Example 3. *Prove that* $H_n(-x) = (-1)^n H_n(x)$.

Solution. We have

$$H_n(x) = \sum_{r=0}^{n/2} \frac{(-1)^r n!(2x)^{n-2r}}{(n-2r)!\,r!}$$

and
$$H_n(-x) = \sum_{r=0}^{n/2} \frac{(-1)^r n!(-2x)^{n-2r}}{(n-2r)!\,r!}$$

$$= \sum_{r=0}^{n/2} \frac{(-1)^r (-1)^{n-2r} n!(2x)^{n-2r}}{(n-2r)!\,r!}$$

$$= (-1)^n \sum_{r=0}^{n/2} \frac{(-1)^r n!(2x)^{n-2r}}{(n-2r)!\,r!} = (-1)^n H_n(x)$$

7.11. Laguerre's Differential Equation

The differential equation

$$xy'' + (1-x)y' + ny = 0 \qquad \ldots(7.188)$$

where n is constant, is known as Laguerre's differential equation. This equation has removable singularity at $x = 0$ hence its solution can be found using series integration method. Let series solution of above equation be

$$y = \sum_{r=0}^{\infty} a_r x^{m+r} \qquad \ldots(7.189)$$

Substitution of (7.189) in (7.188) yields

$$\sum_{r=0}^{\infty} a_r \left[(m+r)^2 x^{m+r-1} - (m+r-n)x^{m+r}\right] = 0 \qquad \ldots(7.190)$$

The indicial equation then is

$$a_0 m^2 = 0$$

giving $m = 0$. Since a_0 is taken as arbitrary constant.

Equating the co-efficient x^{m+r} equal to zero, we have

$$a_{r+1}(m+r+1)^2 = (m+r-n)a_r$$

or
$$a_{r+1} = \frac{m+r-n}{(m+r+1)^2} a_r \,.$$

and since $m = 0$

$$a_{r+1} = \frac{r-n}{(r+1)^2} a_r \qquad \text{...(7.191)}$$

This is recurrence relation for the co-efficients a_r.

Substituting $r = 0, 1, 2, 3$ etc. we get all the co-efficients.

$$a_1 = -na_0 = (-1)^1 na_0 = \frac{(-1)^1}{(1!)^2} n\, a_0$$

$$a_2 = \frac{1-n}{2^2} a_1 = \frac{(-n)(1-n)}{(2)^2} a_0 = \frac{n(n-1)}{(2!)^2} a_0 = \frac{(-1)^2 n(n-1)}{(2!)^2} a_0$$

$$a_3 = \frac{(2-n)}{3^2} a_2 = \frac{(2-n)}{3^2} \frac{(-1)^2 n(n-1)}{(2!)^2} a_0$$

$$= \frac{(-1)^3 n(n-1)(n-2)}{(3!)^2} a_0$$

$$\vdots \qquad \vdots \qquad \vdots$$

$$a_r = \frac{(-1)^r n(n-1)(n-2)......(n-r+1)}{(r!)^2} a_0$$

$$y = \sum_{r=0}^{\infty} a_r x^r = a_0 \left[1 - nx + \frac{n(n-1)}{(2!)^2} x^2 + ... \right.$$

$$\left. + \frac{(-1)^r n(n-1)......(n-r+1)}{(r!)^2} x^r + ... \right] \qquad \text{...(7.192)}$$

If n is a positive integer the series terminate when $r = n+1$ and if we put $a_0 = n\,!$[1], then solution y representing equation (7.192) becomes Laguerre polynomial $L_n(x)$

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1. Some authors use $a_0 = 1$ and Laguerre polynomial is

$$L_n(x) = \sum_{r=0}^{n} \frac{(-1)^r n!}{(r!)^2 (n-r)!} x^r$$

The two expressions merely differ by a constant factor.

$$L_n(x) = (-1)^n \left[x^n - \frac{x^2}{1!} x^{n-1} + \frac{x^2 (n-1)^2}{2!} x^{n-2} + \ldots\ldots + (-1)^n n! \right]$$

$$= \sum_{r=0}^{n} \frac{(-1)^r (n!)^2}{(r!)^2 (n-r)!} x^r \qquad \ldots(7.193)$$

Thus a Laguerre's polynomial is the solution of Laguerre differential equation.

Generating Function for Laguerre Polynomials

The generating function for Laguerre polynomials is

$$f(x, t) = \frac{e^{-xt/(1-t)}}{1-t} = \sum_{n=0}^{\infty} \frac{L_n(x) t^n}{n!} \text{ for } |t| < 1 \qquad \ldots(7.194)$$

We have

$$(1-t)^{-1} e^{-xt/(1-t)} = \sum_{r=0}^{\infty} \frac{(-1)^r \cdot t^r x^r}{(1-t)^{r+1}} \frac{1}{r!}$$

$$= \sum_{r=0}^{\infty} \frac{(-1)^r}{r!} t^r x^r (1-t)^{-(r+1)}$$

$$= \sum_{r=0}^{\infty} \frac{(-1)^r}{r!} x^r t^r \sum_{s=0}^{\infty} \frac{(r+1)(r+2)\ldots(r+s)}{s!} t^s$$

$$= \sum_{r=0}^{\infty} \frac{(-1)^r}{r!} x^r t^r \sum_{s=0}^{\infty} \frac{(r+s)!}{r!\, s!} t^s$$

$$= \sum_{r,s=0}^{\infty} \frac{(-1)^r (r+s)!}{(r!)^2 s!} x^r t^{s+r} \qquad \ldots(7.195)$$

The co-efficient of t^n (for fixed value of r) on R.H.S. is obtained by putting $r + s = n$ *i.e.* $s = n - r$ and is given by

$$(-1)^r \frac{n!}{(r!)^2 (n-r)!} x^r.$$

The net co-efficient of t^n is obtained by summing over all allowed values of r.