general Voltera and Fredholm integral equations which are basically the integral equations are arise from a single differential equation, depending on which sort of conditions are applied at the boundary of the domain of its solution.

INTRODUCTION:

The theory and applications of integral equations, or, as it is often called, of the inversion of **"definite integrals"**, have come suddenly into prominence and have held during the last half dozen years a central place in the attention of mathematicians. By an integral equation is understood as an equation in which the **"unknown function"** occurs under one or more signs of definite integration.

Mathematicians have so far devoted their attention mainly to two peculiarly simple types of integral equations, the linear equations of the first and second kinds, and we shall not in this tract attempt to go beyond these cases. We shall also restrict ourselves to equations in which only simple (as distinguished from multiple) integrals occur. This restriction however, is quite an unessent-ial one made solely to avoid unprofitable complications at the start, since the results we shall obtain usually admit of an obvious extension to the case of multiple integrals without the introduction of any new difficulties. In this respect integral equations are in striking contrast to the closely related differential equations where the passage from ordinary to partial differential equations is attended with very serious complications.

The theory of integral equations may be regarded as dating back atleast as far as the discovery by Fourier of the theorem concerning integrals which bears his name; for, though this was not the point of view of Fourier, this theorem may be regarded as a statement of the solution of a certain integral equation of the first kind. Abel and Liouville, however, and after them others began the treatment of special integral equations in a perfectly conscious way, and many of them perceived clearly what an important place the theory was destined to fill.

As we shall not, except in one relatively, unimportant case, take up any of the applications of the subject. Such applications, together with its relations to other branches of analysis are what give the subject its great importance

LITERATURE REVIEW:

In 1823, Abel proposed a generalization tautochrone problem, a tautochrone also called as isochrone curve is the curve for which the time taken by an object sliding without friction in uniform gravity to its lowest point is independent of its starting point on the curve, whose solution involved the solution of an integral equation which has more recently been designated as an integral equation of the **first kind**, and in 1877, Liouville showed that the determination of a particular solution of a linear differential equation of the **second kind** could be effected by solving an integral equation of a different type, called the integral equation of the second kind. The ripple of mathematical interest which had its origin in these investigations increased but slowly.

Recently, however stimulated by the researches of Volterra, Fredholm and Hilbert in the period between 1896 and the present time, that which seemed at first only a ripple has grown into a formidable wave which bids fair to carry the integral equation theory into a place beside the most important of the mathematical disciplines. Notwithstanding the rapidly multiplying investigations in integral equations and the numerous applications of them which have been made, the sources of information concerning the theory have remained widely scattered and none too easily accessible to any but the specialist in the subject. Consequently, we feel that significant opportunities exist for the applied mathematician and scientist to cooperate on the numerical

Solution of integral equations

INTEGRAL EQUATION:

Definition:

"Integral equations are equations in which an unknown function appears under an integral sign. There is a close connection between differential equation and integral equation"

Integral equations can be divided into two classes

Linear Integral Equation

Non-Linear Integral Equation

The Integral Equation is represented as.

$$h(x)u(x) = f(x) + \int_{a}^{b(x)} K(x,\xi)\, u(\xi)\, d\xi \quad \text{— (1)}$$

So equation (1) is said to be **linear integral equation** when $u_1(x)$ and $u_2(x)$ are solutions to its associated homogeneous case in $u(x)$, then their linear combination $c_1u_1(x) + c_2u_2(x)$ is also a solution to that homogeneous integral equation, so it is termed linear in $u(x)$ and its example is given as for the torsion of wire.

For example:

For the torsion of a wire.

$$m(t) = h\omega(t) + \int_{-\infty}^{t} \phi(t,\tau)\,\omega(\tau)\,d\tau$$

is linear in $\omega(t)$

and equation (1) is said to be **non-linear integral equation** if when $U_1(x)$ and $U_2(x)$ are solutions to its associated homogeneous case in $U(x)$, and their linear combination $c_1U_1(x)$ and $c_2U_2(x)$ is not a solution to that homogeneous integral equation

For example:

The integral equation

$$U(x) = \int_a^b K(x,t)\,U^2(t)\,dt$$

is non-linear in $U(t)$.

Similarly the integral equation is said to be **homogeneous integral equation** when $f(x) = 0$, so equation (1) can be written as

$$h(x)U(x) = 0 + \int_{a}^{b(x)} K(x,\xi)\,U(\xi)\,d\xi$$

$$h(x)U(x) = \int_a^{b(x)} K(x,\xi)\,U(\xi)\,d\xi$$

and integral equation is said to be **non-homogeneous**

integral equation when $f(x) \neq 0$.
when $f(x)=0$, so eqn (1) in its homogeneous form is given as

$$h(x)u(x) = 0 + \int_a^{b(x)} K(x,\xi)\, u(\xi)\, d\xi$$

$$h(x)u(x) = \int_a^{b(x)} K(x,\xi)\, u(\xi).$$

In the above explanation of equation (1), $K(x,\xi)$ is the **Kernel**.

Structure of an integral equation:

upper limit of integration

unknown

$$h(x)u(x) = f(x) + \int_{a}^{b(x)} K(x,\xi)\, u(\xi)\, d\xi$$

unknown

Kernel

known

lower limit of integration

This structure for integration equation gives us a clear point of view about its definition

TYPES:

Integral equations can be of two types according to the upper limit of integration in the integral $\int_a^{b(x)}$

1/- volterra integral equation.
2/- Fredholm integral equation.

volterra Integral equation:

An integral equation with variable limits of integration is called as volterra integral equation

For example:

$$\rightarrow n(t) = n_0 f(t) + K\int_0^t f(t-\xi) n(\xi) d\xi.$$

$$\rightarrow m(t) = hw(t) + \int_{-\infty}^{t} \phi(t,\xi) U(\xi) d\xi.$$

$$\rightarrow U(x) = 1 + \lambda\int_0^x (x-\xi) U(\xi) d\xi + \int_0^x (x-\xi) g(\xi) d\xi.$$

and equation 1 is called as volterra integral equation when $b(x) = x$
i-e

$$h(x)U(x) = f(x) + \int_a^x K(x,\xi) U(\xi) d\xi. \quad \text{--- (2)}$$

Classification of volterra integral equation:

The volterra integral equation can be classified as

1/- volterra integral equation of first kind.

2/- volterra integral equation of second kind.

* Volterra I.E of first kind:

Volterra integral equation which is represented by equ (2) is said to be volterra integral equation of first kind when $h(x)=0$.

For example:

The Abel integral equation

$$-\sqrt{2g}\, f(x) = \int_0^y \frac{\phi(x)}{\sqrt{y-\eta}}\, d\eta$$

is called as the volterra integral equation of first kind.

* Volterra I.E of second kind:

Volterra integral equation which is represented by equ (2) is said to be volterra integral equation equation of second kind when $h(x)=1$

For example:

$$m(t) = hw(t) + \int_{-\infty}^{t} \phi(t,\xi)\, w(\xi)\, d\xi$$

is called as the volterra integral equation of Second kind.

→ **Fredholm Integral equation:**

An integral equation with fixed limits of integration is called as fredholm integral equation.

For example:

$$\rightarrow f(x) = \int_0^l G(x,\xi) f(\xi) d\xi, \; l > 0 \text{ and fixed.}$$

$$\rightarrow y(x) = \omega^2 \int_0^l \rho(\xi) F(x,\xi) y(\xi) d\xi.$$

$$\rightarrow U(x) = e^x - \lambda \int_0^1 G(x,\xi) U(\xi) d\xi.$$

$$\rightarrow U(x) = \lambda \int_a^b K(x,\xi) U(\xi) d\xi$$

and equation (1) is called as fredholm integral equation when $b(x) = b$

i-e

$$h(x) U(x) = f(x) + \int_a^b K(x,\xi) U(\xi) d\xi \quad (3)$$

- **Classification of fredholm integral equation:**

The fredholm integral equation can be classified as

1/ - Fredholm integral equation of first kind.

2// Fredholm integral equation of Second kind.

* **Fredholm I.E of first kind:**
Fredholm integral equation which is represented by equ (3) is said to be fredholm integral equation of first kind when $h(x)=0$.

For example:
The integral equation

$$U(S) = \int_0^{\infty} e^{-Sx} U(x)\,dx$$ is known as fredholm integral equation of first kind.

* **Fredholm I.E of second kind:**
Fredholm integral equation which is represented by equ (3) is said to be fredholm integral equation of second kind when $h(x)=1$.

For example:
The integral equation of deflection of a rotating shaft

$$y(x) = \omega^2 \int_0^{l} F(x,\xi)\,\rho(\xi)\,y(\xi)\,d\xi$$

is called as fredholm integral equation of second kind.

Now, we have to introduce and define Laplace transformation.

LAPLACE TRANSFORMATION:

History:

Laplace transform, in mathematics, a particular integral transform invented by the French mathematician **Pierre-Simon Laplace (1749–1827)** and systematically developed by the British physicist **Oliver Heaviside (1850–1925)** to simplify the solution of many differential equation that describe physical processes. Today it is used most frequently by electrical engineers in the solution of various electronic circuit problems

Definition:

"The Laplace transform of the function $f(x)$ defined on $(0,\infty)$ is also denoted by $\mathcal{L}\{f(x)\}$ or $F(S)$ as

$$F(S) = \mathcal{L}\{f(x)\} = \int_0^\infty e^{-Sx} f(x)\,dx"$$

involving the exponential parameter S in the Kernel $K = e^{-Sx}$

Stepwise Methodology:

Volterra integral equation of first kind:

We will assume that the Kernel $K(x,t)$ is a convolution type Kernel that can be expressed by difference Kernel $(x-t)$. The linear volterra integral equation of first kind can thus be expressed as

$$f(x) = \int_0^x K(x-t)u(t)dt.$$

Applying the Laplace transform to both sides, we have

$$\mathcal{L}\{f(x)\} = \mathcal{L}\left\{\int_0^x K(x-t)u(t)dt\right\}.$$

Using convolution theorem of Laplace, we have

$$\mathcal{L}\{f(x)\} = \mathcal{L}\{K(x)u(x)\}.$$

$$\mathcal{L}\{f(x)\} = \mathcal{L}\{K(x)\}\mathcal{L}\{u(x)\}.$$

$$\Rightarrow \mathcal{L}\{u(x)\} = \frac{\mathcal{L}\{f(x)\}}{\mathcal{L}\{K(x)\}}.$$

$$\Rightarrow \mathcal{L}\{u(x)\} = \mathcal{L}\left\{\frac{f(x)}{K(x)}\right\}$$

$$\Rightarrow U(S) = \mathcal{L}\left\{\frac{f(x)}{K(x)}\right\}.$$

Operating the inverse Laplace transform on both sides, we have

$$\mathcal{L}^{-1}\{U(S)\} = \mathcal{L}^{-1}\left\{\frac{\mathcal{L}(f(x))}{\mathcal{L}(K(x))}\right\}.$$

$$U(x) = \mathcal{L}^{-1}\left\{\frac{\mathcal{L}(f(x))}{\mathcal{L}(K(x))}\right\}.$$

→ **volterra integral equation of Second kind:**

In the convolution theorem of the Laplace transform, it is stated that if the Kernel $K(x,t)$ of the integral equation

$$U(x) = f(x) + \int_0^x K(x,t)U(t)dt.$$

depends upon the difference $x-t$ then it is called as difference Kernel.

So equation becomes

$$U(x) = f(x) + \int_0^x K(x-t)\,U(t)dt \quad \text{—(1)}.$$

Consider two function $f_1(x)$ and $f_2(x)$ that possesses the conditions needed for the existence of Laplace transform for each. Let the Laplace transform of functions $f_1(x)$ and $f_2(x)$ are given by

$$\mathcal{L}\{f_1(x)\} = F_1(S)$$
$$\mathcal{L}\{f_2(x)\} = F_2(S).$$

The Laplace convolution product

of these two functions are defined by

$$(f_1 * f_2)(x) = \int_0^x f_1(x-t)\, f_2(t)\,dt.$$

$$(f_2 * f_1)(x) = \int_0^x f_2(x-t)\, f_1(t)\,dt.$$

Recall that

$$(f_1 * f_2)(x) = (f_2 * f_1)(x)$$

So the Laplace transform of the convolution theorem is

$$\mathcal{L}\{(f_1 * f_2)(x)\} = \mathcal{L}\left\{\int_0^x f_1(x-t)\, f_2(t)\right\}$$

$$= F_1(S)F_2(S).$$

So taking Laplace transform of eqn ① gives

$$\mathcal{L}\{U(x)\} = \mathcal{L}\left\{f(x) + \int_0^x K(x-t)\, U(t)\,dt\right\}$$

$$U(S) = F(S) + K(S)U(S)$$

Taking common $U(S)$

$$U(S) - K(S)U(S) = F(S)$$

$$U(S)[1 - K(S)] = F(S)$$

$$\Rightarrow U(S) = \frac{F(S)}{1-K(S)}$$

Taking inverse Laplace transform

$$\mathcal{L}^{-1}\{U(S)\} = \mathcal{L}^{-1}\left\{\frac{F(S)}{1-K(S)}\right\}$$

$$\Rightarrow U(x) = \mathcal{L}^{-1}\left\{\frac{F(S)}{1-K(S)}\right\}.$$

EXAMPLES

→ **volterra integral equation of first kind.**

Q: Solve the volterra integral equation of first kind by using Laplace transformation method.

$$\int_0^t \frac{f(u)}{(t-u)^{1/2}}\,du = 1+t+t^2.$$

Solution:

$$\int_0^t (t-u)^{-1/2} f(u)\,du = 1+t+t^2$$

By using the convolution theorem the L·H·S of the above equation becomes

$$t^{-1/2} * f(t) = 1+t+t^2$$

Now applying Laplace transformation on both sides of the above equation gives

$$\mathcal{L}\{t^{-1/2} * f(t)\} = \mathcal{L}\{1+t+t^2\}$$

$$\Rightarrow \mathcal{L}\{t^{-1/2}\}\,\mathcal{L}\{f(t)\} = \mathcal{L}\{1+t+t^2\}$$

Scanned with CamScanner

Now by using formula.

$$\mathcal{L}\{t^n\} = \frac{\Gamma n+1}{t^{n+1}}$$

we get

$$\frac{\Gamma -1/2+1}{S^{-1/2+1}} \quad F(S) = \frac{1}{S} + \frac{1}{S^2} + \frac{2}{S^3}$$

$$\frac{\Gamma 1/2}{S^{1/2}} \quad F(S) = \frac{1}{S} + \frac{1}{S^2} + \frac{2}{S^3}$$

and we know that

$$\Gamma 1/2 = \sqrt{\pi}$$

So,

$$\frac{\sqrt{\pi}}{S^{1/2}} \, F(S) = \frac{1}{S} + \frac{1}{S^2} + \frac{2}{S^3}$$

$$\sqrt{\pi}\, F(S) = \frac{S^{1/2}}{S} + \frac{S^{1/2}}{S^2} + \frac{2S^{1/2}}{S^3}$$

$$\sqrt{\pi}\, F(S) = S^{+1/2-1} + S^{+1/2-2} - 2S^{1/2-3}$$

$$\sqrt{\pi}\, F(S) = S^{-1/2} + S^{-3/2} + 2S^{-5/2}$$

$$\Rightarrow \sqrt{\pi}\, F(S) = \frac{1}{S^{1/2}} + \frac{1}{S^{3/2}} + \frac{2}{S^{5/2}}$$

$$\Rightarrow F(S) = \frac{1}{\sqrt{\pi}\, S^{1/2}} + \frac{1}{\sqrt{\pi}\, S^{3/2}} + \frac{2}{\sqrt{\pi}\, S^{3/2}}$$

Now, taking Laplace inverse on both sides, we get

$$\mathcal{L}^{-1}\{F(S)\} = \mathcal{L}^{-1}\left\{\frac{1}{\sqrt{\pi}\, S^{1/2}} + \frac{1}{\sqrt{\pi}\, S^{3/2}} + \frac{2}{\sqrt{\pi}\, S^{3/2}}\right\}$$

$$\mathcal{L}^{-1}\{F(S)\} = \mathcal{L}^{-1}\left\{\frac{1}{\sqrt{\pi}\, S^{1/2}}\right\} + \mathcal{L}^{-1}\left\{\frac{1}{\sqrt{\pi}\, S^{3/2}}\right\} + \mathcal{L}^{-1}\left\{\frac{2}{\sqrt{\pi}\, S^{3/2}}\right\}$$

$$f(x) = \frac{1}{\sqrt{\pi}}\mathcal{L}^{-1}\left\{\frac{1}{S^{1/2}}\right\} + \frac{1}{\sqrt{\pi}}\mathcal{L}^{-1}\left\{\frac{1}{S^{3/2}}\right\} + \frac{2}{\sqrt{\pi}}\mathcal{L}^{-1}\left\{\frac{1}{S^{3/2}}\right\}.$$

by using formula

$$\mathcal{L}^{-1}\left\{\frac{1}{S^{n+1}}\right\} = \frac{t^n}{\Gamma(n+1)}$$

$\because 5/2 = 3/2 + 1$

$\because 1/2 = 1/2 - 1 + 1 = 1/2$

$\because 3/2 = 1/2 + 1$

So,

$$f(x) = \frac{1}{\sqrt{\pi}}\left(\frac{t^{-1/2}}{\Gamma(-1/2+1)}\right) + \frac{1}{\sqrt{\pi}}\left(\frac{t^{+1/2}}{\Gamma(1/2+1)}\right) + \frac{2}{\sqrt{\pi}}\left(\frac{t^{3/2}}{\Gamma(5/2)}\right).$$

$$f(x) = \frac{t^{-1/2}}{\sqrt{\pi}\,\Gamma 1/2} + \frac{t^{1/2}}{\sqrt{\pi}\,\Gamma 3/2} + \frac{2t^{3/2}}{\sqrt{\pi}\,\Gamma 5/2}$$

Since by formula.

$$\Gamma n+1 = n\Gamma n$$

So

$$\Gamma 3/2 = 1/2\,\Gamma 1/2$$
$$\Gamma 5/2 = 3/2\,\Gamma 3/2$$

Hence

$$f(x) = \frac{t^{-1/2}}{\sqrt{\pi}\sqrt{\pi}} + \frac{t^{1/2}}{\sqrt{\pi}\,1/2\,\Gamma 1/2} + \frac{2t^{3/2}}{\sqrt{\pi}\,3/2 \cdot 1/2\,\Gamma 1/2}$$

$$f(x) = \frac{t^{-1/2}}{\pi} + \frac{t^{1/2}}{\sqrt{\pi}\sqrt{\pi}\,1/2} \cdot \frac{2\,t^{3/2}}{\sqrt{\pi}\sqrt{\pi}\,3/4}$$

$$f(x) = \frac{t^{-1/2}}{\pi} + \frac{2t^{+1/2}}{\pi} + \frac{8t^{3/2}}{3\pi}$$

$$f(x) = \frac{1}{\pi}\left[t^{-1/2} + 2t^{1/2} + \frac{8}{3}t^{3/2}\right]$$

Q: Solve volterra integral equation of first kind by applying Laplace transformation method.

$$x = \int_0^x e^{-(x-t)}\,u(t)\,dt.$$

Solution:

Applying convolution theorem the R.H.S of the question becomes.

$$x = e^{-x}\, U(x)$$

Now applying Laplace Transformation on both sides, we get

$$\mathcal{L}\{x\} = \mathcal{L}\{e^{-x}\, U(x)\}$$

$$\mathcal{L}\{x\} = \mathcal{L}\{e^{-x}\}\,\mathcal{L}\{U(x)\}$$

$$\frac{1}{S^2} = \frac{1}{S+1} \cdot U(S)$$

$$\Rightarrow \quad U(S) = \frac{S+1}{S^2}$$

$$U(S) = \frac{1}{S} + 1$$

$$U(S) = \frac{1}{S^2} + \frac{1}{S}$$

Now applying Laplace inverse on both sides, we get

$$\mathcal{L}^{-1}\{U(S)\} = \mathcal{L}^{-1}\left\{\frac{1}{S^2} + \frac{1}{S}\right\}$$

$$U(x) = \mathcal{L}^{-1}\left\{\frac{1}{S}\right\} + \mathcal{L}^{-1}\{1/S^2\}$$

$$U(x) = 1 + x$$

Q: Solve volterra integral equation of first kind by applying Laplace transformation

$$x = \int_0^x U(t)\,dt$$

Solution:

$$x - \int_0^x U(t)\,dt = 0$$

By applying the convolution theorem we get

$$x - 1 * U(x) = 0$$

Now applying Laplace transformation on both sides, we get

$$\mathcal{L}\{x - 1\,U(x)\} = \mathcal{L}\{0\}$$

$$\mathcal{L}\{x\} - \mathcal{L}\{1\}\mathcal{L}\{U(x)\} = \mathcal{L}\{0\}$$

$$\frac{1}{S^2} - \frac{1}{S}\,U(S) = 0$$

$$U(S)\frac{1}{S} = \frac{1}{S^2}$$

$$U(S) = \frac{S}{S^2}$$

$$U(S) = \frac{1}{S}$$

Applying Laplace inverse on both sides, we get

$$\mathcal{L}^{-1}\{U(S)\} = \mathcal{L}^{-1}\{1/S\}$$

$$U(x) = 1$$

Q- Solve volterra integral equation of first kind by applying Laplace transformation.

$$\cos x - J_0(x) = -\int_0^x J_1(x-t)\,U(t)\,dt$$

Solution:

Applying Laplace transformation

on both sides, we get

$$\mathcal{L}\{\cos x\} - \mathcal{L}\{J_0(x)\} = -\mathcal{L}\left\{\int_0^x J_1(x-t)\, U(t)\, dt\right\}.$$

By convolution theorem R.H.S becomes

$$\mathcal{L}\{\cos x\} - \mathcal{L}\{J_0(x)\} = -\mathcal{L}\{J_1(x)U(x)\}$$

$$\frac{S}{S^2+1} - \mathcal{L}\{J_0(x)\} = -\mathcal{L}\{J_1(x)\}\mathcal{L}\{U(x)\}$$

Since

$$\mathcal{L}\{J_0(x)\} = \frac{1}{\sqrt{1+S^2}}$$

$$\mathcal{L}\{J_1(x)\} = 1 - \frac{S}{\sqrt{1+S^2}}.$$

So

$$\frac{S}{S^2+1} - \frac{1}{\sqrt{1+S^2}} = -\left(1 - \frac{S}{\sqrt{1+S^2}}\right)U(S)$$

$$\frac{S}{S^2+1} - \frac{1}{\sqrt{1+S^2}} = \left(-1 + \frac{S}{\sqrt{1+S^2}}\right)U(S)$$

$$\frac{S - (1+S^2)^{+1/2}}{(S^2+1)} = \left(\frac{-\sqrt{1+S^2} + S}{\sqrt{1+S^2}}\right)U(S)$$

$$\frac{S - (1+S^2)^{1/2}}{(S^2+1)} = \left(\frac{S - (1+S^2)^{1/2}}{(1+S^2)^{1/2}}\right)U(S)$$

$$\Rightarrow U(S) = \frac{\cancel{(S - (1+S^2)^{1/2})}(1+S^2)^{1/2}}{(1+S^2)\cancel{(S - (1+S^2)^{1/2})}}$$

$$U(S) = \frac{(1+S^2)^{1/2}}{(1+S^2)} = (1+S^2)^{1/2-1}$$

Scanned with CamScanner

$$U(S) = (1+S^2)^{-1/2}.$$

$$U(S) = \frac{1}{\sqrt{1+S^2}}$$

Apply Laplace inverse on both sides, we get

$$\mathcal{L}^{-1}\{U(S)\} = \mathcal{L}^{-1}\left\{\frac{1}{\sqrt{1+S^2}}\right\}$$

$$U(x) = J_0(x).$$

Q:- Solve the volterra integral equation of first kind by applying Laplace transformation

$$1 - J_0(x) = \int_0^x U(t)\,dt$$

Applying Laplace transformation on both sides, we get

$$\mathcal{L}\{1 - J_0(x)\} = \mathcal{L}\left\{\int_0^x U(t)\,dt\right\}.$$

$$\mathcal{L}\{1\} - \mathcal{L}\{J_0(x)\} = \mathcal{L}\left\{\int_0^x U(t)\,dt\right\}.$$

By convolution theorem, R.H.S of above equation becomes

$$\mathcal{L}\{1\} - \mathcal{L}\{J_0(x)\} = \mathcal{L}\{1 * U(x)\}.$$

$$\mathcal{L}\{1\} - \mathcal{L}\{J_0(x)\} = \mathcal{L}\{1\}\mathcal{L}\{U(x)\}$$

$$\frac{1}{S} - \frac{1}{\sqrt{1+S^2}} = \frac{1}{S}U(S)$$

$$\Rightarrow U(S) = \frac{S}{S} - \frac{S}{\sqrt{1+S^2}}$$

$$U(S) = 1 - \frac{S}{\sqrt{1+S^2}}$$

Applying Laplace inverse on both sides, we get

$$\mathcal{L}^{-1}\{U(s)\} = \mathcal{L}^{-1}\left\{1 - \frac{s}{\sqrt{1+s^2}}\right\}$$

$$U(x) = J_1(x)$$

Q Solve volterra integral equation of first kind by using Laplace Transformation method.

$$\sin x = \int_0^x U(x-t)\,U(t)\,dt$$

$$\sin x - \int_0^x U(x-t)\,U(t)\,dt = 0$$

Applying Laplace transform on both sides, we get

$$\mathcal{L}\{\sin x\} = \mathcal{L}\left\{-\int_0^x U(x-t)\,U(t)\,dt\right\}$$

$$\mathcal{L}\{\sin x\} = -\mathcal{L}\left\{\int_0^x U(x-t)\,U(t)\,dt\right\}$$

By convolution theorem R.H.S becomes

$$\mathcal{L}\{\sin x\} = -\mathcal{L}\{U(x)\,U(x)\}$$

$$\frac{1}{1+s^2} = \left(\mathcal{L}\{U(x)\}\right)^2$$

$$\mathcal{L}\{U(x)\} = \pm\frac{1}{\sqrt{1+s^2}}$$

$$U(S) = \pm \frac{1}{\sqrt{1+S^2}}$$

Applying Laplace inverse on both sides, we get

$$\mathcal{L}^{-1}\{U(S)\} = \mathcal{L}^{-1}\left\{\pm \frac{1}{\sqrt{1+S^2}}\right\}.$$

$$U(x) = \pm \mathcal{L}^{-1}\left\{\frac{1}{\sqrt{1+S^2}}\right\}$$

$$U(x) = \pm J_0.$$

→ **Volterra integral equation of Second Kind:**

Q Solve the Volterra integral equation of Second Kind by Laplace transformation

$$y(t) - \int_0^t (t-v)\, y(v)\, dv = 1$$

Solution:

Taking Laplace transform on both sides, we get

$$\mathcal{L}\left\{y(t) - \int_0^t (t-v)\, y(v)\, dv\right\} = \mathcal{L}\{1\}$$

$$\mathcal{L}\{y(t)\} - \mathcal{L}\left\{\int_0^t (t-v)\, y(v)\, dv\right\} = \mathcal{L}\{1\}.$$

By convolution Theorem, we get

$$Y(S) - \mathcal{L}\{t * y(t)\} = 1/S$$

$Y(S) - \mathcal{L}\{t\}\,\mathcal{L}\{y(t)\} = 1/S$

$Y(S) = \frac{1}{S^2}\,Y(S) \qquad = 1/S$

Taking common $Y(S)$ on L.H.S, we get

$$Y(S)\left[1 - \frac{1}{S^2}\right] = 1/S$$

$$Y(S)\left[\frac{S^2-1}{S^2}\right] = 1/S$$

$$Y(S) = \frac{1}{\cancel{S}} \cdot \frac{\cancel{S^2}}{(S^2-1)}$$

$$Y(S) = \frac{S}{S^2-1}$$

Taking Laplace inverse on both sides, we get

$$\mathcal{L}^{-1}\{Y(S)\} = \mathcal{L}^{-1}\left\{\frac{S}{S^2-1}\right\}$$

$$y(t) = \cosh(t)$$

Q. Solve volterra integral of second kind by Laplace transformation method.

$$U(x) = 1 + \int_0^x U(t)\,dt$$

Solution:

$$U(x) - \int_0^x U(t)\,dt = 1$$

Taking Laplace transform on both sides, we get

$$\mathcal{L}\left\{U(x) - \int_0^x U(t)\,dt\right\} = \mathcal{L}\{1\}$$

By convolution theorem, gives

$$\mathcal{L}\{U(x) - (1 * U(t))\} = \frac{1}{S}$$

$\mathcal{L}\{U(x)\} - \mathcal{L}\{1\}\mathcal{L}\{U(t)\} = 1/S$

$$U(S) - \frac{1}{S}U(S) = 1/S.$$

Taking common $U(S)$ on L.H.S

$$U(S)\left[1 - \frac{1}{S}\right] = 1/S.$$

$$U(S) = \frac{1/S}{1-\frac{1}{S}} = \frac{1/S}{(S-1)/S}$$

$$U(S) = 1/S \Big/ \frac{(S-1)}{S}$$

$$U(S) = \frac{\frac{1}{S}\cdot S}{(S-1)} = \frac{1}{(S-1)}$$

Taking Laplace inverse, we get

$$\mathcal{L}^{-1}\{U(S)\} = \mathcal{L}^{-1}\{1/(S-1)\}$$

$$U(x) = e^{x}.$$

Q Solve the volterra integral equation of Second kind by Laplace transformation.

$$U(x) = 1 - \int_0^x (x-t)\,u(t)\,dt$$

Solution:

$$U(x) + \int_0^x (x-t)\,U(t)\,dt = 1$$

Taking Laplace transform given

$$\mathcal{L}\{U(x) + \int_0^x (x-t)\,U(t)\,dt\} = \mathcal{L}\{1\}$$

By convolution theorem, given

$$\mathcal{L}\{U(x)\} + \mathcal{L}\{U(x) * x\} = 1/S$$

$$U(S) + \frac{1}{S^2}U(S) = 1/S$$

Taking common $U(S)$ at L.H.S

$$U(S)\left[1+\frac{1}{S^2}\right] = 1/S$$

$$U(S)\left[\frac{S^2+1}{S^2}\right] = 1/S$$

$$U(S) = \frac{1}{S}\cdot\frac{S^2}{S^2+1}$$

$$U(S) = \frac{S}{S^2+1}.$$

Taking Laplace inverse, gives

$$\mathcal{L}^{-1}\{U(S)\} = \mathcal{L}^{-1}\{S/S^2+1\}$$

$$U(x) = \cos x.$$

Q: **Solve volterra integral equation of second kind by Laplace Transformation.**

$$U(x) = \sin x + \cos x + 2\int_0^x \sin(x-t)U(t)dt$$

Solution:

Taking Laplace transform gives

$$\mathcal{L}\{U(x)\} = \mathcal{L}\{\sin x + \cos x + 2\int_0^x \sin(x-t)U(t)\,dt\}.$$

$$\mathcal{L}\{U(x)\} - 2\mathcal{L}\{\int_0 \sin(x-t)U(t)dt\} = \mathcal{L}\{\sin x + \cos x\}$$

By convolution theorem, gives

$$\mathcal{L}\{U(x)\} - 2\mathcal{L}\{\sin x * U(x)\} = \mathcal{L}\{\sin x\} + \mathcal{L}\{\cos x\}.$$

$$U(S) - 2\left(\frac{1}{S^2+1}\cdot U(S)\right) = \frac{1}{S^2+1} + \frac{S}{S^2+1}$$

$$U(S) - \frac{2U(S)}{S^2+1} = \frac{1+S}{S^2+1}$$

Taking common $U(S)$, we get

$$U(S)\left[1-\frac{2}{S^2+1}\right]=\frac{1+S}{S^2+1}$$

$$U(S)\left[\frac{S^2+1-2}{S^2+1}\right]=\frac{1+S}{S^2+1}$$

$$U(S)=\frac{(1+S)}{(1+\cancel{S^2})}\cdot\frac{(1+\cancel{S^2})}{(S^2-1)}$$

$$U(S)=\frac{(1+\cancel{S})}{(S+\cancel{1})(S-1)}=\frac{1}{S-1}$$

Taking Laplace inverse

$$\mathcal{L}^{-1}\{U(S)\}=\mathcal{L}^{-1}\{1/(S-1)\}$$

$$U(x)=e^x$$

Q Solve volterra integral equation of second kind by Laplace transformation.

$$U(x)=x^3-\int_0^x (x-t)U(t)\,dt$$

Solution:

$$U(x)+\int_0^x (x-t)\,U(t)\,dt=x^3.$$

Taking Laplace Transform, we get

$$\mathcal{L}\left\{U(x)+\int_0^x (x-t)U(t)\,dt\right\}=\mathcal{L}\{x^3\}$$

$$U(S)+\mathcal{L}\int_0^x (x-t)\,U(t)\,dt=1/S^4$$

By convolution theorem gives

$$U(S)+\mathcal{L}\{x*U(x)\}=1/S^4$$

$$U(S)+1/S^2\cdot U(S)=1/S^4$$

Taking common $U(S)$ at L.H.S

$$U(S)\left[1+\frac{1}{S^2}\right]=\frac{1}{S^4}$$

$$U(S)\left[\frac{S^2+1}{S^2}\right]=1/S^4.$$

$$U(S) = \frac{1}{S^4} \cdot \frac{S^2}{(S^2+1)}$$

$$U(S) = \frac{1}{S^2(S^2+1)}$$

Now applying Laplace inverse, gives

$$\mathcal{L}^{-1}\{U(S)\} = \mathcal{L}^{-1}\left\{\frac{1}{S^2(S^2+1)}\right\}$$

$$U(x) = \mathcal{L}^{-1}\{1/S^2(S^2+1)\}$$

Now by partial fraction gives

$$\frac{1}{S^2(S^2+1)} = \frac{A}{S^2} + \frac{BS+C}{S^2+1}$$

$$1 = A(S^2+1) + (BS+C)(S^2)$$

$$1 = AS^2+A+BS^3+CS^2$$

Now by comparing coefficient

S^3 : $0 = B$

S^2 : $0 = A+C$

constant : $A = 1$

So $C = -1$

Thus.

$$U(x) = \mathcal{L}^{-1}\left\{\frac{1}{S^2} - \frac{1}{S^2+1}\right\}$$

$$U(x) = \mathcal{L}^{-1}\{1/S^2\} - \mathcal{L}^{-1}\{1/S^2+1\}$$

$$U(x) = x - \sin(x)$$

Q Solve volterra integral equation of Second Kind by Laplace transformation.

(Solution)

$$y(t) = 1+t+\int_0^t (t-u)\,y(u)\,du.$$

Solution:

$$y(t) - \int_0^t (t-u)\,y(u)\,du = 1+t$$

Apply Laplace transform, gives

$$\mathcal{L}\left\{y(t) - \int_0^t (t-u)\,y(u)\,du\right\} = \mathcal{L}\{1+t\}.$$

$$\mathcal{L}\{y(t)\} - \mathcal{L}\left\{\int_0^t (t-u)\,y(u)\,du\right\} = \mathcal{L}\{1\} + \mathcal{L}\{t\}$$

$$Y(S) - \mathcal{L}\left\{\int_0^t (t-u)\,y(u)\,du\right\} = 1/S + 1/S^2$$

By convolution theorem

$$Y(S) - \mathcal{L}\{t * y(t)\} = \frac{S+1}{S^2}$$

$$Y(S) - \mathcal{L}\{t\}\mathcal{L}\{y(t)\} = \frac{(S+1)}{S^2}$$

$$Y(S) - \frac{1}{S^2}Y(S) = \frac{(S+1)}{S^2}$$

Taking common $Y(S)$ at L.H.S

$$Y(S)\left[1 - \frac{1}{S^2}\right] = \frac{(S+1)}{S^2}$$

$$Y(S)\left[\frac{S^2-1}{S^2}\right] = \frac{(S+1)}{S^2}$$

$$\Rightarrow Y(S) = \frac{(S+1)}{\cancel{S^2}} \cdot \frac{\cancel{S^2}}{(S^2-1)}$$

$$Y(S) = \frac{\cancel{(S+1)}}{\cancel{(S+1)}(S-1)}$$

$$Y(S) = \frac{1}{(S-1)}$$

Taking Laplace inverse on both sides gives

$$\mathcal{L}^{-1}\{Y(S)\} = \mathcal{L}^{-1}\left\{\frac{1}{S-1}\right\}$$

$$y(t) = e^t$$

Scanned with CamScanner

Fredholm Integral equation:

We cannot apply Laplace transformation on Fredholm integral equation of first and second kind.

CONCLUSION:-

We have concluded from the above discussion that we have successfully used the Laplace transformation for the solution of first and second linear volterra integral equation. The given method or application show that the exact solution have been obtained using very less computational work and spending a very little time