

# Chapter 18

## Higher-Order Difference Equations

The economic models in Chap. 17 involve difference equations that relate  $P_t$  and  $P_{t-1}$  to each other. As the  $P$  value in one period can uniquely determine the  $P$  value in the next, the time path of  $P$  becomes fully determinate once an initial value  $P_0$  is specified. It may happen, however, that the value of an economic variable in period  $t$  (say,  $y_t$ ) depends not only on  $y_{t-1}$  but also on  $y_{t-2}$ . Such a situation will give rise to a difference equation of the second order.

Strictly speaking, a *second-order difference equation* is one that involves an expression  $\Delta^2 y_t$ , called the *second difference* of  $y_t$ , but contains no differences of order higher than 2. The symbol  $\Delta^2$ , the discrete-time counterpart of the symbol  $d^2/dt^2$ , is an instruction to “take the second difference” as follows:

$$\begin{aligned}\Delta^2 y_t &= \Delta(\Delta y_t) = \Delta(y_{t+1} - y_t) && \text{[by (17.1)]} \\ &= (y_{t+2} - y_{t+1}) - (y_{t+1} - y_t) && \text{[again by (17.1)]}^\dagger \\ &= y_{t+2} - 2y_{t+1} + y_t\end{aligned}$$

Thus a second difference of  $y_t$  is transformable into a sum of terms involving a two-period time lag. Since expressions like  $\Delta^2 y_t$  and  $\Delta y_t$  are quite cumbersome to work with, we shall simply redefine a second-order difference equation as one involving a two-period time lag in the variable. Similarly, a third-order difference equation is one that involves a three-period time lag, etc.

Let us first concentrate on the method of solving a second-order difference equation, leaving the generalization to higher-order equations in Section 18.4. To keep the scope of discussion manageable, we shall only deal with linear difference equations with constant coefficients in the present chapter. However, both the constant-term and variable-term varieties will be examined.

<sup>†</sup> That is, we first move the subscripts in the  $(y_{t+1} - y_t)$  expression forward by one period, to get a new expression  $(y_{t+2} - y_{t+1})$ , and then we subtract from the latter the original expression. Note that, since the resulting difference may be written as  $\Delta y_{t+1} - \Delta y_t$ , we may infer the following rule of operation:

$$\Delta(y_{t+1} - y_t) = \Delta y_{t+1} - \Delta y_t$$

## 18.1 Second-Order Linear Difference Equations with Constant Coefficients and Constant Term

A simple variety of second-order difference equations takes the form

$$y_{t+2} + a_1 y_{t+1} + a_2 y_t = c \quad (18.1)$$

You will recognize this equation to be linear, nonhomogeneous, and with constant coefficients ( $a_1, a_2$ ) and constant term  $c$ .

### Particular Solution

As before, the solution of (18.1) may be expected to have two components: a particular solution  $y_p$  representing the intertemporal equilibrium level of  $y$ , and a complementary function  $y_c$  specifying, for every time period, the deviation from the equilibrium. The particular solution, defined as any solution of the complete equation, can sometimes be found simply by trying a solution of the form  $y_t = k$ . Substituting this constant value of  $y$  into (18.1), we obtain

$$k + a_1 k + a_2 k = c \quad \text{and} \quad k = \frac{c}{1 + a_1 + a_2}$$

Thus, so long as  $(1 + a_1 + a_2) \neq 0$ , the particular integral is

$$y_p (= k) = \frac{c}{1 + a_1 + a_2} \quad (\text{case of } a_1 + a_2 \neq -1) \quad (18.2)$$

### Example 1

Find the particular integral of  $y_{t+2} - 3y_{t+1} + 4y_t = 6$ . Here we have  $a_1 = -3$ ,  $a_2 = 4$ , and  $c = 6$ . Since  $a_1 + a_2 \neq -1$ , the particular solution can be obtained from (18.2) as follows:

$$y_p = \frac{6}{1 - 3 + 4} = 3$$

In case  $a_1 + a_2 = -1$ , then the trial solution  $y_t = k$  breaks down, and we can try  $y_t = kt$  instead. Substituting the latter into (18.1) and bearing in mind that we now have  $y_{t+1} = k(t+1)$  and  $y_{t+2} = k(t+2)$ , we find that

$$k(t+2) + a_1 k(t+1) + a_2 kt = c$$

$$\text{and} \quad k = \frac{c}{(1 + a_1 + a_2)t + a_1 + 2} = \frac{c}{a_1 + 2} \quad [\text{since } a_1 + a_2 = -1]$$

Thus we can write the particular solution as

$$y_p (= kt) = \frac{c}{a_1 + 2} t \quad (\text{case of } a_1 + a_2 = -1; a_1 \neq -2) \quad (18.2')$$

### Example 2

Find the particular solution of  $y_{t+2} + y_{t+1} - 2y_t = 12$ . Here,  $a_1 = 1$ ,  $a_2 = -2$ , and  $c = 12$ . Obviously, formula (18.2) is not applicable, but (18.2') is. Thus,

$$y_p = \frac{12}{1 + 2} t = 4t$$

This particular solution represents a moving equilibrium.

If  $a_1 + a_2 = -1$ , but at the same time  $a_1 = -2$  (that is, if  $a_1 = -2$  and  $a_2 = 1$ ), then we can adopt a trial solution of the form  $y_t = kt^2$ , which implies  $y_{t-1} = k(t+1)^2$ , etc. As you may verify, in this case the particular solution turns out to be

$$y_p = kt^2 = \frac{c}{2}t^2 \quad (\text{case of } a_1 = -2; a_2 = 1) \quad (18.2'')$$

However, since this formula applies only to the unique case of the difference equation  $y_{t+2} - 2y_{t+1} + y_t = c$ , its usefulness is rather limited.

## Complementary Function

To find the complementary function, we must concentrate on the reduced equation

$$y_{t+2} + a_1y_{t+1} + a_2y_t = 0 \quad (18.3)$$

Our experience with first-order difference equations has taught us that the expression  $Ab^t$  plays a prominent role in the general solution of such an equation. Let us therefore try a solution of the form  $y_t = Ab^t$ , which naturally implies that  $y_{t+1} = Ab^{t+1}$ , and so on. It is our task now to determine the values of  $A$  and  $b$ .

Upon substitution of the trial solution into (18.3), the equation becomes

$$Ab^{t+2} + a_1Ab^{t+1} + a_2Ab^t = 0$$

or, after canceling the (nonzero) common factor  $Ab^t$ ,

$$b^2 + a_1b + a_2 = 0 \quad (18.3')$$

This quadratic equation—the *characteristic equation* of (18.3) or of (18.1)—which is comparable to (16.4''), possesses the two *characteristic roots*

$$b_1, b_2 = \frac{-a_1 \pm \sqrt{a_1^2 - 4a_2}}{2} \quad (18.4)$$

each of which is acceptable in the solution  $Ab^t$ . In fact, both  $b_1$  and  $b_2$  should appear in the general solution of the homogeneous difference equation (18.3) because, just as in the case of differential equations, this general solution must consist of two *linearly independent* parts, each with its own multiplicative arbitrary constant.

Three possible situations may be encountered in regard to the characteristic roots, depending on the square-root expression in (18.4). You will find these parallel very closely the analysis of second-order differential equations in Sec. 16.1.

**Case 1 (distinct real roots)** When  $a_1^2 > 4a_2$ , the square root in (18.4) is a real number, and  $b_1$  and  $b_2$  are real and distinct. In that event,  $b_1^t$  and  $b_2^t$  are linearly independent, and the complementary function can simply be written as a linear combination of these expressions; that is,

$$y_c = A_1b_1^t + A_2b_2^t \quad (18.5)$$

You should compare this with (16.7).

### Example 3

Find the solution of  $y_{t+2} + y_{t+1} - 2y_t = 12$ . This equation has the coefficients  $a_1 = 1$  and  $a_2 = -2$ ; from (18.4), the characteristic roots can be found to be  $b_1, b_2 = 1, -2$ . Thus, the complementary function is

$$y_c = A_1(1)^t + A_2(-2)^t = A_1 + A_2(-2)^t$$

Since, in Example 2, the particular solution of the given difference equation has already been found to be  $y_p = 4t$ , we can write the general solution as

$$y_t = y_c + y_p = A_1 + A_2(-2)^t + 4t$$

There are still two arbitrary constants  $A_1$  and  $A_2$  to be definitized; to accomplish this, two initial conditions are necessary. Suppose that we are given  $y_0 = 4$  and  $y_1 = 5$ . Then, since by letting  $t = 0$  and  $t = 1$  successively in the general solution we find

$$\begin{aligned} y_0 &= A_1 + A_2 && (= 4 \text{ by the first initial condition}) \\ y_1 &= A_1 - 2A_2 + 4 && (= 5 \text{ by the second initial condition}) \end{aligned}$$

the arbitrary constants can be definitized to  $A_1 = 3$  and  $A_2 = 1$ . The definite solution then can finally be written as

$$y_t = 3 + (-2)^t + 4t$$

**Case 2 (repeated real roots)** When  $a_1^2 = 4a_2$ , the square root in (18.4) vanishes, and the characteristic roots are repeated:

$$b (= b_1 = b_2) = -\frac{a_1}{2}$$

Now, if we express the complementary function in the form of (18.5), the two components will collapse into a single term:

$$A_1 b_1^t + A_2 b_2^t = (A_1 + A_2) b^t \equiv A_3 b^t$$

This will not do, because we are now short of one constant.

To supply the missing component—which, we recall, should be linearly independent of the term  $A_3 b^t$ —the old trick of multiplying  $b^t$  by the variable  $t$  will again work. The new component term is therefore to take the form  $A_4 t b^t$ . That this is linearly independent of  $A_3 b^t$  should be obvious, for we can never obtain the expression  $A_4 t b^t$  by attaching a constant coefficient to  $A_3 b^t$ . That  $A_4 t b^t$  does indeed qualify as a solution of the homogeneous equation (18.3), just as  $A_3 b^t$  does, can easily be verified by substituting  $y_t = A_4 t b^t$  [and  $y_{t+1} = A_4(t+1)b^{t+1}$ , etc.] into (18.3)<sup>†</sup> and seeing that the latter will reduce to an identity  $0 = 0$ .

The complementary function for the repeated-root case is therefore

$$y_c = A_3 b^t + A_4 t b^t \quad (18.6)$$

which you should compare with (16.9).

#### Example 4

Find the complementary function of  $y_{t-2} + 6y_{t+1} + 9y_t = 4$ . The coefficients being  $a_1 = 6$  and  $a_2 = 9$ , the characteristic roots are found to be  $b_1 = b_2 = -3$ . We therefore have

$$y_c = A_3(-3)^t + A_4 t(-3)^t$$

If we proceed a step further, we can easily find  $y_p = \frac{1}{4}$ , so the general solution of the given difference equation is

$$y_t = A_3(-3)^t + A_4 t(-3)^t + \frac{1}{4}$$

Given two initial conditions,  $A_3$  and  $A_4$  can again be assigned definite values.

<sup>†</sup> In this substitution it should be kept in mind that we have in the present case  $a_1^2 = 4a_2$  and  $b = -a_1/2$ .

**Case 3 (complex roots)** Under the remaining possibility of  $a_1^2 < 4a_2$ , the characteristic roots are conjugate complex. Specifically, they will be in the form

$$b_1, b_2 = h \pm vi$$

where

$$h = -\frac{a_1}{2} \quad \text{and} \quad v = \frac{\sqrt{4a_2 - a_1^2}}{2} \quad (18.7)$$

The complementary function itself thus becomes

$$y_c = A_1 b_1^t + A_2 b_2^t = A_1 (h + vi)^t + A_2 (h - vi)^t$$

As it stands,  $y_c$  is not easily interpreted. But fortunately, thanks to De Moivre's theorem, given in (16.23'), this complementary function can easily be transformed into trigonometric terms, which we have learned to interpret.

According to the said theorem, we can write

$$(h \pm vi)^t = R^t (\cos \theta t \pm i \sin \theta t)$$

where the value of  $R$  (always taken to be positive) is, by (16.10),

$$R = \sqrt{h^2 + v^2} = \sqrt{\frac{a_1^2 + 4a_2 - a_1^2}{4}} = \sqrt{a_2} \quad (18.8)$$

and  $\theta$  is the radian measure of the angle in the interval  $[0, 2\pi)$ , which satisfies the conditions

$$\cos \theta = \frac{h}{R} = \frac{-a_1}{2\sqrt{a_2}} \quad \text{and} \quad \sin \theta = \frac{v}{R} = \sqrt{1 - \frac{a_1^2}{4a_2}} \quad (18.9)$$

Therefore, the complementary function can be transformed as follows:

$$\begin{aligned} y_c &= A_1 R^t (\cos \theta t + i \sin \theta t) + A_2 R^t (\cos \theta t - i \sin \theta t) \\ &= R^t [(A_1 + A_2) \cos \theta t + (A_1 - A_2) i \sin \theta t] \\ &= R^t (A_5 \cos \theta t + A_6 \sin \theta t) \end{aligned} \quad (18.10)$$

where we have adopted the shorthand symbols

$$A_5 \equiv A_1 + A_2 \quad \text{and} \quad A_6 \equiv (A_1 - A_2)i$$

The complementary function (18.10) differs from its differential-equation counterpart (16.24') in two important respects. First, the expressions  $\cos \theta t$  and  $\sin \theta t$  have replaced the previously used  $\cos vt$  and  $\sin vt$ . Second, the multiplicative factor  $R^t$  (an exponential with base  $R$ ) has replaced the natural exponential expression  $e^{ht}$ . In short, we have switched from the Cartesian coordinates ( $h$  and  $v$ ) of the complex roots to their polar coordinates ( $R$  and  $\theta$ ). The values of  $R$  and  $\theta$  can be determined from (18.8) and (18.9) once  $h$  and  $v$  become known. It is also possible to calculate  $R$  and  $\theta$  directly from the parameter values  $a_1$  and  $a_2$  via (18.8) and (18.9), provided we first make certain that  $a_1^2 < 4a_2$  and that the roots are indeed complex.

**Example 5**

Find the general solution of  $y_{t+2} + \frac{1}{4}y_t = 5$ . With coefficients  $a_1 = 0$  and  $a_2 = \frac{1}{4}$ , this constitutes an illustration of the complex-root case of  $a_1^2 < 4a_2$ . By (18.7), the real and imaginary parts of the roots are  $h = 0$  and  $v = \frac{1}{2}$ . It follows from (18.8) that

$$R = \sqrt{0 + \left(\frac{1}{2}\right)^2} = \frac{1}{2}$$

Since the value of  $\theta$  is that which can satisfy the two equations

$$\cos \theta = \frac{h}{R} = 0 \quad \text{and} \quad \sin \theta = \frac{v}{R} = 1$$

it may be concluded from Table 16.1 that

$$\theta = \frac{\pi}{2}$$

Consequently, the complementary function is

$$y_c = \left(\frac{1}{2}\right)^t \left( A_5 \cos \frac{\pi}{2}t + A_6 \sin \frac{\pi}{2}t \right)$$

To find  $y_p$ , let us try a constant solution  $y_t = k$  in the complete equation. This yields  $k = 4$ ; thus,  $y_p = 4$ , and the general solution can be written as

$$y_t = \left(\frac{1}{2}\right)^t \left( A_5 \cos \frac{\pi}{2}t + A_6 \sin \frac{\pi}{2}t \right) + 4 \quad (18.11)$$

**Example 6**

Find the general solution of  $y_{t+2} - 4y_{t+1} + 16y_t = 0$ . In the first place, the particular solution is easily found to be  $y_p = 0$ . This means that the general solution  $y_t (= y_c + y_p)$  will be identical with  $y_c$ . To find the latter, we note that the coefficients  $a_1 = -4$  and  $a_2 = 16$  do produce complex roots. Thus we may substitute the  $a_1$  and  $a_2$  values directly into (18.8) and (18.9) to obtain

$$R = \sqrt{16} = 4$$

$$\cos \theta = \frac{4}{2 \cdot 4} = \frac{1}{2} \quad \text{and} \quad \sin \theta = \sqrt{1 - \frac{16}{4 \cdot 16}} = \sqrt{\frac{3}{4}} = \frac{\sqrt{3}}{2}$$

The last two equations enable us to find from Table 16.2 that

$$\theta = \frac{\pi}{3}$$

It follows that the complementary function—which also serves as the general solution here—is

$$y_c (= y_t) = 4^t \left( A_5 \cos \frac{\pi}{3}t + A_6 \sin \frac{\pi}{3}t \right) \quad (18.12)$$

**The Convergence of the Time Path**

As in the case of first-order difference equations, the convergence of the time path  $y_t$  hinges solely on whether  $y_c$  tends toward zero as  $t \rightarrow \infty$ . What we learned about the various configurations of the expression  $b^t$ , in Fig. 17.1, is therefore still applicable, although in the present context we shall have to consider *two* characteristic roots rather than one.

Consider first the case of distinct real roots:  $b_1 \neq b_2$ . If  $|b_1| > 1$  and  $|b_2| > 1$ , then both component terms in the complementary function (18.5)— $A_1 b_1^t$  and  $A_2 b_2^t$ —will be

explosive, and thus  $y_c$  must be divergent. In the opposite case of  $|b_1| < 1$  and  $|b_2| < 1$ , both terms in  $y_c$  will converge toward zero as  $t$  is indefinitely increased, as will  $y_c$  also. What if  $|b_1| > 1$  but  $|b_2| < 1$ ? In this intermediate case, it is evident that the  $A_2b_2^t$  term tends to “die down,” while the other term tends to deviate farther from zero. It follows that the  $A_1b_1^t$  term must eventually dominate the scene and render the path divergent.

Let us call the root with the higher *absolute* value the *dominant root*. Then it appears that it is the dominant root  $b_1$  which really sets the tone of the time path, at least with regard to its ultimate convergence or divergence. Such is indeed the case. We may state, thus, that a *time path will be convergent—whatever the initial conditions may be—if and only if the dominant root is less than 1 in absolute value*. You can verify that this statement is valid for the cases where both roots are greater than or less than 1 in absolute value (discussed previously), and where one root has an absolute value of 1 exactly (*not* discussed previously). Note, however, that even though the eventual convergence depends on the dominant root alone, the *non-dominant* root will exert a definite influence on the time path, too, at least in the beginning periods. Therefore, the exact configuration of  $y_t$  is still dependent on both roots.

Turning to the repeated-root case, we find the complementary function to consist of the terms  $A_3b^t$  and  $A_4tb^t$ , as shown in (18.6). The former is already familiar to us, but a word of explanation is still needed for the latter, which involves a multiplicative  $t$ . If  $|b| > 1$ , the  $b^t$  term will be explosive, and the multiplicative  $t$  will simply serve to intensify the explosiveness as  $t$  increases. If  $|b| < 1$ , on the other hand, the  $b^t$  part (which tends to zero as  $t$  increases) and the  $t$  part will run counter to each other; i.e., the value of  $t$  will offset rather than reinforce  $b^t$ . Which force will prove the stronger? The answer is that the damping force of  $b^t$  will always win over the exploding force of  $t$ . For this reason, the basic requirement for convergence in the repeated-root case is still that the root be less than 1 in absolute value.

### **Example 7**

Analyze the convergence of the solutions in Examples 3 and 4. For Example 3, the solution is

$$y_t = 3 + (-2)^t + 4t$$

where the roots are 1 and  $-2$ , respectively [ $3(1)^t = 3$ ], and where there is a moving equilibrium  $4t$ . The dominant root being  $-2$ , the time path is divergent.

For Example 4, where the solution is

$$y_t = A_3(-3)^t + A_4t(-3)^t + \frac{1}{4}$$

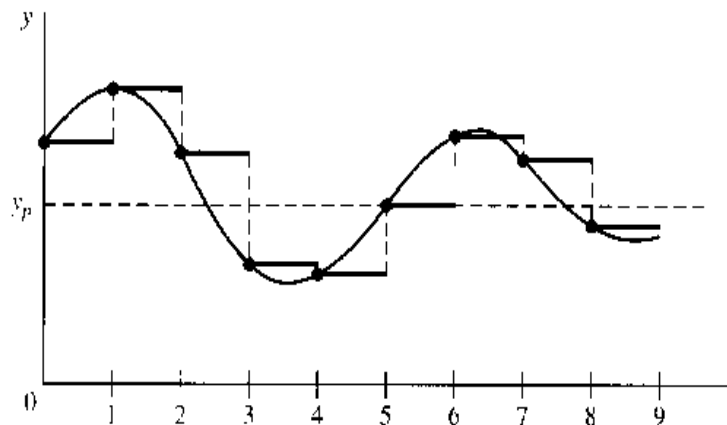
and where  $|b| = 3$ , we also have divergence.

Let us now consider the complex-root case. From the general form of the complementary function in (18.10),

$$y_c = R^t(A_5 \cos \theta t + A_6 \sin \theta t)$$

it is clear that the parenthetical expression, like the one in (16.24'), will produce a fluctuating pattern of a periodic nature. However, since the variable  $t$  can only take integer values 0, 1, 2, . . . in the present context, we shall catch and utilize only a subset of the points on the graph of a circular function. The  $y$  value at each such point will always prevail for a whole period, till the next relevant point is reached. As illustrated in Fig. 18.1, the resulting path is neither the usual oscillatory type (not alternating between values above and below

FIGURE 18.1



$y_p$  in consecutive periods), nor the usual fluctuating type (not smooth); rather, it displays a sort of *stepped* fluctuation. As far as convergence is concerned, though, the decisive factor is really the  $R^t$  term, which, like the  $e^{ht}$  term in (16.24'), will dictate whether the stepped fluctuation is to be intensified or mitigated as  $t$  increases. In the present case, the fluctuation can be gradually narrowed down if and only if  $R < 1$ . Since  $R$  is by definition the absolute value of the conjugate complex roots ( $h \pm vi$ ), the condition for convergence is again that the characteristic roots be less than unity in absolute value.

To summarize: For all three cases of characteristic roots, the time path will converge to a (stationary or moving) intertemporal equilibrium—regardless of what the initial conditions may happen to be—if and only if the absolute value of every root is less than 1.

### Example 8

Are the time paths (18.11) and (18.12) convergent? In (18.11) we have  $R = \frac{1}{2}$ ; therefore the time path will converge to the stationary equilibrium ( $= 4$ ). In (18.12), on the other hand, we have  $R = 4$ , so the time path will not converge to the equilibrium ( $= 0$ ).

### EXERCISE 18.1

- Write out the characteristic equation for each of the following, and find the characteristic roots:
  - $y_{t+2} - y_{t+1} + \frac{1}{2}y_t = 2$
  - $y_{t+2} - 4y_{t+1} + 4y_t = 7$
  - $y_{t+2} + \frac{1}{2}y_{t-1} - \frac{1}{2}y_t = 5$
  - $y_{t+2} - 2y_{t+1} + 3y_t = 4$
- For each of the difference equations in Prob. 1 state on the basis of its characteristic roots whether the time path involves oscillation or stepped fluctuation, and whether it is explosive.
- Find the particular solutions of the equations in Prob. 1. Do these represent stationary or moving equilibria?
- Solve the following difference equations:
  - $y_{t+2} + 3y_{t+1} - \frac{7}{4}y_t = 9$  ( $y_0 = 6$ ;  $y_1 = 3$ )
  - $y_{t+2} - 2y_{t+1} + 2y_t = 1$  ( $y_0 = 3$ ;  $y_1 = 4$ )
  - $y_{t+2} - y_{t+1} + \frac{1}{4}y_t = 2$  ( $y_0 = 4$ ;  $y_1 = 7$ )
- Analyze the time paths obtained in Prob. 4.



## 18.2 Samuelson Multiplier-Acceleration Interaction Model

As an illustration of the use of second-order difference equations in economics, let us cite a classic work of Professor Paul Samuelson, the first economist to win the Nobel Prize. We refer to his classic *interaction* model, which seeks to explore the dynamic process of income determination when the acceleration principle is in operation along with the Keynesian multiplier.<sup>†</sup> Among other things, that model serves to demonstrate that the mere interaction of the multiplier and the accelerator is capable of generating cyclical fluctuations endogenously.

### The Framework

Suppose that national income  $Y_t$  is made up of three component expenditure streams: consumption  $C_t$ , investment  $I_t$ , and government expenditure  $G_t$ . Consumption is envisaged as a function not of current income but of the income of the prior period,  $Y_{t-1}$ ; for simplicity, it is assumed that  $C_t$  is strictly proportional to  $Y_{t-1}$ . Investment, which is of the “induced” variety, is a function of the prevailing trend of consumer spending. It is through this induced investment, of course, that the acceleration principle enters into the model. Specifically, we shall assume  $I_t$  to bear a fixed ratio to the consumption increment  $\Delta C_{t-1} \equiv C_t - C_{t-1}$ . The third component,  $G_t$ , on the other hand, is taken to be exogenous; in fact, we shall assume it to be a constant and simply denote it by  $G_0$ .

These assumptions can be translated into the following set of equations:

$$\begin{aligned} Y_t &= C_t + I_t + G_0 \\ C_t &= \gamma Y_{t-1} & (0 < \gamma < 1) \\ I_t &= \alpha(C_t - C_{t-1}) & (\alpha > 0) \end{aligned} \quad (18.13)$$

where  $\gamma$  (the Greek letter gamma) represents the marginal propensity to consume, and  $\alpha$  stands for the accelerator (short for *acceleration coefficient*). Note that, if induced investment is expunged from the model, we are left with a first-order difference equation which embodies the dynamic multiplier process (cf. Example 2 of Sec. 17.2). With induced investment included, however, we have a second-order difference equation that depicts the interaction of the multiplier and the accelerator.

By virtue of the second equation, we can express  $I_t$  in terms of income as follows:

$$I_t = \alpha(\gamma Y_{t-1} - \gamma Y_{t-2}) = \alpha\gamma(Y_{t-1} - Y_{t-2})$$

Upon substituting this and the  $C_t$  equation into the first equation in (18.13) and rearranging, the model can be condensed into the single equation

$$Y_t - \gamma(1 + \alpha)Y_{t-1} + \alpha\gamma Y_{t-2} = G_0$$

or, equivalently (after shifting the subscripts forward by two periods),

$$Y_{t+2} - \gamma(1 + \alpha)Y_{t+1} + \alpha\gamma Y_t = G_0 \quad (18.14)$$

Because this is a second-order linear difference equation with constant coefficients and constant term, it can be solved by the method just learned.

<sup>†</sup> Paul A. Samuelson, “Interactions between the Multiplier Analysis and the Principle of Acceleration,” *Review of Economic Statistics*, May 1939, pp. 75–78; reprinted in American Economic Association, *Readings in Business Cycle Theory*, Richard D. Irwin, Inc., Homewood, Ill., 1944, pp. 261–269.

### The Solution

As the particular solution, we have, by (18.2),

$$Y_p = \frac{G_0}{1 - \gamma(1 + \alpha) + \alpha\gamma} = \frac{G_0}{1 - \gamma}$$

It may be noted that the expression  $1/(1 - \gamma)$  is merely the multiplier that would prevail in the absence of induced investment. Thus  $G_0/(1 - \gamma)$ —the exogenous expenditure item times the multiplier—should give us the equilibrium income  $Y^*$  in the sense that this income level satisfies the equilibrium condition “national income = total expenditure” [cf. (3.24)]. Being the particular solution of the model, however, it also gives us the intertemporal equilibrium income  $\bar{Y}$ .

With regard to the complementary function, there are three possible cases. Case 1 ( $\alpha^2 > 4a_2$ ), in the present context, is characterized by

$$\gamma^2(1 + \alpha)^2 > 4\alpha\gamma \quad \text{or} \quad \gamma(1 + \alpha)^2 > 4\alpha$$

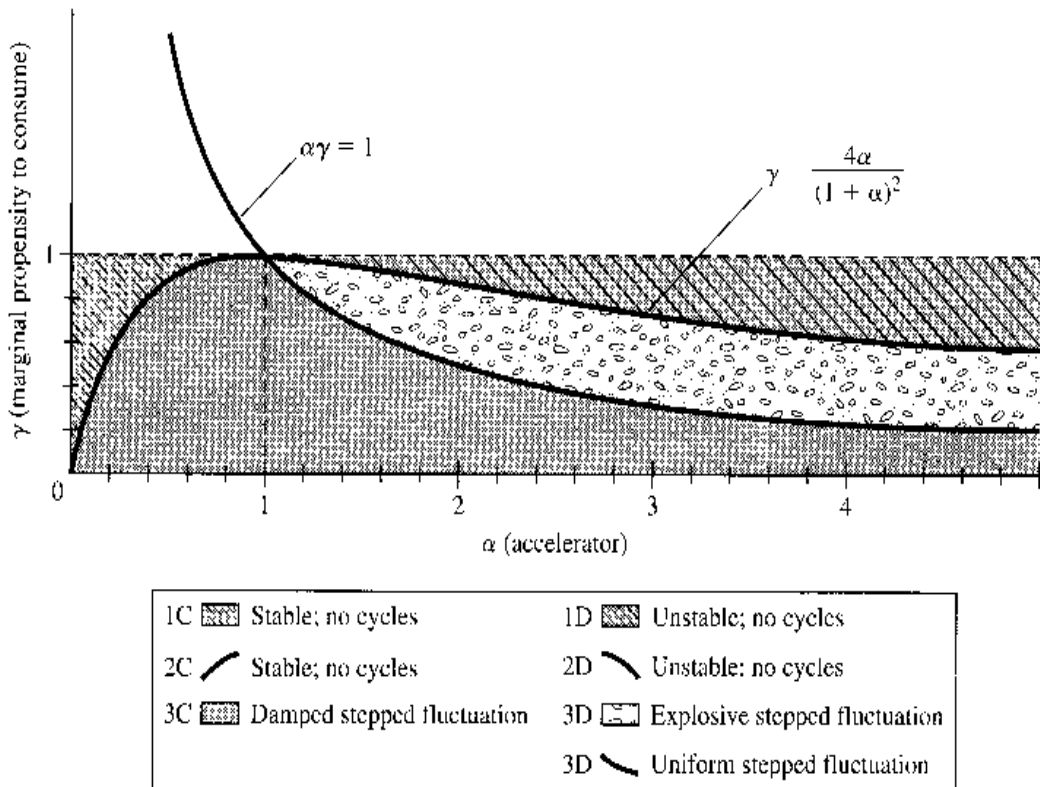
or

$$\gamma > \frac{4\alpha}{(1 + \alpha)^2}$$

Similarly, to characterize Cases 2 and 3, we only need to change the  $>$  sign in the last inequality to  $=$  and  $<$ , respectively. In Fig. 18.2, we have drawn the graph of the equation  $\gamma = 4\alpha/(1 + \alpha)^2$ . According to the preceding discussion, the  $(\alpha, \gamma)$  pairs that are located exactly *on* this curve pertain to Case 2. On the other hand, the  $(\alpha, \gamma)$  pairs lying *above* this curve (involving higher  $\gamma$  values) have to do with Case 1, and those lying *below* the curve with Case 3.

This tripartite classification, with its graphical representation in Fig. 18.2, is of interest because it reveals clearly the conditions under which cyclical fluctuations can emerge

FIGURE 18.2



endogenously from the interaction of the multiplier and the accelerator. But this tells nothing about the convergence or divergence of the time path of  $Y$ . It remains, therefore, for us to distinguish, under each case, between the *damped* and the *explosive* subcases. We could, of course, take the easy way out by simply illustrating such subcases by citing specific numerical examples. But let us attempt the more rewarding, if also more arduous, task of delineating the general conditions under which convergence and divergence will prevail.

### Convergence versus Divergence

The difference equation (18.14) has the characteristic equation

$$b^2 - \gamma(1 + \alpha)b + \alpha\gamma = 0$$

which yields the two roots

$$b_1, b_2 = \frac{\gamma(1 + \alpha) \pm \sqrt{\gamma^2(1 + \alpha)^2 - 4\alpha\gamma}}{2}$$

Since the question of convergence versus divergence depends on the values of  $b_1$  and  $b_2$ , and since  $b_1$  and  $b_2$ , in turn, depend on the values of the parameters  $\alpha$  and  $\gamma$ , the conditions for convergence and divergence should be expressible in terms of the values of  $\alpha$  and  $\gamma$ . To do this, we can make use of the fact that—by (16.6)—the two characteristic roots are always related to each other by the following two equations:

$$b_1 + b_2 = \gamma(1 + \alpha) \quad (18.15)$$

$$b_1 b_2 = \alpha\gamma \quad (18.15')$$

On the basis of these two equations, we may observe that

$$\begin{aligned} (1 - b_1)(1 - b_2) &= 1 - (b_1 + b_2) + b_1 b_2 \\ &= 1 - \gamma(1 + \alpha) + \alpha\gamma = 1 - \gamma \end{aligned} \quad (18.16)$$

In view of the model specification that  $0 < \gamma < 1$ , it becomes necessary to impose on the two roots the condition

$$0 < (1 - b_1)(1 - b_2) < 1 \quad (18.17)$$

Let us now examine the question of convergence under Case 1, where the roots are real and distinct. Since, by assumption,  $\alpha$  and  $\gamma$  are both positive, (18.15') tells us that  $b_1 b_2 > 0$ , which implies that  $b_1$  and  $b_2$  possess the same algebraic sign. Furthermore, since  $\gamma(1 + \alpha) > 0$ , (18.15) indicates that both  $b_1$  and  $b_2$  must be positive. Hence, the time path  $Y_t$  cannot have oscillations in Case 1.

Even though the signs of  $b_1$  and  $b_2$  are now known, there actually exist under Case 1 as many as five possible combinations of  $(b_1, b_2)$  values, each with its own implication regarding the corresponding values for  $\alpha$  and  $\gamma$ :

$$(i) \quad 0 < b_2 < b_1 < 1 \quad \Rightarrow \quad 0 < \gamma < 1; \alpha\gamma < 1$$

$$(ii) \quad 0 < b_2 < b_1 = 1 \quad \Rightarrow \quad \gamma = 1$$

$$(iii) \quad 0 < b_2 < 1 < b_1 \quad \Rightarrow \quad \gamma > 1$$

$$(iv) \quad 1 = b_2 < b_1 \quad \Rightarrow \quad \gamma = 1$$

$$(v) \quad 1 < b_2 < b_1 \quad \Rightarrow \quad 0 < \gamma < 1; \alpha\gamma > 1$$

Possibility *i*, where both  $b_1$  and  $b_2$  are positive fractions, duly satisfies condition (18.17) and hence conforms to the model specification  $0 < \gamma < 1$ . The product of the two roots must also be a positive fraction under this possibility, and this, by (18.15'), implies that  $\alpha\gamma < 1$ . In contrast, the next three possibilities all violate condition (18.17) and result in inadmissible  $\gamma$  values (see Exercise 18.2-3). Hence they must be ruled out. But Possibility *v* may still be acceptable. With both  $b_1$  and  $b_2$  greater than one, (18.17) may still be satisfied if  $(1 - b_1)(1 - b_2) < 1$ . But this time we have  $\alpha\gamma > 1$  (rather than  $< 1$ ) from (18.15'). The upshot is that there are only two admissible subcases under Case 1. The first—Possibility *i*—involves fractional roots  $b_1$  and  $b_2$ , and therefore yields a convergent time path of  $Y$ . The other subcase—Possibility *v*—features roots greater than one, and thus produces a divergent time path. As far as the values of  $\alpha$  and  $\gamma$  are concerned, however, the question of convergence and divergence only hinges on whether  $\alpha\gamma < 1$  or  $\alpha\gamma > 1$ . This information is summarized in the top part of Table 18.1, where the convergent subcase is labeled 1C, and the divergent subcase 1D.

The analysis of Case 2, with repeated roots, is similar in nature. The roots are now  $b = \gamma(1 + \alpha)/2$ , with a positive sign because  $\alpha$  and  $\gamma$  are positive. Thus, there is again no oscillation. This time we may classify the value of  $b$  into three possibilities only:

- (vi)  $0 < b < 1 \Rightarrow \gamma < 1; \alpha\gamma < 1$
- (vii)  $b = 1 \Rightarrow \gamma = 1$
- (viii)  $b > 1 \Rightarrow \gamma < 1; \alpha\gamma > 1$

Under Possibility *vi*,  $b (= b_1 = b_2)$  is a positive fraction; thus the implications regarding  $\alpha$  and  $\gamma$  are entirely identical with those of Possibility *i* under Case 1. In an analogous manner, Possibility *viii*, with  $b (= b_1 = b_2)$  greater than one, can satisfy (18.17) only if  $1 < b < 2$ ; if so, it yields the same results as Possibility *v*. On the other hand, Possibility *vii* violates (18.17) and must be ruled out. Thus there are again only two admissible subcases. The first—Possibility *vi*—yields a convergent time path, whereas the other—Possibility *viii*—gives a divergent one. In terms of  $\alpha$  and  $\gamma$ , the convergent and divergent subcases are again associated, respectively, with  $\alpha\gamma < 1$  and  $\alpha\gamma > 1$ . These results are listed in the middle part of Table 18.1, where the two subcases are labeled 2C (convergent) and 2D (divergent).

**TABLE 18.1**  
Cases and Subcases of the Samuelson Model

Case	Subcase	Values of $\alpha$ and $\gamma$	Time Path $Y_t$
1. Distinct real roots			
	$\gamma > \frac{4\alpha}{(1 + \alpha)^2}$	1C: $0 < b_2 < b_1 < 1$ 1D: $1 < b_2 < b_1$	$\alpha\gamma < 1$ $\alpha\gamma > 1$
2. Repeated real roots			
	$\gamma = \frac{4\alpha}{(1 + \alpha)^2}$	2C: $0 < b < 1$ 2D: $b > 1$	$\alpha\gamma < 1$ $\alpha\gamma > 1$
3. Complex roots			
	$\gamma < \frac{4\alpha}{(1 + \alpha)^2}$	3C: $R < 1$ 3D: $R \geq 1$	$\alpha\gamma < 1$ $\alpha\gamma \geq 1$

Finally, in Case 3, with complex roots, we have stepped fluctuation, and hence endogenous business cycles. In this case, we should look to the absolute value  $R = \sqrt{a_2}$  [see (18.8)] for the clue to convergence and divergence, where  $a_2$  is the coefficient of the  $y_t$  term in the difference equation (18.1). In the present model, we have  $R = \sqrt{\alpha\gamma}$ , which gives rise to the following three possibilities:

$$(ix) \quad R < 1 \quad \Rightarrow \quad \alpha\gamma < 1$$

$$(x) \quad R = 1 \quad \Rightarrow \quad \alpha\gamma = 1$$

$$(xi) \quad R > 1 \quad \Rightarrow \quad \alpha\gamma > 1$$

Even though all of these happen to be admissible (see Exercise 18.2-4), only the  $R < 1$  possibility entails a convergent time path and qualifies as Subcase 3C in Table 18.1. The other two are thus collectively labeled as Subcase 3D.

In sum, we may conclude from Table 18.1 that a convergent time path can occur if and only if  $\alpha\gamma < 1$ .

## A Graphical Summary

The preceding analysis has resulted in a somewhat complex classification of cases and subcases. It would help to have a visual representation of the classificatory scheme. This is supplied in Fig. 18.2.

The set of all admissible  $(\alpha, \gamma)$  pairs in the model is shown in Fig. 18.2 by the variously shaded rectangular area. Since the values of  $\gamma = 0$  and  $\gamma = 1$  are excluded, as is the value  $\alpha = 0$ , the shaded area is a sort of rectangle without sides. We have already graphed the equation  $\gamma = 4\alpha/(1 + \alpha)^2$  to mark off the three major cases of Table 18.1: The points on that curve pertain to Case 2; the points lying to the north of the curve (representing higher  $\gamma$  values) belong to Case 1; those lying to the south (with lower  $\gamma$  values) are of Case 3. To distinguish between the convergent and divergent subcases, we now add the graph of  $\alpha\gamma = 1$  (a rectangular hyperbola) as another demarcation line. The points lying to the north of this rectangular hyperbola satisfy the inequality  $\alpha\gamma > 1$ , whereas those located below it correspond to  $\alpha\gamma < 1$ . It is then possible to mark off the subcases easily. Under Case 1, the broken-line shaded region, being below the hyperbola, corresponds to Subcase 1C, but the solid-line shaded region is associated with Subcase 1D. Under Case 2, which relates to the points lying on the curve  $\gamma = 4\alpha/(1 + \alpha)^2$ , Subcase 2C covers the upward-sloping portion of that curve, and Subcase 2D, the downward-sloping portion. Finally, for Case 3, the rectangular hyperbola serves to separate the dot-shaded region (Subcase 3C) from the pebble-shaded region (Subcase 3D). The latter, you should note, also includes the points located *on* the rectangular hyperbola itself, because of the *weak inequality* in the specification  $\alpha\gamma \geq 1$ .

Since Fig. 18.2 is the repository of all the qualitative conclusions of the model, given any ordered pair  $(\alpha, \gamma)$ , we can always find the correct subcase graphically by plotting the ordered pair in the diagram.

### Example 1

If the accelerator is 0.8 and the marginal propensity to consume is 0.7, what kind of interaction time path will result? The ordered pair (0.8, 0.7) is located in the dot-shaded region, Subcase 3C; thus the time path is characterized by damped stepped fluctuation.

### Example 2

What kind of interaction is implied by  $\alpha = 2$  and  $\gamma = 0.5$ ? The ordered pair (2, 0.5) lies exactly on the rectangular hyperbola, under Subcase 3D. The time path of  $Y$  will again display stepped fluctuation, but it will be neither explosive nor damped. By analogy to the cases of

uniform oscillation and uniform fluctuation, we may term this situation as “uniform stepped fluctuation.” However, the uniformity feature in this latter case cannot in general be expected to be a perfect one, because, similarly to what was done in Fig. 18.1, we can only accept those points on a sine or cosine curve that correspond to integer values of  $t$ , but these values of  $t$  may hit an entirely different set of points on the curve in each period of fluctuation.

### EXERCISE 18.2

- By consulting Fig. 18.2, find the subcases to which the following sets of values of  $\alpha$  and  $\gamma$  pertain, and describe the interaction time path qualitatively.
  - $\alpha = 3.5; \gamma = 0.8$
  - $\alpha = 2; \gamma = 0.7$
  - $\alpha = 0.2; \gamma = 0.9$
  - $\alpha = 1.5; \gamma = 0.6$
- From the values of  $\alpha$  and  $\gamma$  given in parts (a) and (c) of Prob. 1, find the numerical values of the characteristic roots in each instance, and analyze the nature of the time path. Do your results check with those obtained earlier?
- Verify that Possibilities *ii*, *iii*, and *iv* in Case 1 imply inadmissible values of  $\gamma$ .
- Show that in Case 3 we can never encounter  $\gamma \geq 1$ .

## 18.3 Inflation and Unemployment in Discrete Time

The interaction of inflation and unemployment, discussed earlier in the continuous-time framework, can also be couched in discrete time. Using essentially the same economic assumptions, we shall illustrate in this section how that model can be reformulated as a difference-equation model.

### The Model

The earlier continuous-time formulation (Sec. 16.5) consisted of three differential equations:

$$p = \alpha - T - \beta U + g\pi \quad \begin{array}{l} \text{[expectations-augmented} \\ \text{Phillips relation]} \end{array} \quad (16.33)$$

$$\frac{d\pi}{dt} = j(p - \pi) \quad \text{[adaptive expectations]} \quad (16.34)$$

$$\frac{dU}{dt} = -k(m - p) \quad \text{[monetary policy]} \quad (16.35)$$

Three endogenous variables are present:  $p$  (actual rate of inflation),  $\pi$  (expected rate of inflation), and  $U$  (rate of unemployment). As many as six parameters appear in the model; among these, the parameter  $m$ —the rate of growth of nominal money (or, the rate of monetary expansion)—differs from the others in that its magnitude is set as a policy decision.

When cast into the period-analysis mold, the Phillips relation (16.33) simply becomes

$$p_t = \alpha - T - \beta U_t + g\pi_t \quad (\alpha, \beta > 0; 0 < g \leq 1) \quad (18.18)$$

In the adaptive-expectations equation, the derivative must be replaced by a difference expression:

$$\pi_{t+1} - \pi_t = j(p_t - \pi_t) \quad (0 < j \leq 1) \quad (18.19)$$

By the same token, the monetary-policy equation should be changed to<sup>†</sup>

$$U_{t+1} - U_t = -k(m - p_{t+1}) \quad (k > 0) \quad (18.20)$$

These three equations constitute the new version of the inflation-unemployment model.

### The Difference Equation in $p$

As the first step in the analysis of this new model, we again try to condense the model into a single equation in a single variable. Let that variable be  $p$ . Accordingly, we shall focus our attention on (18.18). However, since (18.18)—unlike the other two equations—does not by itself describe a pattern of change, it is up to us to create such a pattern. This is accomplished by *differencing*  $p_t$ , i.e., by taking the first difference of  $p_t$ , according to the definition

$$\Delta p_t \equiv p_{t+1} - p_t$$

Two steps are involved in this. First, we shift the time subscripts in (18.18) forward one period, to get

$$p_{t+1} = \alpha - T - \beta U_{t+1} + g\pi_{t+1} \quad (18.18')$$

Then we subtract (18.18) from (18.18'), to obtain the first difference of  $p_t$  that gives the desired pattern of change:

$$\begin{aligned} p_{t+1} - p_t &= -\beta(U_{t+1} - U_t) + g(\pi_{t+1} - \pi_t) \\ &= \beta k(m - p_{t+1}) + gj(p_t - \pi_t) \quad [\text{by (18.20) and (18.19)}] \end{aligned} \quad (18.21)$$

Note that, on the second line of (18.21), the patterns of change of the other two variables as given in (18.19) and (18.20) have been incorporated into the pattern of change of the  $p$  variable. Thus (18.21) now embodies all the information in the present model.

However, the  $\pi_t$  term is extraneous to the study of  $p$  and needs to be eliminated from (18.21). To that end, we make use of the fact that

$$g\pi_t = p_t - (\alpha - T) + \beta U_t \quad [\text{by (18.18)}] \quad (18.22)$$

Substituting this into (18.21) and collecting terms, we obtain

$$(1 + \beta k)p_{t+1} - [1 - j(1 - g)]p_t + j\beta U_t = \beta km + j(\alpha - T) \quad (18.23)$$

But there now appears a  $U_t$  term to be eliminated. To do that, we difference (18.23) to get a  $(U_{t+1} - U_t)$  term and then use (18.20) to eliminate the latter. Only after this rather lengthy process of substitutions, do we get the desired difference equation in the  $p$  variable alone, which, when duly normalized, takes the form

$$p_{t+2} - \underbrace{\frac{1 + gj + (1 - j)(1 + \beta k)}{1 + \beta k}}_{a_1} p_{t+1} + \underbrace{\frac{1 - j(1 - g)}{1 + \beta k}}_{a_2} p_t = \underbrace{\frac{j\beta km}{1 + \beta k}}_c \quad (18.24)$$

<sup>†</sup> We have assumed that the change in  $U_t$  depends on  $(m - p_{t+1})$ , the rate of growth of real money in period  $(t + 1)$ . As an alternative, it is possible to make it depend on the rate of growth of real money in period  $t$ ,  $(m - p_t)$  (see Exercise 18.3-4).

## The Time Path of $p$

The intertemporal equilibrium value of  $p$ , given by the particular integral of (18.24), is

$$\bar{p} = \frac{c}{1 + a_1 + a_2} = \frac{j\beta km}{\beta kj} = m \quad [\text{by (18.2)}]$$

As in the continuous-time model, therefore, the equilibrium rate of inflation is exactly equal to the rate of monetary expansion.

As to the complementary function, there may arise either distinct real roots (Case 1), or repeated real roots (Case 2), or complex roots (Case 3), depending on the relative magnitudes of  $a_1^2$  and  $4a_2$ . In the present model,

$$a_1^2 \begin{array}{l} \geq \\ \leq \end{array} 4a_2 \quad \text{iff} \quad [1 + gj + (1 - j)(1 + \beta k)]^2 \begin{array}{l} \geq \\ \leq \end{array} 4[1 - j(1 - g)](1 + \beta k) \quad (18.25)$$

If  $g = \frac{1}{2}$ ,  $j = \frac{1}{3}$  and  $\beta k = 5$ , for instance, then  $a_1^2 = (5\frac{1}{6})^2$  whereas  $4a_2 = 20$ ; thus Case 1 results. But if  $g = j = 1$ , then  $a_1^2 = 4$  while  $4a_2 = 4(1 + \beta k) > 4$ , and we have Case 3 instead. In view of the larger number of parameters in the present model, however, it is not feasible to construct a classificatory graph like Fig. 18.2 in the Samuelson model.

Nevertheless, the analysis of convergence can still proceed along the same line as in Sec. 18.2. Specifically, we recall from (16.6) that the two characteristic roots  $b_1$  and  $b_2$  must satisfy the following two relations:

$$b_1 + b_2 = -a_1 = \frac{1 + gj}{1 + \beta k} + 1 - j > 0 \quad (18.26)$$

[see (18.24)]

$$b_1 b_2 = a_2 = \frac{1 - j(1 - g)}{1 + \beta k} \in (0, 1) \quad (18.26')$$

Furthermore, we have in the present model

$$(1 - b_1)(1 - b_2) = 1 - (b_1 + b_2) + b_1 b_2 = \frac{\beta j k}{1 + \beta k} > 0 \quad (18.27)$$

Now consider Case 1, where the two roots  $b_1$  and  $b_2$  are real and distinct. Since their product  $b_1 b_2$  is positive,  $b_1$  and  $b_2$  must take the same sign. Because their sum is positive, moreover,  $b_1$  and  $b_2$  must both be positive, implying that no oscillation can occur. From (18.27), we can infer that neither  $b_1$  nor  $b_2$  can be equal to one; for otherwise  $(1 - b_1)(1 - b_2)$  would be zero, in violation of the indicated inequality. This means that, in terms of the various possibilities of  $(b_1, b_2)$  combinations enumerated in the Samuelson model, Possibilities *ii* and *iv* cannot arise here. It is also unacceptable to have one root greater, and the other root less, than one; for otherwise  $(1 - b_1)(1 - b_2)$  would be negative. Thus Possibility *iii* is ruled out as well. It follows that  $b_1$  and  $b_2$  must be *either* both greater than one, *or* both less than one. If  $b_1 > 1$  and  $b_2 > 1$  (Possibility *v*), however, (18.26') would be violated. Hence the only viable eventuality is Possibility *i*, with  $b_1$  and  $b_2$  both being positive fractions, so that the time path of  $p$  is convergent.

The analysis of Case 2 is basically not much different. By practically identical reasoning, we can conclude that the repeated root  $b$  can only turn out to be a positive fraction in this model; that is, Possibility *vi* is feasible, but not Possibilities *vii* and *viii*. The time path of  $p$  in Case 2 is again nonoscillatory and convergent.



For Case 3, convergence requires that  $R$  (the absolute value of the complex roots) be less than one. By (18.8),  $R = \sqrt{a_2}$ . Inasmuch as  $a_2$  is a positive fraction [see (18.26')], we do have  $R < 1$ . Thus the time path of  $p$  in Case 3 is also convergent, although this time there will be stepped fluctuation.

### The Analysis of $U$

If we wish to analyze instead the time path of the rate of unemployment, we may take (18.20) as the point of departure. To get rid of the  $p$  term in that equation, we first substitute (18.18') to get

$$(1 + \beta k)U_{t+1} - U_t = k(\alpha - T - m) + kg\pi_{t+1} \quad (18.28)$$

Next, to prepare for the substitution of the other equation, (18.19), we difference (18.28) to find that

$$(1 + \beta k)U_{t+2} - (2 + \beta k)U_{t+1} + U_t = kg(\pi_{t-2} - \pi_{t+1}) \quad (18.29)$$

In view of the presence of a difference expression in  $\pi$  on the right, we can substitute for it a forward-shifted version of the adaptive-expectations equation. The result of this,

$$(1 + \beta k)U_{t+2} - (2 + \beta k)U_{t+1} + U_t = kgj(p_{t+1} - \pi_{t+1}) \quad (18.30)$$

is the embodiment of all the information in the model.

However, we must eliminate the  $p$  and  $\pi$  variables before a proper difference equation in  $U$  will emerge. For this purpose, we note from (18.20) that

$$kp_{t+1} = U_{t+1} - U_t + km \quad (18.31)$$

Moreover, by multiplying (18.22) through by  $(-kj)$  and shifting the time subscripts, we can write

$$\begin{aligned} -kjg\pi_{t-1} &= -kj p_{t+1} + kj(\alpha - T) - \beta kj U_{t+1} \\ &= -j(U_{t+1} - U_t + km) + kj(\alpha - T) - \beta kj U_{t+1} \\ &\quad \text{[by (18.31)]} \\ &= -j(1 + \beta k)U_{t+1} + jU_t + kj(\alpha - T - m) \end{aligned} \quad (18.32)$$

These two results express  $p_{t+1}$  and  $\pi_{t+1}$  in terms of the  $U$  variable and can thus enable us, on substitution into (18.30), to obtain—at long last!—the desired difference equation in the  $U$  variable alone:

$$\begin{aligned} U_{t+2} - \frac{1 + gj + (1 - j)(1 + \beta k)}{1 + \beta k} U_{t+1} + \frac{1 - j(1 - g)}{1 + \beta k} U_t \\ = \frac{kj[\alpha - T - (1 - g)m]}{1 + \beta k} \end{aligned} \quad (18.33)$$

It is noteworthy that the two constant coefficients on the left ( $a_1$  and  $a_2$ ) are identical with those in the difference equation for  $p$  [i.e., (18.24)]. As a result, the earlier analysis of the complementary function of the  $p$  path should be equally applicable to the present context. But the constant term on the right of (18.33) does differ from that of (18.24). Consequently, the particular solutions in the two situations will be different. This is as it should be, for, coincidence aside, there is no inherent reason to expect the intertemporal equilibrium rate of unemployment to be the same as the equilibrium rate of inflation.

## The Long-Run Phillips Relation

It is readily verified that the intertemporal equilibrium rate of unemployment is

$$\bar{U} = \frac{1}{\beta}[\alpha - T - (1 - g)m]$$

But since the equilibrium rate of inflation has been found to be  $\bar{p} = m$ , we can link  $\bar{U}$  to  $\bar{p}$  by the equation

$$\bar{U} = \frac{1}{\beta}[\alpha - T - (1 - g)\bar{p}] \quad (18.34)$$

Because this equation is concerned only with the *equilibrium* rates of unemployment and inflation, it is said to depict the *long-run* Phillips relation.

A special case of (18.34) has received a great deal of attention among economists: the case of  $g = 1$ . If  $g = 1$ , the  $\bar{p}$  term will have a zero coefficient and thus drop out of the picture. In other words,  $\bar{U}$  will become a constant function of  $\bar{p}$ . In the standard Phillips diagram, where the rate of unemployment is plotted on the horizontal axis, this outcome gives rise to a vertical long-run Phillips curve. The  $\bar{U}$  value in this case, referred to as the *natural rate of unemployment*, is then consistent with any equilibrium rate of inflation, with the notable policy implication that, in the long run, there is no trade-off between the twin evils of inflation and unemployment as exists in the short run.

But what if  $g < 1$ ? In that event, the coefficient of  $\bar{p}$  in (18.34) will be negative. Then the long-run Phillips curve will turn out to be downward-sloping, thereby still providing a trade-off relation between inflation and unemployment. Whether the long-run Phillips curve is vertical or negatively sloped is, therefore, critically dependent on the value of the  $g$  parameter, which, according to the expectations-augmented Phillips relation, measures the extent to which the expected rate of inflation can work its way into the wage structure and the actual rate of inflation. All of this may sound familiar to you. This is because we discussed the topic in Example 1 in Sec. 16.5, and you have also worked on it in Exercise 16.5-4.

### EXERCISE 18.3

1. Supply the intermediate steps leading from (18.23) to (18.24).
2. Show that if the model discussed in this section is condensed into a difference equation in the variable  $\pi$ , the result will be the same as (18.24) except for the substitution of  $\pi$  for  $p$ .
3. The time paths of  $p$  and  $U$  in the model discussed in this section have been found to be consistently convergent. Can divergent time paths arise if we drop the assumption that  $g \leq 1$ ? If yes, which divergent "possibilities" in Cases 1, 2, and 3 will now become feasible?
4. Retain equations (18.18) and (18.19), but change (18.20) to
 
$$U_{t-1} - U_t = -k(m - p_t)$$
  - (a) Derive a new difference equation in the variable  $p$ .
  - (b) Does the new difference equation yield a different  $\bar{p}$ ?
  - (c) Assume that  $j = g = 1$ . Find the conditions under which the characteristic roots will fall under Cases 1, 2, and 3, respectively.
  - (d) Let  $j = g = 1$ . Describe the time path of  $p$  (including convergence or divergence) when  $\beta k = 3, 4$ , and  $5$ , respectively.