1 Generating Function for $P_n(x)$:

The Legendre polynomial $P_n(x)$ is the co-efficient of z^n in the expansion of $(1-2xz+z^2)^{-1/2}$ in ascending powers of "x"

i.e:

$$\sum_{n=0}^{\infty} P_n(x)\, z^n = (1-2xz+z^2)^{-1/2} \quad \text{--- Eq 1}$$

Let, us consider only the R.H.S & expanding it.

R.H.S =

$$(1-2xz+z^2)^{-1/2} = (1-(2xz-z^2))^{-1/2}$$

Let, $2xz - z^2 = h$

Then

$$R.H.S = (1-2xz+z^2)^{-1/2}$$

$$R.H.S = (1-h)^{-1/2} \quad \text{--- Eq 2}$$

By using binomial expansion.

$$(1-h)^{-1/2} = 1 + \left(\frac{-1}{2}\right)(-h) + \left(\frac{-1}{2}\right)\left(\frac{-3}{2}\right)\frac{h^2}{2!} + \left(\frac{-1}{2}\right)\left(\frac{-3}{2}\right)\left(\frac{-5}{2}\right)\frac{(-h)^3}{3!}$$

$$+ \cdots + \left(\frac{-1}{2}\right)\left(\frac{-3}{2}\right)\left(\frac{-5}{2}\right)\cdots\left(\frac{-(2n-1)}{2}\right)\frac{(-h)^n}{n!} + \cdots$$

In summation form

$$(1-h)^{-1/2} = \sum_{n=0}^{\infty} (+1/2)(+3/2)(+5/2)\cdots\left(\frac{+(2n-1)}{2}\right)\frac{h^n}{n!}$$

$$= \sum_{n=0}^{\infty} \cancel{(-1)} \frac{1\cdot 3\cdot 5\cdots(2n-1)}{2^n\, n!} h^n$$

$$(1-h)^{-1/2} = \sum_{n=0}^{\infty} \cancel{(-1)} \frac{1\cdot 3\, 5\cdots(2n-1)}{2^n\, n!} (2xz-z^2)^n \quad \text{--- Eq 3}$$

$$(2xz-z^2)^n = z^n(2x-z)^n$$

By using binomial theorem,

$$z^n(2x-z)^n = z^n\left[\binom{n}{0}(2x)^n(-z)^0 + \binom{n}{1}(2x)^{n-1}(-z)^1 + \binom{n}{2}(2x)^{n-2}(-z)^2 + \cdots + \binom{n}{k}(2x)^{n-k}(-z)^k + \cdots\right]$$

$$z^n(2x-z)^n = z^n\left[\sum_{k=0}^{\infty}\binom{n}{k}(2x)^{n-k}(-z)^k\right]$$

So, Eq 3 become,

$$(1-2xz+z^2)^{-1/2} = \sum_{n=0}^{\infty}\cancel{(-1)}\left(\frac{1\cdot3\cdot5\cdots(2n-1)}{2^n n!}\right)\left[z^n\sum_{k=0}^{\infty}\frac{n!(2x)^{n-k}(-z)^k}{k!(n-k)!}\right]$$

$$=\sum_{n=0}^{\infty}\sum_{k=0}^{\infty}(-1)^k\left(\frac{1\cdot3\cdot5\cdots(2n-1)}{2^n\cancel{n!}}\right)\left(\frac{\cancel{n!}(2x)^{n-k}z^{n+k}}{k!(n-k)!}\right)$$

$$=\sum_{n=0}^{\infty}\sum_{k=0}^{\infty}(-1)^k\left(\frac{1\cdot3\cdot5\cdots(2n-1)}{2^k k!(n-k)!}\right)\left(x^{n-k}z^{n+k}\right)$$

(×)ing & (÷)ing first term by $2\cdot4\cdot6\cdots2n = 2^n n!$

We get,

$$(1-2xz+z^2)^{-1/2} = \sum_{n=0}^{\infty}\sum_{k=0}^{\infty}(-1)^k\left(\frac{1\cdot2\cdot3\cdot4\cdots(2n-1)2n}{2^{n+k}n!k!}\right)\frac{x^{n-k}z^{n+k}}{(n-k)!}$$

$$(1-2xz+z^2)^{-1/2} = \sum_{n=0}^{\infty}\left[\sum_{k=0}^{\infty}(-1)^k\left(\frac{(2n)!}{2^{n+k}}\frac{x^{n-k}}{n!k!(n-k)!}\right)\right]z^{n+k}$$

Replacing n by m

$$(1-2xz+z^2)^{-1/2} = \sum_{m=0}^{\infty}\left[\sum_{k=0}^{\infty}\frac{(-1)^k(2m)!\,x^{m-k}}{2^{m+k}m!k!(m-k)!}\right]z^{m+k}$$

where m, n, K are dummy constants.
Now, replacing, $m+K$ by n

i.e. $m = n-K$

$$(1-2xz+z^2)^{-1/2} = \sum_{n=0}^{\infty}\left[\sum_{K=0}^{\infty} \frac{(-1)^K\, 2(n-K)!\; x^{(n-K)-K}}{2^n\,(n-K)!\,K!\,(n-K-K)!}\right] z^n$$

Since,

$$= \sum_{n=0}^{\infty}\left[\sum_{k=0}^{N} (-1)^k \frac{(2n-2k)!}{2^n\, k!\,(n-k)!\,(n-2k)!}\, x^{n-2k}\right] z^n$$

$$\because \sum_{k=0}^{N} (-1)^k \frac{(2n-2k)!}{2^n\, k!\,(n-k)!\,(n-2k)!}\, x^{n-2k} = P_n(x)$$

So,

$$L.H.S = \sum_{n=0}^{\infty} P_n(x)\, z^n = R.H.S$$

Hence, by expanding the function "$(1-2xz+z^2)^{-1/2}$" in the binomial expansion in ascending power of "x", we obtain the Legendre polynomial different orders as the co-efficients of corresponding powers of "z".

Proofs:

(i) $P_n(-1) = (-1)^n$

By Using Generating Function.

$$(1-2xz+z^2)^{-1/2} = \sum_{n=0}^{\infty} P_n(x)\, z^n$$

Putting $x = -1$

$$(1+2z+z^2)^{-1/2} = \sum_{n=0}^{\infty} P_n(-1)\, z^n$$