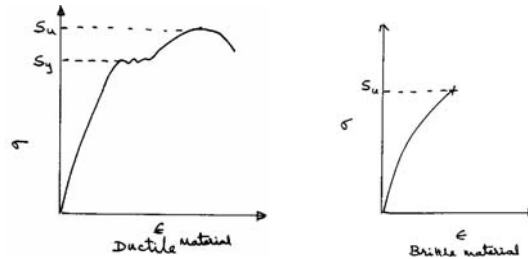


# Theories of Failure

Strength of a material or failure of the material is deduced generally from uni-axial tests from which stress strain characteristics of the material are obtained.

The typical stress-strain curves for ductile and brittle materials are shown below.



Material Strength parameters are  $S_y$  OR  $S_u$

# Theories of Failure

In the case of multidimensional stress at a point we have a more complicated situation present. Since it is impractical to test every material and every combination of stresses  $\sigma_1$ ,  $\sigma_2$ , and  $\sigma_3$ , a failure theory is needed for making predictions on the basis of a material's performance on the tensile test., of how strong it will be under any other conditions of static loading.

The “theory” behind the various failure theories is that *whatever is responsible for failure in the standard tensile test will also be responsible for failure under all other conditions of static loading.*

# Theories of Failure

The microscopic yielding mechanism in ductile material is understood to be due to relative sliding of materials atoms within their lattice structure. This sliding is caused by shear stresses and is accompanied by distortion of the shape of the part. Thus the yield strength in shear  $S_{sy}$  is strength parameter of the ductile material used for design purposes.

Generally used theories for Ductile Materials are:

- Maximum shear stress theory
- Maximum distortion energy theory.  
(von Mises-Hencky's theory).

# Theories of Failure

## The Maximum - Shear - Stress Theory

The Maximum Shear Stress theory states that *failure occurs when the maximum shear stress from a combination of principal stresses equals or exceeds the value obtained for the shear stress at yielding in the uniaxial tensile test.*

At yielding, in an uni-axial test, the principal stresses are

$$\sigma_1 = S_y; \sigma_2 = 0 \text{ and } \sigma_3 = 0.$$

Therefore the shear strength at yielding

$$S_{sy} = [\sigma_1 - (\sigma_2 \text{ or } \sigma_3 = 0)]/2. \text{ Therefore } S_{sy} = S_y/2$$

# Theories of Failure

## (Maximum Shear Stress theory)

To use this theory for either two or three-dimensional static stress in homogeneous, isotropic, ductile materials, first compute the three principal stresses ( $\sigma_1, \sigma_2, \sigma_3$ ) and the maximum shear stress  $\tau_{13}$  as

$$\tau_{\max} = \frac{(\sigma_1 - \sigma_2)}{2} = \frac{(\sigma_{p\max} - \sigma_{p\min})}{2}$$

Then compare the maximum shear stress to the failure criterion.

$$\tau_{\max} \leq S_{sy} \quad \text{OR} \quad \frac{(\sigma_{p\max} - \sigma_{p\min})}{2} \leq S_{sy}$$

The safety factor for the maximum shear-stress theory is given by

$$N = \frac{S_{sy}}{\tau_{\max}}$$

# Theories of Failure

## Distortion-Energy Theory OR The von Mises - Hencky Theory

It has been observed that a solid under hydro-static, external pressure (e.g. volume element subjected to three equal normal stresses) can withstand very large stresses.

When there is also energy of distortion or shear to be stored, as in the tensile test, the stresses that may be imposed are limited.

Since, it was recognized that engineering materials could withstand enormous amounts of hydro-static pressures without damage, it was postulated that a given material has a definite limited capacity to absorb energy of distortion and that any attempt to subject the material to greater amounts of distortion energy result in yielding failure.

Total Strain Energy: Assuming that the stress-strain curve is essentially linear up to the yield point, we can express the total strain energy at any point in that range as.

$$U = \frac{1}{2} \sigma \epsilon \quad (a)$$

Extending this to three dimensional case

$$U = \frac{1}{2} (\sigma_1 \epsilon_1 + \sigma_2 \epsilon_2 + \sigma_3 \epsilon_3) \quad (b)$$

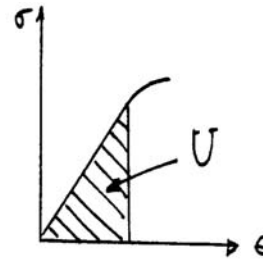
where  $\sigma_1$ ,  $\sigma_2$ , and  $\sigma_3$  are principal stresses and  $\epsilon_1$ ,  $\epsilon_2$  and  $\epsilon_3$  are principal strains.

Expressing strains in terms of stresses as.

$$\begin{aligned} \epsilon_1 &= 1/E [\sigma_1 - \nu (\sigma_2 + \sigma_3)] \\ \epsilon_2 &= 1/E [\sigma_2 - \nu (\sigma_1 + \sigma_3)] \\ \epsilon_3 &= 1/E [\sigma_3 - \nu (\sigma_2 + \sigma_1)] \end{aligned} \quad (c)$$

U, the total energy can be written by substituting (c) into (b)

$$U = (1/2E) [\sigma_1^2 + \sigma_2^2 + \sigma_3^2 - 2\nu (\sigma_1\sigma_2 + \sigma_2\sigma_3 + \sigma_3\sigma_1)] \quad (d)$$



Let  $U_h$  be energy due to volume change and  $U_d$  be energy due to distortion. Then we can express each of the principal stresses in terms of hydrostatic component ( $\sigma_h$ ), common to all the faces of volume element and distortion component ( $\sigma_{id}$ ) that is unique to each face.

$$U = U_h + U_d \quad (e)$$

$$\sigma_1 = \sigma_h + \sigma_{1d}$$

$$\sigma_2 = \sigma_h + \sigma_{2d} \quad (f)$$

$$\sigma_3 = \sigma_h + \sigma_{3d}$$

Adding the three principal stresses, gives

$$\sigma_1 + \sigma_2 + \sigma_3 = 3\sigma_h + (\sigma_{1d} + \sigma_{2d} + \sigma_{3d}) \quad (g)$$

For volumetric change with no distortion, the terms in the bracket of eqn (g) must be zero. Thus, we have

$$\sigma_h = 1/3 (\sigma_1 + \sigma_2 + \sigma_3) \quad (h)$$

Now,  $U_h$  can be obtained by replacing principal stresses in eqn (d) as

$$\begin{aligned} U_h &= (1/2E) [\sigma_h^2 + \sigma_h^2 + \sigma_h^2 - 2\nu (\sigma_h^2 + \sigma_h^2 + \sigma_h^2)] \\ &= 3/2 (1-2\nu)/E \sigma_h^2 \quad \text{.....(i)} \end{aligned}$$

Substituting eqn (h) into eqn (i), we obtain

$$U_h = (1-2\nu)/E [\sigma_1^2 + \sigma_2^2 + \sigma_3^2 - 2(\sigma_1\sigma_2 + \sigma_2\sigma_3 + \sigma_3\sigma_1)] \quad (j)$$

$$U = U_h + U_d$$

Distortion Energy

$$U_d = U - U_h$$

substituting (j) and (d) into (k), we obtain

$$U_d = (1 + \nu)/3E [\sigma_1^2 + \sigma_2^2 + \sigma_3^2 - \sigma_1\sigma_2 + \sigma_2\sigma_3 + \sigma_3\sigma_1]$$

In uniaxial stress-state at yield

$$\sigma_1 = S_y; \quad \sigma_2 = \sigma_3 = 0.$$

Therefore energy of distortion in uni axial stress state

$$U_{du} = (1 + \nu)/3E S_y^2 \quad (m)$$

Failure criterion according to distortion energy is

$$(1 + \nu)/3E S_y^2 = (1 + \nu)/3E [\sigma_1^2 + \sigma_2^2 + \sigma_3^2 - \sigma_1\sigma_2 - \sigma_2\sigma_3 - \sigma_3\sigma_1]$$

$$S_y = [\sigma_1^2 + \sigma_2^2 + \sigma_3^2 - \sigma_1\sigma_2 - \sigma_2\sigma_3 - \sigma_3\sigma_1]^{1/2}$$

.....

## Theories of Failure

“Distortion energy theory states that failure by yielding under a combination of stresses occurs when the energy of distortion equals or exceeds the energy of distortion in the tensile test when the yield strength is reached.”

According to theory failure criteria is

$$S_y = [\sigma_1^2 + \sigma_2^2 + \sigma_3^2 - \sigma_1\sigma_2 - \sigma_2\sigma_3 - \sigma_3\sigma_1]^{1/2}$$

For two dimensional stress state ( $\sigma_2 = 0$ ), the equations reduces to

$$S_y = [\sigma_1^2 + \sigma_3^2 - \sigma_3\sigma_1]^{1/2}$$

# Theories of Failure

It is often convenient in situations involving combined tensile and shear stresses acting at a point to define an effective stress that can be used to represent the stress combination.

**The von-Mises effective stress** ( $\sigma_e$ ) also sometimes referred to as **equivalent stress** is defined as the uniaxial tensile stress that would create the same distortion energy as is created by the actual combination of applied stresses.

$$\sigma_e = \left[ \sigma_1^2 + \sigma_2^2 + \sigma_3^2 - \sigma_1\sigma_2 - \sigma_2\sigma_3 - \sigma_3\sigma_1 \right]^{1/2}$$

$$\sigma_e = \left[ \sigma_1^2 + \sigma_3^2 - \sigma_3\sigma_1 \right]^{1/2}$$

*In terms of applied stresses in coordinate directions*

$$\sigma_e = \left[ \sigma_{xx}^2 + \sigma_{yy}^2 - \sigma_{xx}\sigma_{yy} + 3\tau_{xy}^2 \right]^{1/2}$$

$$\text{Safety factor } N = \frac{S_y}{\sigma_e}$$

## Static Failure Theories for Brittle Materials

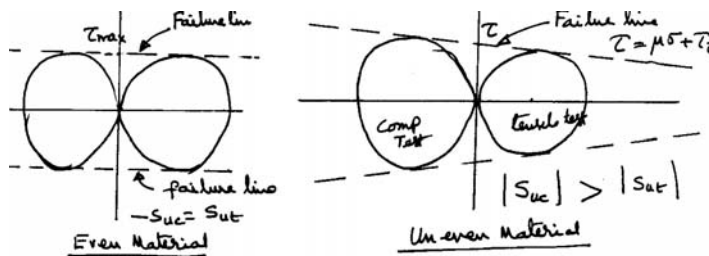
- Brittle materials fracture than yield.
- Brittle Fracture in tension is considered to be due to normal tensile stress alone and thus the maximum normal-stress theory is applicable.
- Brittle fracture in compression is due to some combination of normal compressive stress and shear stress.

### Even and Uneven Materials

- Some wrought materials, such as fully hardened tool steel, can be brittle. These materials tend to have compressive strength equal to their tensile strengths. They are called EVEN materials.
- Many cast materials, such as gray cast iron, are brittle but have compressive strengths much greater than their tensile strengths. These are called UNEVEN materials.

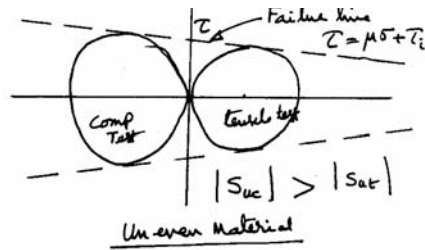
- For uneven materials; tensile strength is due to the presence of microscopic flaws in the castings, which when subjected to tensile loading, serve as nuclei for crack formation.
- when subjected to compressive stress, these flaws are pressed together, increasing the resistance to slippage from shear stresses.
- Gray cast irons typically have compressive strengths 3 to 4 times their tensile strengths and ceramics have even larger ratios.
- Another characteristics of some cast, brittle materials is that their shear strength can be greater than their tensile strength, falling between their compressive and tensile strengths.

Mohr's circles for both compression and tensile tests of an even and uneven materials are shown below.



The lines tangent to these circles constitute failure lines for all combinations of applied stress between the two circles. The area enclosed by the circles and the failure lines represent a safe zone.

In the case of even material, the failure lines are independent of the normal stresses and are defined by the maximum shear strength of the material. This is consistent with the maximum shear stress theory.

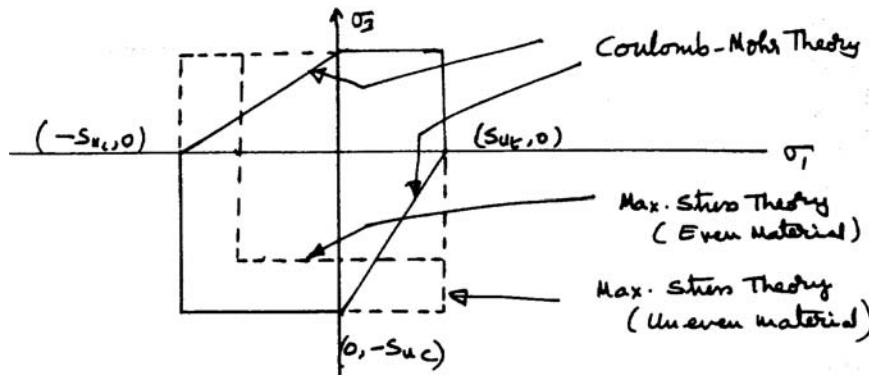


- For the uneven material, the failure lines are a function of both normal stresses and shear stresses. For compressive regime, as the normal stress component becomes increasingly negative (i.e. more compression) the material's resistance to shear stress increases.
- The interdependence between shear and normal stress is confirmed by experiment for cases where the compressive stress is dominant, specifically where the principal stress having the largest absolute value is compressive.
- However, experiments also show that in tensile-stress-dominated situations with uneven, brittle materials, failure is due to tensile stress alone. The shear stress appears not to be a factor in uneven materials if the largest absolute value is tensile.

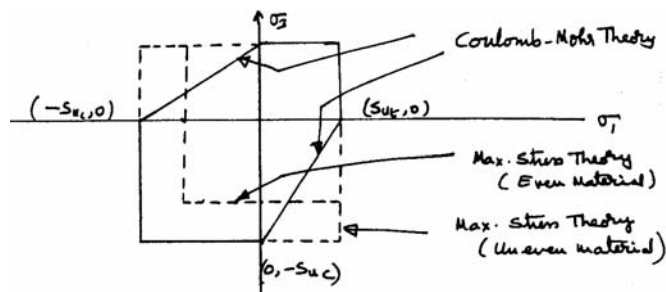
## Maximum Normal Stress Theory

The maximum normal stress theory, shown for even materials could be used as the failure criterion for brittle materials in static loading if compressive and tensile strengths were equal (even material).

The maximum-normal stress theory envelope for an uneven material as the asymmetric square of half-dimensions  $S_{ut}$ ,  $-S_{uc}$  is also shown. This failure envelope is only valid in the first and third quadrants as it does not account for the interdependence of normal and shear stresses which affects second and fourth quadrants.





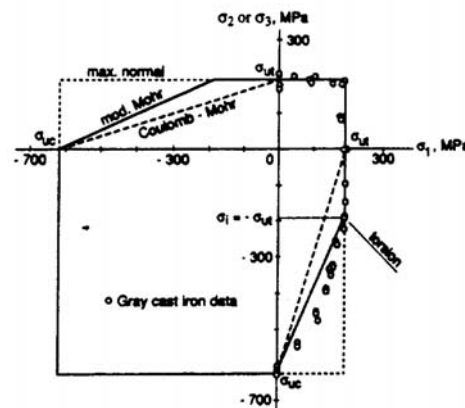


### Coulomb-Mohr theory

The Coulomb-Mohr envelope attempts to account for the interdependence by connecting opposite corners of these quadrants with diagonals.

The Figure shows some gray cast-iron experimental test data superposed on the theoretical failure envelopes.

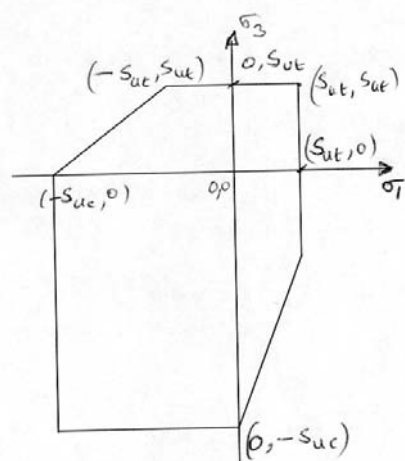
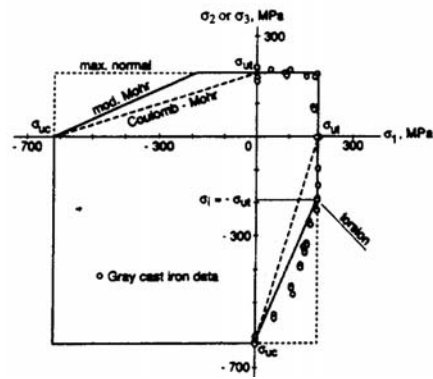
- The failures in the first quadrant fit the maximum normal-stress theory line.
- The failures in the fourth quadrant fall **inside** the maximum normal-stress line (indicating its unsuitability)
- Also experimental data fall **outside** the Coulomb-Mohr line (indicating its conservative nature).



This observation leads to a modification of the Coulomb-Mohr theory to make it better fit the observed data.

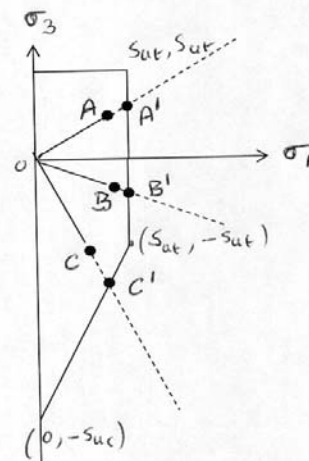
The actual failure data in the above figure follow the even material' maximum normal stress theory envelop down to a point  $S_{ut}$ ,  $-S_{ut}$  below the  $\sigma_1$  axis and then follow a straight line to 0,  $-S_{uc}$ . The set of lines shown by a solid line is the **modified-Mohr failure theory envelop**.

It is the preferred failure theory for uneven, brittle materials in static loading.



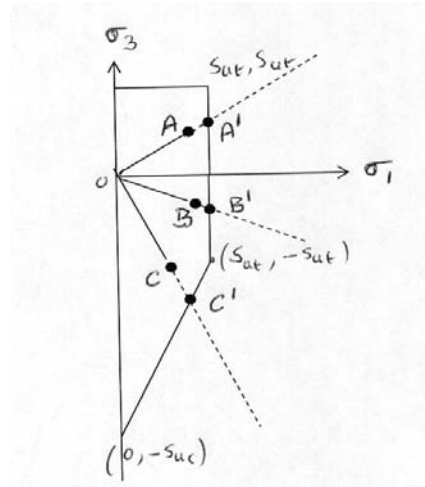
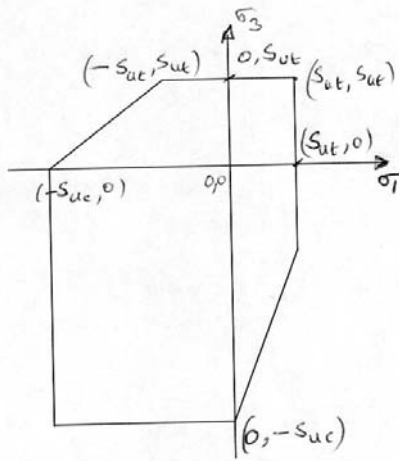
If the 2-D principal stresses are ordered  $\sigma_1 > \sigma_3$ ,  $\sigma_2 = 0$ , then only the first and fourth quadrants need to be drawn as shown in Figure

the figure depicts three plane stress conditions labeled A, B, and C.

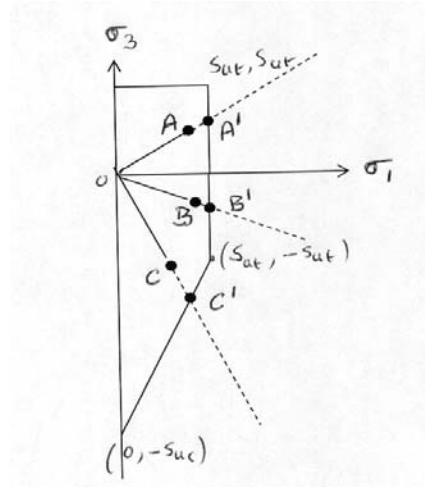
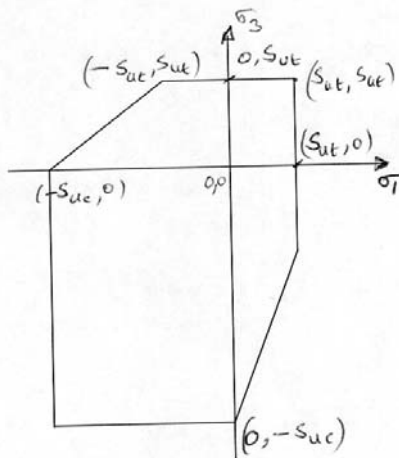


Point A represents any stress state in which the two non zero principal stresses  $\sigma_1$ ,  $\sigma_3$  are positive. Failure will occur when the load line OA crosses the failure envelop at  $A'$ . The safety factor for this situation can be expressed as

$$N = S_{ut}/\sigma_1$$



If the two nonzero principal stresses have opposite sign, then two possibilities exist for failure, as depicted by points B and C. The only difference between these two points is the relative values of their two stress components  $\sigma_1$  and  $\sigma_3$ . The load line OB exits the failure envelop at  $B'$  above the point  $(S_{ut}, -S_{ut})$  and the safety factor for this case is the same as the previous equation.



If the stress state is as depicted by point C, then the intersection of the load line OC and the failure envelop occurs at  $C'$  below the point  $(S_{ut}, -S_{ut})$ . The safety factor can be found by solving for the intersection between the load line OC and the failure line and is given by

$$N = \frac{S_{ut} S_{uc}}{S_{uc} \sigma_1 - S_{ut} (\sigma_1 + \sigma_3)}$$

If the stress state is in the fourth quadrant both of these equations should be checked and the resulting smaller safety factor used.

