Lec 01-04-2020

Proposition: Let X be a complete M.S and let $S, T : X \to CB(X)$ be two contraction mappings each having Lipschitz constant $k<1$,

i.e. $H(Sx, Sy) \leq k\, d(x, y)$

$H(Tx, Ty) \leq k\, d(x, y)$

$\forall\, x, y \in X$. Then

$$H(F(S), F(T)) \leq (1-k)^{-1} \sup_{x \in X} H(Sx, Tx).$$

Proof:

Let $\epsilon > 0$ and $c > 0$ be such that $c \sum_{n=1}^{\infty} n k^n < 1$. Let $x_0 \in F(S)$ and $x_1 \in Tx_0$ such that

$$d(x_0, x_1) \leq H(Sx_0, Tx_0) + \epsilon \quad \text{—(1)}$$

Now, choose $x_2 \in Tx_1$ such that

$$d(x_1, x_2) \leq H(Tx_0, Tx_1) + \frac{c\epsilon k}{1-k}$$

$$\leq k\, d(x_0, x_1) + \frac{c\epsilon k}{1-k}. \quad \text{—(2)}$$

In the same way, we construct a sequence (x_n) by $x_{n+1} \in Tx_n$ and

$$d(x_{n+1}, x_n) \leq k\, d(x_n, x_{n-1}) + \frac{c\epsilon k^n}{1-k} \quad \text{—(3)}$$

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Put $\eta = \frac{c\epsilon}{1-k}$. Now from (3)

$$d(x_{n+1}, x_n) \leq k d(x_n, x_{n-1}) + \eta k^n$$
$$\leq k\left(k d(x_{n-1}, x_{n-2}) + \eta k^{n-1}\right) + \eta k^n$$
$$= k^2 d(x_{n-1}, x_{n-2}) + 2\eta k^n$$
$$---$$
$$\leq k^n d(x_0, x_1) + n\eta k^n. \quad -(4)$$

Now since $\sum_{n=1}^{\infty} k^n < \infty$ and $\sum_{n=1}^{\infty} n k^n < \infty$, it follows that (x_n) is Cauchy sequence in X and since X is complete ~~so~~ so ~~ff~~ there is $u \in X$ such that $x_n \to u$. Continuity of T implies that

$$\lim_{n \to \infty} H(Tx_n, Tu) = 0$$

Hence ~~x_n~~ $u \in F(T)$ because $x_{n+1} \in Tx_n$.

Now,

$$d(x_0, u) \leq d(x_0, x_1) + d(x_1, x_2) + ---$$
$$= \sum_{n=0}^{\infty} d(x_n, x_{n+1})$$
$$\leq \sum_{n=0}^{\infty} k^n d(x_0, x_1) + \eta \sum_{n=0}^{\infty} n k^n$$
$$\leq (1-k)^{-1} d(x_0, x_1) + \eta \sum_{n=0}^{\infty} n k^n$$

$$\leq (1-k)^{-1}\left[d(x_0, x_1) + \epsilon\right] \because c\sum_{n=1}^{\infty} n k^{n} < 1.$$

$$\leq (1-k)^{-1}\left[H(Sx_0, Tx_0) + 2\epsilon\right] \quad \text{--- (5)}$$

$$\Rightarrow d(x_0, u) \leq (1-k)^{-1}\left[H(Sx_0, Tx_0) + 2\epsilon\right] \quad \text{--- (6)}$$

Now, interchanging the role of S & T, we have for each $y_0 \in F(T)$, there exist $y_1 \in Sy_0$ and $u \in F(S)$ such that

$$d(y_0, u) \leq (1-k)^{-1}\left[H(Sy_0, Ty_0) + 2\epsilon\right] \quad \text{--- (7)}$$

Since $\epsilon > 0$ is arbitrary, we have

$$H\left(F(S), F(T)\right) \leq (1-k)^{-1} \sup_{x \in X} H(Sx, Tx). \quad \text{--- (8)}$$

Theorem:

Let X be a Complete metric Space and let $T_n : X \to CB(X)$ $(n = 1, 2, -)$ be Contraction mappings with Lipschitz Constants $k < 1$,

i.e. $H(T_n x, T_n y) \leq k d(x, y) \; \forall x, y \in X$ — (9)

and $n \in \mathbb{N}$.

If $\lim_{n \to \infty} H(T_n x, T_0 x) = 0$ uniformly for $x \in X$, then

$$\lim_{n \to \infty} H(F(T_n), F(T_0)) = 0 \quad \text{— (10)}$$

Proof: Since (T_n) Converges to T_0 uniformly, and T_n is multivalued Contraction, then

$$H(T_n x, T_n y) \leq k d(x, y) \quad \text{— (11)}$$

implies

$$\exists \; H(T_0 x, T_0 y) \leq k d(x, y) \quad \text{— (12)}$$

i.e. T_0 is multivalued Contraction. Therefore there exist $n_0 \in \mathbb{N}$ Such that

$$\sup_{x \in X} H(T_n x, T_0 x) < (1-k)\epsilon \;\; \forall n \geq n_0 \quad \text{— (13)}$$

$$\Rightarrow (1-k)^{-1} \sup_{x \in X} H(T_n x, T_0 x) < \epsilon$$

$$\Rightarrow \quad H(F(T_n), F(T_0)) < \epsilon \ \forall n \geq n_0$$

q —— d

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