There will now be n characteristic roots b_i $(i = 1, 2, \ldots, n)$, all of which should enter into the complementary function thus:

$$y_c = \sum_{i=1}^{n} A_i b_i^t \tag{18.39}$$

provided, of course, that the roots are all real and distinct. In case there are repeated real roots (say, $b_1 = b_2 = b_3$), then the first three terms in the sum in (18.39) must be modified to

$$A_1 b_1^t + A_2 t b_1^t + A_3 t^2 b_1^t \qquad \text{[cf. (18.6)]}$$

Moreover, if there is a pair of conjugate complex roots—say, b_{n-1}, b_n—then the last two terms in the sum in (18.39) are to be combined into the expression

$$R^t(A_{n-1} \cos \theta t + A_n \sin \theta t)$$

A similar expression can also be assigned to any other pair of complex roots. In case of two *repeated* pairs, however, one of the two must be given a multiplicative factor of tR^t instead of R^t.

After y_p and y_c are both found, the general solution of the complete difference equation (18.37) is again obtained by summing; that is,

$$y_t = y_p + y_c$$

But since there will be a total of n arbitrary constants in this solution, no less than n initial conditions will be required to definitize it.

Example 3

Find the general solution of the third-order difference equation

$$y_{t+3} - \frac{7}{8} y_{t+2} + \frac{1}{8} y_{t+1} + \frac{1}{32} y_t = 9$$

By trying the solution $y_t = k$, the particular solution is easily found to be $y_p = 32$. As for the complementary function, since the cubic characteristic equation

$$b^3 - \frac{7}{8} b^2 + \frac{1}{8} b + \frac{1}{32} = 0$$

can be factored into the form

$$\left(b - \frac{1}{2}\right)\left(b - \frac{1}{2}\right)\left(b + \frac{1}{8}\right) = 0$$

the roots are $b_1 = b_2 = \frac{1}{2}$ and $b_3 = -\frac{1}{8}$. This enables us to write

$$y_c = A_1 \left(\frac{1}{2}\right)^t + A_2 t \left(\frac{1}{2}\right)^t + A_3 \left(-\frac{1}{8}\right)^t$$

Note that the second term contains a multiplicative t; this is due to the presence of repeated roots. The general solution of the given difference equation is then simply the sum of y_c and y_p.

In this example, all three characteristic roots happen to be less than 1 in their absolute values. We can therefore conclude that the solution obtained represents a time path which converges to the stationary equilibrium level 32.

Convergence and the Schur Theorem

When we have a high-order difference equation that is not easily solved, we can nonetheless determine the convergence of the relevant time path qualitatively without having to struggle with its actual quantitative solution. You will recall that the time path can converge if and only if every root of the characteristic equation is less than 1 in absolute value.

In view of this, the following theorem—known as the *Schur theorem*†—becomes directly applicable:

> The roots of the nth-degree polynomial equation
>
> $$a_0 b^n + a_1 b^{n-1} + \cdots + a_{n-1} b + a_n = 0$$
>
> will all be less than unity in absolute value if and only if the following n determinants
>
> $$\Delta_1 = \left|\begin{array}{c|c} a_0 & a_n \\ \hline a_n & a_0 \end{array}\right| \qquad \Delta_2 = \left|\begin{array}{cc|cc} a_0 & 0 & a_n & a_{n-1} \\ a_1 & a_0 & 0 & a_n \\ \hline a_n & 0 & a_0 & a_1 \\ a_{n-1} & a_n & 0 & a_0 \end{array}\right| \qquad \cdots$$
>
> $$\Delta_n = \left|\begin{array}{cccc|cccc} a_0 & 0 & \cdots & 0 & a_n & a_{n-1} & \cdots & a_1 \\ a_1 & a_0 & \cdots & 0 & 0 & a_n & \cdots & a_2 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ a_{n-1} & a_{n-2} & \cdots & a_0 & 0 & 0 & \cdots & a_n \\ \hline a_n & 0 & \cdots & 0 & a_0 & a_1 & \cdots & a_{n-1} \\ a_{n-1} & a_n & \cdots & 0 & 0 & a_0 & \cdots & a_{n-2} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ a_1 & a_2 & \cdots & a_n & 0 & 0 & \cdots & a_0 \end{array}\right|$$
>
> are all positive.

Note that, since the condition in the theorem is given on the "if and only if" basis, it is a necessary-and-sufficient condition. Thus the Schur theorem is a perfect difference-equation counterpart of the Routh theorem introduced earlier in the differential-equation framework.

The construction of these determinants is based on a simple procedure. This is best explained with the aid of the dashed lines which partition each determinant into four *areas*. Each area of the kth determinant, Δ_k, always consists of a $k \times k$ subdeterminant. The *upper-left* area has a_0 alone in the diagonal, zeros above the diagonal, and progressively larger subscripts for the successive coefficients in each column below the diagonal elements. When we transpose the elements of the upper-left area, we obtain the *lower-right* area. Turning to the *upper-right* area, we now place the a_n coefficient alone in the diagonal, with zeros below the diagonal, and progressively smaller subscripts for the successive coefficients as we go up each column from the diagonal. When the elements of this area are transposed, we get the *lower-left* area.

The application of this theorem is straightforward. Since the coefficients of the characteristic equation are the same as those appearing on the left side of the original difference equation, we can introduce them directly into the determinants cited. Note that, in our context, we always have $a_0 = 1$.

Example 4

Does the time path of the equation $y_{t+2} + 3y_{t+1} + 2y_t = 12$ converge? Here we have $n = 2$, and the coefficients are $a_0 = 1$, $a_1 = 3$, and $a_2 = 2$. Thus we get

$$\Delta_1 = \begin{vmatrix} a_0 & a_2 \\ a_2 & a_0 \end{vmatrix} = \begin{vmatrix} 1 & 2 \\ 2 & 1 \end{vmatrix} = -3 < 0$$

† For a discussion of this theorem and its history, see John S. Chipman, *The Theory of Inter-Sectoral Money Flows and Income Formation*, The Johns Hopkins Press, Baltimore, 1951, pp. 119–120.

Since this already violates the convergence condition, there is no need to proceed to Δ_2.

Actually, the characteristic roots of the given difference equation are easily found to be $b_1, b_2 = -1, -2$, which indeed imply a divergent time path.

Example 5 Test the convergence of the path of $y_{t+2} + \frac{1}{6}y_{t+1} - \frac{1}{6}y_t = 2$ by the Schur theorem. Here the coefficients are $a_0 = 1$, $a_1 = \frac{1}{6}$, $a_2 = -\frac{1}{6}$ (with $n = 2$). Thus we have

$$\Delta_1 = \begin{vmatrix} a_0 & a_2 \\ a_2 & a_0 \end{vmatrix} = \begin{vmatrix} 1 & -\frac{1}{6} \\ -\frac{1}{6} & 1 \end{vmatrix} = \frac{35}{36} > 0$$

$$\Delta_2 = \begin{vmatrix} a_0 & 0 & a_2 & a_1 \\ a_1 & a_0 & 0 & a_2 \\ a_2 & 0 & a_0 & a_1 \\ a_1 & a_2 & 0 & a_0 \end{vmatrix} = \begin{vmatrix} 1 & 0 & -\frac{1}{6} & \frac{1}{6} \\ \frac{1}{6} & 1 & 0 & -\frac{1}{6} \\ -\frac{1}{6} & 0 & 1 & \frac{1}{6} \\ \frac{1}{6} & -\frac{1}{6} & 0 & 1 \end{vmatrix} = \frac{1{,}176}{1{,}296} > 0$$

These do satisfy the necessary-and-sufficient condition for convergence.

EXERCISE 18.4

1. Apply the definition of the "differencing" symbol Δ, to find:
   (*a*) Δt (*b*) $\Delta^2 t$ (*c*) Δt^3
   Compare the results of differencing with those of differentiation.
2. Find the particular solution of each of the following:
   (*a*) $y_{t+2} + 2y_{t+1} + y_t = 3^t$
   (*b*) $y_{t+2} - 5y_{t+1} - 6y_t = 2(6)^t$
   (*c*) $3y_{t+2} + 9y_t = 3(4)^t$
3. Find the particular solutions of:
   (*a*) $y_{t+2} - 2y_{t+1} + 5y_t = t$
   (*b*) $y_{t+2} - 2y_{t+1} + 5y_t = 4 + 2t$
   (*c*) $y_{t+2} + 5y_{t+1} + 2y_t = 18 + 6t + 8t^2$
4. Would you expect that, when the variable term takes the form $m^t + t^n$, the trial solution should be $B(m)^t + (B_0 + B_1 t + \cdots + B_n t^n)$? Why?
5. Find the characteristic roots and the complementary function of:
   (*a*) $y_{t+3} - \frac{1}{2}y_{t+2} - y_{t+1} + \frac{1}{2}y_t = 0$
   (*b*) $y_{t+3} - 2y_{t+2} + \frac{5}{4}y_{t+1} - \frac{1}{4}y_t = 1$
   [*Hint:* Try factoring out $(b - \frac{1}{4})$ in both characteristic equations.]
6. Test the convergence of the solutions of the following difference equations by the Schur theorem:
   (*a*) $y_{t+2} + \frac{1}{2}y_{t+1} - \frac{1}{2}y_t = 3$
   (*b*) $y_{t+2} - \frac{1}{9}y_t = 1$
7. In the case of a third-order difference equation
   $y_{t+3} + a_1 y_{t+2} + a_2 y_{t+1} + a_3 y_t = c$
   what are the exact forms of the determinants required by the Schur theorem?