CHAPTER

2 Theories of Failure

In the previous chapter, we have seen that a member is subjected to any of the simple stresses—tensile, compressive, shear or bending stress—then it is easy to predict the failure of the member. But in practice machine members will be subjected to more than one type of stress simultaneously and hence it will be difficult to predict the failure of such machine members using these simple stress theories.

In order to predict the failure of such members subjected to combined stresses, the following theories of failure are being suggested by different people:

(i) Rankine's theory or maximum normal stress theory.
(ii) Guest's theory or maximum shear stress theory.
(iii) Hencky-Von-Mises theory or distortion energy theory or shear energy theory.
(iv) Saint Venant theory or maximum strain theory.

BIAXIAL STRESSES WITH SHEAR STRESS

2.1 RANKINE'S THEORY OR MAXIMUM NORMAL STRESS THEORY

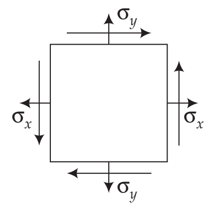

Fig. 2.1

Figure 2.1 shows an element subjected to stresses, σ_x action along x-direction (tensile or compressive), σ_y acting along y-direction $\perp lr$ to x (tensile or compressive) combined with shear stress, τ_{xy}.

According to the maximum normal stress theory or Rankine's theory of failure, equivalent stress

$$\sigma_e = \frac{1}{2}\left[(\sigma_x + \sigma_y) + \sqrt{(\sigma_x - \sigma_y)^2 + 4\tau_{xy}^2}\right] \quad ...\left(\frac{5-20}{\text{P 5.8}}\right)$$

PROBLEMS

Problem 1: A machine element is subjected to the following stresses $\sigma_x = 60$ MPa, $\sigma_y = 45$ MPa, $\tau_{xy} = 30$ MPa. Find the factor of safety if it is made of *C*45 steal having yield stress as 353 MPa, using the following theories of failure.

(i) Maximum principal stress theory,
(ii) Maximum shear stress theory,
(iii) Shear energy theory, and
(iv) Maximum strain theory taking Poisson ratio as 0.3

Given data: $\sigma_x = 60$ MPa, $\sigma_y = 45$ MPa, $\tau_{xy} = 30$ MPa yield stress, $\sigma_{ys} = 353$ MPa
Poisson ratio $v = 0.3$.

(i) According to maximum principal stress or Rankine's theory of equivalent stress

$$\sigma_e = \frac{1}{2} = \left[(\sigma_x + \sigma_y) + \sqrt{(\sigma_x - \sigma_y)^2 + 4\tau_{xy}^2}\right] \quad \text{...(5-20)}$$

$$\sigma_e = \frac{1}{2} = \left[(60 + 45) + \sqrt{(60 - 45)^2 + 4(30)^2}\right] = 83.42 \text{ MPa}$$

$$\therefore \quad \text{FOS} = \frac{\sigma_{ys}}{\sigma_e} = \frac{353}{83.42} = 4.23$$

(ii) According to max. shear stress theory or Guest's theory equivalent stress

$$\tau_e = \frac{1}{2}\sqrt{(\sigma_x - \sigma_y)^2 + 4\tau_{xy}^2} \quad \text{...(5-21)}$$

or

$$\sigma_e = \sqrt{(\sigma_x - \sigma_y)^2 + 4\tau_{xy}^2} \quad \left(\because \tau_e = \frac{\sigma_e}{2}\right)$$

$$= \sqrt{(60 - 45)^2 + 4(30)^2} = 61.85 \text{ MPa}$$

$$\therefore \quad \text{FOS} = \frac{\sigma_{ys}}{\sigma_e} = 353/61.85 = 5.71$$

(iii) According to shear energy theory or Hencky-Von-Mises theory, equivalent stress

$$\sigma_e = \sqrt{\sigma_x^2 + \sigma_y^2 - \sigma_x\sigma_y + 3\tau_{xy}^2} \quad \text{...(5-22)}$$

$$\sigma_e = \sqrt{60^2 + 45^2 - 60 \times 45 + 3 \times 30^2} = 75 \text{ MPa}$$

$$\therefore \quad \text{FOS} = \frac{\sigma_{ys}}{\sigma_e} = \frac{353}{75} = 4.71$$

(iv) According to Max-Strain theory or Saint-Venant theory. Equivalent stress

$$\sigma_e = \frac{1}{2}\left[(1-v)(\sigma_x+\sigma_y)+(1+v)\sqrt{(\sigma_x-\sigma_y)^2+4\tau_{xy}^2}\right] \quad ...(5\text{-}23)$$

$$\sigma_e = \frac{1}{2}\left[(1-0.3)(60+45)+(1+0.3)\sqrt{(60-45)^2+4(30)^2}\right]$$

$$= 76.95 \text{ MPa}$$

$$\therefore \quad \text{FOS} = \frac{\sigma_{ys}}{\sigma_e} = \frac{353}{76.95} = 4.59.$$

Problem 2: A M.S. shaft having yield stress as 232 MPa is subjected to the following stresses. σ_x = 120 MPa, σ_y = –60 MPa and τ_{xy} = 36 MPa. Find the factor of safety using:

(i) Rankine's theory of failure,

(ii) Guest's theory of failure and

(iii) Von-Mises theory of failure.

Given data: Yield stress, σ_{ys} = 232 MPa

σ_x = 120 MPa, σ_y = –60 MPa and τ_{xy} = 36 MPa.

According to Rankine's theory or maximum normal stress theory of failure

$$\sigma_e = \frac{1}{2}\left[(\sigma_x+\sigma_y)+\sqrt{(\sigma_x-\sigma_y)^2+4\tau_{xy}^2}\right]$$

$$\sigma_e = \frac{1}{2}\left[(120-60)+\sqrt{[120-(-60)]^2+4(36)^2}\right] = 126.93 \text{ MPa}$$

$$\text{FOS} = \frac{\sigma_{ys}}{\sigma_e} = \frac{232}{126.93} = 1.828$$

(ii) According to Guest's theory or max shear stress theory of failure

$$\tau_e = \frac{1}{2}\sqrt{(\sigma_x-\sigma_y)^2+4\tau_{xy}^2}$$

or

$$\sigma_e = \sqrt{(\sigma_x-\sigma_y)^2+4\tau_{xy}^2} = \sqrt{[120-(-60)]^4+4(36)^2}$$

$$\sigma_e = 193.87 \text{ MPa}$$

$$\therefore \quad \text{FOS} = \frac{\sigma_{ys}}{\sigma_e} = \frac{232}{193.87} = 1.197$$

(ii) According to Hencky-Von-Mises theory or shear energy theory of failure

$$\sigma_e = \sqrt{\sigma_x^2+\sigma_y^2-\sigma_x\sigma_y+3\tau_{xy}^2}$$

$$= \sqrt{120^2+(-60)^2-120\times(-60)+3(36)^2} = 170.55 \text{ MPa}$$

$$\therefore \quad \text{FOS} = \frac{\sigma_{ys}}{\sigma_e} = \frac{232}{170.55} = 1.36$$

Problem 3: A machine member is subjected to the following stresses $\sigma_x = 150$ MPa, $\tau_{xy} = 24$ MPa. Find the equivalent stress as per the following theories of failure.

(i) Shear stress theory,

(ii) Normal stress theory,

(iii) Von-Mises theory.

Given data: $\sigma_x = 150$ MPa, $\tau_{xy} = 24$ MPa

$(\sigma_y = \text{Not given})$ $(\sigma_y = 0, \text{Not given})$

(i) According to maximum shear stress theory, equivalent stress $s_e = \sqrt{(\sigma_x - \sigma_y)^2 + 4\tau_{xy}^2}$

$$\sigma_e = \sqrt{150^2 + 4 \times 24^2} = 157.49 \text{ MPa}$$

(ii) According to maximum normal stress theory, equivalent stress

$$\sigma_e = \frac{1}{2}\left[(\sigma_x + \sigma_y) + \sqrt{(\sigma_x - \sigma_y)^2 + 4\tau_{xy}^2}\right]$$

$$\sigma_e = \frac{1}{2}\left[150 + \sqrt{150^2 + 4(24)^2}\right] = 153.75 \text{ MPa}$$

(iii) According to Von-Mises theory, equivalent stress

$$\sigma_e = \sqrt{\sigma_x^2 + \sigma_y^2 - \sigma_x\sigma_y + 4\tau_{xy}^2}$$

$$\sigma_e = \sqrt{150^2 + 3(24)^2} = 155.65 \text{ MPa.}$$

Problem 4: Find the diameter of a rod subjected to a bending moment of 3 kNm and a twisting moment of 1.8 kNm according to the following theories of failure, taking normal yield stress as 420 MPa and factor of safety as 3.

(i) Normal stress theory, (ii) Shear stress theory.

Given data: Bending moment, $M_b = 3$ kNm $= 3 \times 10^6$ N-mm

Twisting moment, $M_t = 1.8$ kNm $= 1.8 \times 10^6$ N-mm

Yield stress, $\sigma_{ys} = 420$ MPa FOS = 3

$$\therefore \quad \text{Allowable stress, } \sigma = \sigma_e = \frac{\sigma_{ys}}{\text{FOS}} = \frac{420}{3} = 140 \text{ MPa}$$

$$\text{Bending stress, } \sigma = \frac{M_b \cdot C}{I} = \frac{3 \times 10^6 \times d/2}{(\pi d^4/64)} = \frac{30.56 \times 10^6}{d^3}$$

$$\sigma = \frac{30.56 \times 10^6}{d^3} = \sigma_x$$

$$\text{Shear stress, } \quad \tau = \frac{M_t r}{J} = \frac{1.8 \times 10^6 \times d/2}{(\pi d^4/32)} = \frac{9.167 \times 10^6}{d^3} = \tau_{xy}$$

(i) According to maximum normal stress theory,

$$\sigma_e = \frac{1}{2}\left[(\sigma_x + \sigma_y + \sqrt{(\sigma_x - \sigma_y)^2} + 4\tau_{xy}^2\right]$$

(Here σ_y = 0, no stress in $\perp lr$ direction)

$$140 = \frac{1}{2}\left[\frac{30.56 \times 10^6}{d^3} + \sqrt{\left(\frac{30.56 \times 10^6}{d^3}\right)^2 + 4\left(\frac{9.167 \times 10^6}{d^3}\right)^2}\right]$$

$\therefore$ d = 61.834 mm

(ii) According to maximum shear stress theory

$$\sigma_e = \sqrt{(\sigma_x - \sigma_y)^2 + 4\tau_{xy}^2}$$

$$140 = \sqrt{\left(\frac{30.56 \times 10^6}{d^3}\right)^2 + 4\left(\frac{9.167 \times 10^6}{d^3}\right)^2}$$

$\therefore$ d = 63.376 mm

$\therefore$ Recommended diameter, d = 63.376 $\simeq$ 64 mm. (Take bigger one always).

Problem 5: A bolt is subjected to a tensile load of 18 kN and a shear load of 12 kN. The material has an yield stress of 328.6 MPa. Taking factor of safety as 2.5, determine the core diameter of bolt according to the following theories of failure.

(i) Rankine's theory,
(ii) Shear stress theory,
(iii) Shear energy theory and
(iv) Saint Venant's theory. Take Possion ratio = 0.298

Given data: Tensile load, F_T = 18 kN = 18×10^3 N

Shear load, F_s = 12 kN = 12×10^3 N

Yield stress, σ_{ys} = 328.6 MPa FOS = 2.5

$\therefore$ Allowable stress, $\sigma_e = \dfrac{\sigma_{ys}}{\text{FOS}} = \dfrac{328.6}{2.5}$ = 131.44 MPa.

$$\text{Tensile stress, } \sigma = \frac{F_T}{A} = \frac{18 \times 10^3}{A} = \sigma_x$$

$$\text{Shear stress, } \tau = \frac{F_s}{A} = \frac{12 \times 10^3}{A} = \tau_{xy}$$

(σ_y = 0, not given)

(i) According to Rankine's theory of failure

$$\sigma_e = \frac{1}{2}\left[\sigma_x + \sqrt{\sigma_x^2 + 4\tau_{xy}^2}\right]$$

$$131.44 = \frac{1}{2}\left[\frac{18\times10^3}{A} + \sqrt{\left(\frac{18\times10^3}{A}\right)^2 + 4\left(\frac{12\times10^3}{A}\right)^2}\right]$$

$$A = 182.59 = \frac{\pi d_c^2}{4}$$

$\therefore$ Core dia, dc = 15.25 mm

(ii) According to maximum shear stress theory,

$$\sigma_e = \sqrt{\sigma_x^2 + 4\tau_{xy}^2}$$

$$131.44 = \sqrt{\left(\frac{18\times10^3}{A}\right)^2 + 4\left(\frac{12\times10^3}{A}\right)^2}$$

$\therefore$ $$A = 228.24 = \frac{\pi d_c^2}{4}$$

$\therefore$ Core dia, dc = 17.05 mm

(iii) According to Von-Mises theory of failure

$$\sigma_e = \sqrt{\sigma_x^2 + 3\tau_{xy}^2}$$

$$131.44 = \sqrt{\left(\frac{18\times10^3}{A}\right)^2 + 3\left(\frac{12\times10^3}{A}\right)^2}$$

$$A = 209.19 = \frac{\pi d_c^2}{4}$$

$\therefore$ Core dia, dc = 16.32 mm

(iv) According to Saint Venant's theory of failure

$$\sigma_e = \frac{1}{2}\left[(1-v)(\sigma_x) + (1+v)\sqrt{\sigma_x^2 + 4\tau_{xy}^2}\right]$$

$$131.44 = \frac{1}{2}\left[(1-0.298)\frac{18\times10^3}{A} + (1+0.298)\sqrt{\left(\frac{18\times10^3}{A}\right)^2 + 4\left(\frac{12\times10^3}{A}\right)^2}\right]$$

$$A = 196.196 = \frac{\pi d_c^2}{4}$$

$\therefore$ Core dia, dc = 15.81 mm.

Problem 6: A SAE 1045 steel rod (σ_{ys} = 309.9 MPa) of 80 mm diameter is subjected to a bending moment of 3 kNm and torque T. Taking Factor of safety as 2.5, find the maximum value of torque T that can be safely carried by rod according to:

(i) Maximum normal stress theory,

(ii) Maximum shear stress theory.

Given data: Material SAE 1045.

$$\text{Yield stress, } \sigma_{ys} = 309.9 \text{ MPa}$$

$$\text{FOS} = 2.5 \text{ diameter } d = 80 \text{ mm}$$

$$\therefore \quad \text{Allowable stress, } \sigma_e = \frac{\sigma_{ys}}{\text{FOS}} = \frac{309.9}{2.5} = 123.96 \text{ MPa}$$

$$\text{Bending moment, } M_b = 3 \text{ kNm} = 3 \times 10^6 \text{ N-mm.}$$

$$\therefore \quad \text{Bending stress, } \sigma = \frac{M_b \cdot C}{I} = \frac{3 \times 10^6 (80/2)}{(\pi/64 \times 80^4)} = 59.68 \text{ MPa} = \sigma_x$$

$$\text{Torque, } M_t = T$$

$$\therefore \quad \text{Shear stress, } \tau = \frac{M_t \cdot r}{J} = \frac{T \cdot (80/2)}{(\pi/32 \times 80)^4} = (9.95 \times 10^{-6}) \text{ MPa}$$

$$\therefore \quad \tau = \tau_{xy} = (9.95 \; 10^{-6}) \; T$$

$$(\sigma_y = 0, \text{ not given})$$

(i) According to maximum normal stress theory

$$\sigma_e = \frac{1}{2}\left[\sigma_x + \sqrt{\sigma_x^2 + 4\tau_{xy}^2}\right]$$

$$123.96 = \frac{1}{2}\left[59.68 + \sqrt{59.68^2 + 4(9.95 \times 10^{-6} T)^2}\right]$$

$$\therefore \quad \text{Torque, } T = 8.971 \times 10^6 \text{ N-mm} = 8.971 \text{ kNm}$$

(ii) According to maximum shear stress theory

$$\tau_e = \frac{1}{2}\sqrt{\sigma_x^2 + 4\tau_{xy}^2}$$

Assuming, $\tau_e = 0.5 \; \sigma_e = 0.5 \times 123.96 = 61.98$ MPa

$$61.98 = \frac{1}{2}\sqrt{59.68^2 + 4(9.95 \times 10^{-6} T)^2}$$

$$\text{Torque, } T = 5.46 \times 10^6 \text{ N-mm} = 5.46 \text{ kNm.}$$

Problem 7: A stressed element is loaded as shown in Fig. 2.3. Determine the following:

(i) Von-Mises stress,

(ii) Maximum shear stress,

(iii) Maximum normal stress,

(iv) Octahedral shear stress.

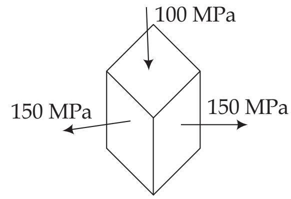

Fig. 2.3

Given data: Arranging in descending order $150 \geq 150 > -100$

$\therefore$ $\sigma_1 = 150$ MPa,

$\sigma_2 = 150$ MPa and $\sigma_3 = -100$ MPa (compressive)

(i) Von-Mises stress

$$\tau_e = \sqrt{\frac{(\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_3 - \sigma_1)^2}{2}}$$

$$= \sqrt{\frac{(150-150)^2 + (150+100)^2 + (-100-150)^2}{2}} = 250 \text{ MPa}$$

(ii) Maximum shear stress

$$\tau_{12} = \frac{\sigma_1 - \sigma_2}{2} = \frac{150-150}{2} = 0$$

$$\tau_{23} = \frac{\sigma_2 - \sigma_3}{2} = \frac{150-(-100)}{2} = 125 \text{ MPa}$$

$$\tau_{13} = \frac{\sigma_1 - \sigma_3}{2} = \frac{150-(-100)}{2} = 125 \text{ MPa}$$

$\therefore$ $\tau_{max} = 125$ MPa (max of these 3 values)

(iii) Maximum normal stress

$$\sigma_1 > \sigma_2 > \sigma_3$$

then $\sigma_{max} = \sigma_1 = 150$ MPa.

(iv) Octahedral shear stress

$$\tau_e = \frac{1}{3}\sqrt{(\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_3 - \sigma_1)^2}$$

$$= \frac{1}{3}\sqrt{(150-150)^2 + (150-100)^2 + (-100-150)^2} = 117.85 \text{ MPa.}$$

Problem 8: A material has a yield strength of 600 MPa. Compute the factor of safety for each of the failure theories for the each of the following stresses:

(i) $\sigma_1 = 420$ MPa, $\sigma_2 = 410$ MPa, $\sigma_3 = 0$,

(ii) $\sigma_1 = 420$ MPa, $\sigma_2 = 180$ MPa, $\sigma_3 = 0$,

(a) Von-mises theory, $\sigma_e = \sqrt{\dfrac{(\sigma_1 - \sigma_2)^{-2} + (\sigma_2 - \sigma_3)^2 + (\sigma_3 - \sigma_1)^2}{2}}$

$$\sigma_e = \sqrt{\frac{(420 - 180)^2 + (180 - 0)^2 + (420 - 0)^2}{2}} = 364.97 \text{ MPa}$$

$$\therefore \quad \text{FOS} = \frac{\tau_{ys}}{\tau_e} = \frac{600}{364.97} = 1.644$$

(b) Max. normal stress theory, $\sigma_e = \sigma_1 = 420$ MPa

$$\therefore \quad \text{FOS} = \frac{\sigma_{ys}}{\sigma_e} = \frac{600}{420} = 1.4286$$

(c) Max. shear stress theory

$$\tau_{12} = \frac{\sigma_1 - \sigma_2}{2} = \frac{420 - 180}{2} = 120 \text{ MPa}$$

$$\tau_{23} = \frac{\sigma_2 - \sigma_3}{2} = \frac{180}{2} = 90 \text{ MPa}$$

$$\tau_{13} = \frac{\sigma_1 - \sigma_3}{2} = \frac{420}{2} = 210 \text{ MPa}$$

$$\therefore \quad \tau_{max} = 210 \text{ MPa} = \tau_e$$

$$\therefore \quad \text{FOS} = \frac{\sigma_{ys}}{2\tau_{max}} = \frac{600}{2 \times 210} = 1.4286$$

(iii) $\sigma_1 = 0$, $\sigma_2 = -180$ MPa, $\sigma_3 = -420$ MPa

(a) Von-Mises theory, $\sigma_e = \sqrt{\dfrac{(0 + 180)^2 + (-180 + 420)^2 + (0 + 420)^2}{2}} = 364.96$

$$\therefore \quad \text{FOS} = \frac{\sigma_{ys}}{\sigma_e} = \frac{600}{364.93} = 1.644$$

(b) Max. normal stress theory, $\sigma_e = \sigma_1 = 0$

$$\therefore \quad \text{FOS} = \frac{600}{0} = \infty$$

But, in compression $\quad \text{FOS} = \dfrac{600}{420} = 1.4286$

(c) Max. shear stress theory

$$\tau_{12} = \frac{\sigma_1 - \sigma_2}{2} = \frac{0 + 180}{2} = 90 \text{ MPa}$$

$$\tau_{23} = \frac{\sigma_2 - \sigma_3}{2} = \frac{-180 + 4.20}{2} = +120 \text{ MPa}$$

$$\tau_{13} = \frac{\sigma_1 - \sigma_3}{2} = \frac{0 + 420}{2} = 210 \text{ MPa} \quad \therefore \tau_{max} = 210 \text{ MPa}$$

$$\text{FOS} = \frac{\sigma_{ys}}{2\tau_{max}} = \frac{600}{2 \times 210} = 1.4286.$$

Problem 9: A hot rolled bar has yield stress of 390 MPa. Compute the factor of safety for the following theories of failure:

(i) Maximum normal stress theory,
(ii) Maximum shear stress theory and
(iii) Distortion energy theory for the following states of stress.

(a) σ_1 = 225 MPa, σ_2 = 225 MPa, σ_3 = 0
(b) σ_1 = 225 MPa, σ_2 = 120 MPa, σ_3 = 0
(c) σ_1 = 225 MPa, σ_2 = 0, σ_3 = –120 MPa.

Given data: Yield stress, σ_{ys} = 390 MPa

$$\text{FOS} = \frac{\sigma_{ys}}{\sigma_e}$$

(a) σ_1 = 225 MPa, σ_2 = 225 MPa, σ_3 = 0

$\sigma_1 > \sigma_2 > \sigma_3$

(i) Maximum normal stress theory, $\sigma_e = \sigma_1$ = 225 MPa

$$\therefore \quad \text{FOS} = \frac{390}{225} = 1.733$$

(ii) Maximum shear stress theory

$$\tau_{12} = \frac{\sigma_1 - \sigma_2}{2} = \frac{225 - 225}{2} = 0$$

$$\tau_{23} = \frac{\sigma_2 - \sigma_3}{2} = \frac{225 - 0}{2} = 112.5 \text{ MPa}$$

$$\tau_{13} = \frac{\sigma_1 - \sigma_3}{2} = \frac{225 - 0}{2} = 112.5 \text{ MPa}$$

$$\therefore \quad \tau_e = \tau_{max} = 112.5 \text{ MPa}$$

and

$$\text{FOS} = \frac{\sigma_{ys}}{2\tau_{max}} = \frac{390}{2 \times 112.5} = 1.733$$