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person answers each query truthfully, we can find x using $\log n$ queries by successively splitting the sets used in each query in half. Ulam's problem, proposed by Stanislaw Ulam in 1976, asks for the number of queries required to find x, supposing that the first person is allowed to lie exactly once.

a) Show that by asking each question twice, given a number x and a set with n elements, and asking one more question when we find the lie, Ulam's problem can be solved using $2 \log n + 1$ queries.

b) Show that by dividing the initial set of n elements into four parts, each with $n/4$ elements, 1/4 of the elements can be eliminated using two queries. [*Hint:* Use two queries, where each of the queries asks whether the element is in the union of two of the subsets with $n/4$ elements and where one of the subsets of $n/4$ elements is used in both queries.]

c) Show from part (b) that if $f(n)$ equals the number of queries used to solve Ulam's problem using the method from part (b) and n is divisible by 4, then $f(n) = f(3n/4) + 2$.

d) Solve the recurrence relation in part (c) for $f(n)$.

e) Is the naive way to solve Ulam's problem by asking each question twice or the divide-and-conquer method based on part (b) more efficient? The most efficient way to solve Ulam's problem has been determined by A. Pelc [Pe87].

In Exercises 29–33, assume that f is an increasing function satisfying the recurrence relation $f(n) = af(n/b) + cn^d$, where $a \geq 1$, b is an integer greater than 1, and c and d are positive real numbers. These exercises supply a proof of Theorem 2.

***29.** Show that if $a = b^d$ and n is a power of b, then $f(n) = f(1)n^d + cn^d \log_b n$.

30. Use Exercise 29 to show that if $a = b^d$, then $f(n)$ is $O(n^d \log n)$.

***31.** Show that if $a \neq b^d$ and n is a power of b, then $f(n) = C_1 n^d + C_2 n^{\log_b a}$, where $C_1 = b^d c/(b^d - a)$ and $C_2 = f(1) + b^d c/(a - b^d)$.

32. Use Exercise 31 to show that if $a < b^d$, then $f(n)$ is $O(n^d)$.

33. Use Exercise 31 to show that if $a > b^d$, then $f(n)$ is $O(n^{\log_b a})$.

34. Find $f(n)$ when $n = 4^k$, where f satisfies the recurrence relation $f(n) = 5f(n/4) + 6n$, with $f(1) = 1$.

35. Give a big-O estimate for the function f in Exercise 34 if f is an increasing function.

36. Find $f(n)$ when $n = 2^k$, where f satisfies the recurrence relation $f(n) = 8f(n/2) + n^2$ with $f(1) = 1$.

37. Give a big-O estimate for the function f in Exercise 36 if f is an increasing function.

8.4 Generating Functions

8.4.1 Introduction

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Generating functions are used to represent sequences efficiently by coding the terms of a sequence as coefficients of powers of a variable x in a formal power series. Generating functions can be used to solve many types of counting problems, such as the number of ways to select or distribute objects of different kinds, subject to a variety of constraints, and the number of ways to make change for a dollar using coins of different denominations. Generating functions can be used to solve recurrence relations by translating a recurrence relation for the terms of a sequence into an equation involving a generating function. This equation can then be solved to find a closed form for the generating function. From this closed form, the coefficients of the power series for the generating function can be found, solving the original recurrence relation. Generating functions can also be used to prove combinatorial identities by taking advantage of relatively simple relationships between functions that can be translated into identities involving the terms of sequences. Generating functions are a helpful tool for studying many properties of sequences besides those described in this section, such as their use for establishing asymptotic formulae for the terms of a sequence.

We begin with the definition of the generating function for a sequence.

Definition 1 The *generating function for the sequence* $a_0, a_1, \ldots, a_k, \ldots$ of real numbers is the infinite series

$$G(x) = a_0 + a_1 x + \cdots + a_k x^k + \cdots = \sum_{k=0}^{\infty} a_k x^k.$$

Remark: The generating function for $\{a_k\}$ given in Definition 1 is sometimes called the **ordinary generating function** of $\{a_k\}$ to distinguish it from other types of generating functions for this sequence.

EXAMPLE 1 The generating functions for the sequences $\{a_k\}$ with $a_k = 3, a_k = k + 1$, and $a_k = 2^k$ are $\sum_{k=0}^{\infty} 3x^k$, $\sum_{k=0}^{\infty}(k + 1)x^k$, and $\sum_{k=0}^{\infty} 2^k x^k$, respectively. ◀

Extra Examples

We can define generating functions for finite sequences of real numbers by extending a finite sequence $a_0, a_1, \ldots, a_n$ into an infinite sequence by setting $a_{n+1} = 0, a_{n+2} = 0$, and so on. The generating function $G(x)$ of this infinite sequence $\{a_n\}$ is a polynomial of degree n because no terms of the form $a_j x^j$ with $j > n$ occur, that is,

$$G(x) = a_0 + a_1 x + \cdots + a_n x^n.$$

EXAMPLE 2 What is the generating function for the sequence 1, 1, 1, 1, 1, 1?

Solution: The generating function of 1, 1, 1, 1, 1, 1 is

$$1 + x + x^2 + x^3 + x^4 + x^5.$$

By Theorem 1 of Section 2.4 we have

$$(x^6 - 1)/(x - 1) = 1 + x + x^2 + x^3 + x^4 + x^5$$

when $x \neq 1$. Consequently, $G(x) = (x^6 - 1)/(x - 1)$ is the generating function of the sequence 1, 1, 1, 1, 1, 1. [Because the powers of x are only place holders for the terms of the sequence in a generating function, we do not need to worry that $G(1)$ is undefined.] ◀

EXAMPLE 3 Let m be a positive integer. Let $a_k = C(m, k)$, for $k = 0, 1, 2, \ldots, m$. What is the generating function for the sequence $a_0, a_1, \ldots, a_m$?

Solution: The generating function for this sequence is

$$G(x) = C(m, 0) + C(m, 1)x + C(m, 2)x^2 + \cdots + C(m, m)x^m.$$

The binomial theorem shows that $G(x) = (1 + x)^m$. ◀

8.4.2 Useful Facts About Power Series

When generating functions are used to solve counting problems, they are usually considered to be **formal power series**. As such, they are treated as algebraic objects; questions about their convergence are ignored. However, when formal power series are convergent, valid operations carry over to their use as formal power series. We will take advantage of the power series of particular functions around $x = 0$. These power series are unique and have a positive radius of convergence. Readers familiar with calculus can consult textbooks on this subject for details about power series, including the convergence of the series we use here.

We will now state some widely important facts about infinite series used when working with generating functions. These facts can be found in calculus texts.

EXAMPLE 4 The function $f(x) = 1/(1 - x)$ is the generating function of the sequence $1, 1, 1, 1, \ldots$, because

$$1/(1 - x) = 1 + x + x^2 + \cdots$$

for $|x| < 1$. ◀

EXAMPLE 5 The function $f(x) = 1/(1 - ax)$ is the generating function of the sequence $1, a, a^2, a^3, \ldots$, because

$$1/(1 - ax) = 1 + ax + a^2x^2 + \cdots$$

when $|ax| < 1$, or equivalently, for $|x| < 1/|a|$ for $a \neq 0$. ◀

We also will need some results on how to add and how to multiply two generating functions. Proofs of these results can be found in calculus texts.

THEOREM 1 Let $f(x) = \sum_{k=0}^{\infty} a_k x^k$ and $g(x) = \sum_{k=0}^{\infty} b_k x^k$. Then

$$f(x) + g(x) = \sum_{k=0}^{\infty} (a_k + b_k)x^k \quad \text{and} \quad f(x)g(x) = \sum_{k=0}^{\infty} \left(\sum_{j=0}^{k} a_j b_{k-j} \right) x^k.$$

Remark: Theorem 1 is valid only for power series that converge in an interval, as all series considered in this section do. However, the theory of generating functions is not limited to such series. In the case of series that do not converge, the statements in Theorem 1 can be taken as definitions of addition and multiplication of generating functions.

We will illustrate how Theorem 1 can be used with Example 6.

EXAMPLE 6 Let $f(x) = 1/(1 - x)^2$. Use Example 4 to find the coefficients $a_0, a_1, a_2, \ldots$ in the expansion $f(x) = \sum_{k=0}^{\infty} a_k x^k$.

Solution: From Example 4 we see that

$$1/(1 - x) = 1 + x + x^2 + x^3 + \cdots.$$

Hence, from Theorem 1, we have

$$1/(1 - x)^2 = \sum_{k=0}^{\infty} \left(\sum_{j=0}^{k} 1 \right) x^k = \sum_{k=0}^{\infty} (k + 1)x^k.$$

◀

Remark: This result also can be derived from Example 4 by differentiation. Taking derivatives is a useful technique for producing new identities from existing identities for generating functions.

To use generating functions to solve many important counting problems, we will need to apply the binomial theorem for exponents that are not positive integers. Before we state an extended version of the binomial theorem, we need to define extended binomial coefficients.

Definition 2 Let u be a real number and k a nonnegative integer. Then the *extended binomial coefficient* $\binom{u}{k}$ is defined by

$$\binom{u}{k} = \begin{cases} u(u-1)\cdots(u-k+1)/k! & \text{if } k > 0, \\ 1 & \text{if } k = 0. \end{cases}$$

EXAMPLE 7 Find the values of the extended binomial coefficients $\binom{-2}{3}$ and $\binom{1/2}{3}$.

Solution: Taking $u = -2$ and $k = 3$ in Definition 2 gives us

$$\binom{-2}{3} = \frac{(-2)(-3)(-4)}{3!} = -4.$$

Similarly, taking $u = 1/2$ and $k = 3$ gives us

$$\begin{aligned}\binom{1/2}{3} &= \frac{(1/2)(1/2-1)(1/2-2)}{3!} \\ &= (1/2)(-1/2)(-3/2)/6 \\ &= 1/16.\end{aligned}$$

◀

Example 8 provides a useful formula for extended binomial coefficients when the top parameter is a negative integer. It will be useful in our subsequent discussions.

EXAMPLE 8 When the top parameter is a negative integer, the extended binomial coefficient can be expressed in terms of an ordinary binomial coefficient. To see that this is the case, note that

$$\begin{aligned}\binom{-n}{r} &= \frac{(-n)(-n-1)\cdots(-n-r+1)}{r!} && \text{by definition of extended binomial coefficient} \\ &= \frac{(-1)^r n(n+1)\cdots(n+r-1)}{r!} && \text{factoring out } -1 \text{ from each term in the numerator} \\ &= \frac{(-1)^r (n+r-1)(n+r-2)\cdots n}{r!} && \text{by the commutative law for multiplication} \\ &= \frac{(-1)^r (n+r-1)!}{r!(n-1)!} && \text{multiplying both the numerator and denominator by } (n-1)! \\ &= (-1)^r \binom{n+r-1}{r} && \text{by the definition of binomial coefficients} \\ &= (-1)^r C(n+r-1, r) && \text{using alternative notation for binomial coefficients.}\end{aligned}$$

◀

We now state the extended binomial theorem.

THEOREM 2 **THE EXTENDED BINOMIAL THEOREM** Let x be a real number with $|x| < 1$ and let u be a real number. Then

$$(1+x)^u = \sum_{k=0}^{\infty} \binom{u}{k} x^k.$$

Theorem 2 can be proved using the theory of Maclaurin series. We leave its proof to the reader with a familiarity with this part of calculus.

Remark: When u is a positive integer, the extended binomial theorem reduces to the binomial theorem presented in Section 6.4, because in that case $\binom{u}{k} = 0$ if $k > u$.

Example 9 illustrates the use of Theorem 2 when the exponent is a negative integer.

EXAMPLE 9 Find the generating functions for $(1+x)^{-n}$ and $(1-x)^{-n}$, where n is a positive integer, using the extended binomial theorem.

Solution: By the extended binomial theorem, it follows that

$$(1+x)^{-n} = \sum_{k=0}^{\infty} \binom{-n}{k} x^k.$$

Using Example 8, which provides a simple formula for $\binom{-n}{k}$, we obtain

$$(1+x)^{-n} = \sum_{k=0}^{\infty} (-1)^k C(n+k-1, k) x^k.$$

Replacing x by $-x$, we find that

$$(1-x)^{-n} = \sum_{k=0}^{\infty} C(n+k-1, k) x^k.$$

◀

Table 1 presents a useful summary of some generating functions that arise frequently.

Remark: Note that the second and third formulae in this table can be deduced from the first formula by substituting ax and x^r for x, respectively. Similarly, the sixth and seventh formulae can be deduced from the fifth formula using the same substitutions. The tenth and eleventh can be deduced from the ninth formula by substituting $-x$ and ax for x, respectively. Also, some of the formulae in this table can be derived from other formulae using methods from calculus (such as differentiation and integration). Students are encouraged to know the core formulae in this table (that is, formulae from which the others can be derived, perhaps the first, fourth, fifth, eighth, ninth, twelfth, and thirteenth formulae) and understand how to derive the other formulae from these core formulae.