Definition 2.4 Given $\epsilon > 0$, a metric space (X, ρ) is said to be ϵ-chainable if and only if given $u, v \in X$ there is an ϵ-chain from u and v, i.e., there exists a finite set of points $x_0, x_1, \ldots, x_n$ with $u = x_0$ and $v = x_n$ such that $\rho(x_{i-1}, x_i) < \epsilon$ for all $i = 1, 2, \ldots, n$.

A set-valued mapping $T : (X, \rho) \to CB(X)$ is said to be (ϵ, α)-uniformly locally contractive with $\epsilon > 0$ and $0 < \alpha < 1$ provided that $h(T(x), T(y)) \leq \alpha\rho(x, y)$ whenever $x, y \in X$ with $\rho(x, y) < \epsilon$.

Theorem 2.29 *Let (X, ρ) be a complete ϵ-chainable metric space and $T : X \to K(X)$ an (ϵ, α)-uniformly locally contractive set-valued mapping. Then T has a fixed point.*

Proof. For any $x, y \in X \times X$, we define

$$\rho_\epsilon(x, y) = \inf\{\sum_{i=1}^{n} \rho(x_{i-1}, x_i) : x_0 = x, x_1, \cdots . x_n = y \text{ is an } \epsilon\text{-chain from } x \text{ to } y\}.$$

We can easily verify that ρ_ϵ is a metric on X. ρ_ϵ also satisfies

(2.7.3) $$\rho(x, y) \leq \rho_\epsilon(x, y) \quad \text{for all } x, y \in X;$$

and

(2.7.4) $$\rho(x, y) = \rho_\epsilon(x, y) \quad \text{for all } x, y \in X: \text{ with } \rho(x, y) < \epsilon.$$

(2.7.3) follows from the triangle inequality:

$$\rho(x,y) \leq \rho(x,x_1) + \rho(x_1,x_2) + \cdots + \rho(x_{n-1},x_n).$$

It also follows from (2.7.3), (2.7.4) and the completeness of (X,ρ) that (X,ρ_ϵ) is complete. Let h_ϵ be the Hausdorff metric on $K(X)$ derived from ρ_ϵ. We can easily see that if $A, B \in K(X)$ such that $h(A,B) < \epsilon$ then $h_\epsilon(A,B) = h(A,B)$. Now we will show that $T : X \to K(X)$ is a set-valued contraction mapping of (X,ρ_ϵ) into $(K(X), h_\epsilon)$ with contraction constant α. Let $x, y \in X$ and $x_0 = x, x_1, \ldots, x_n = y$ be an ϵ-chain from x to y. Since $\rho(x_{i-1},x_i) < \epsilon$ for all $i = 1, 2\ldots, n$, we have

$$h(T(x_{i-1}),T(x_i)) \leq \alpha\rho(x_{i-1},x_i) < \epsilon \quad \text{for all } i = 1,2\ldots,n.$$

Hence

$$h_\epsilon(T(x),T(y)) \leq \sum_{i=1}^{n} h_\epsilon(T(x_{i-1}),T(x_i)) = \sum_{i=1}^{n} h(T(x_{i-1}),T(x_i)) \leq \alpha.$$

Now since $x_0 = x, x_1, \ldots, x_n = y$ is an arbitrary ϵ-chain, it follows that $h_\epsilon(T(x),T(y)) \leq \alpha\rho_\epsilon(x,y)$ for all $x, y \in X$. Thus T is a set-valued contraction mapping of (X,ρ_ϵ) into $(K(X),h_\epsilon)$ with contraction constant α. Hence by Theorem 2.28, T has a fixed point. □

Remark 2.20 Similar results for a single-valued mapping was first obtained by Edelstein (1961).

Definition 2.5 Let X and Y be topological spaces and $T : X \to 2^Y$ a set-valued mapping with $T(x) \neq \emptyset$ for each $x \in X$, i.e.,$T : X \to 2^Y \setminus \{\emptyset\}$. T is said to be upper semi-continuous at $x_0 \in X$ if given an open set G containing $f(x_0)$, there exists an open neighborhood $U(x_0)$ of x_0 such that $T(U(x_0) \subset G$, where for any subset A of X, $T(A) = \cup_{x\in A} T(x)$. T is said to be upper semi-continuous if T is upper semi-continuous at each point $x \in X$.

A set-valued mapping $T : X \to 2^Y \setminus \{\emptyset\}$ is said to be lower semi-continuous at $x_0 \in X$ if given an open set G in Y with $T(x_0) \cap G \neq \emptyset$, there exists an open neighborhood $U(x_0)$ of x_0 such that $T(x) \cap G \neq \emptyset$ for each $x \in U(x_0)$. T is said to be lower semi-continuous if T is lower semi-continuous at each point $x \in X$.

Lemma 2.18 *Let X and Y be non-empty sets and $T : X \to 2^Y \setminus \{\emptyset\}$ a set-valued mapping. Then for any non-empty set A of Y,*

$$X \setminus \{\cup_{y\in A} T^{-1}(y)\} = \{x \in X : T(x) \subset Y \setminus A\}.$$

Proof. Let u belong to the left-hand side (of the above expression). Then $u \notin T^{-1}(y)$ for any $y \in A$. This implies that $y \notin T(u)$ for any $y \in A$. Thus $T(u) \subset Y \setminus A$ which implies that u belongs to the right-hand side.

Next let u belong to the right-hand side. Then $T(u) \subset Y \setminus A$. It follows that $u \notin T^{-1}(y)$ for any $y \in A$. This implies that $u \notin \cup_{y\in A} T^{-1}(y)$; i.e., u belongs to the left-hand side. □

For any subset A of Y, let $T_+(A) = \{x \in X : T(x) \cap A \neq \emptyset\}$.

Lemma 2.19 *For any subset A of Y, $T_+(A) = \cup_{y \in A} T^{-1}(y)$.*

Proof. Let $x \in A$. Then $T(x) \cap A \neq \emptyset$. Let $y \in T(x) \cap A$. This implies that $x \in T^{-1}(y)$ with $y \in A$, i.e., $x \in T^{-1}(y) \subset \cup_{y \in A} T^{-1}(y)$. Next, let $u \in \cup_{y \in A} T^{-1}(y)$. Then $u \in T^{-1}(y)$ for some $y \in A$, i.e., $y \in T(u)$ with $y \in A$, i.e., $u \in T_+(A)$. □

Theorem 2.30 *Let X and Y be topological spaces and $T : X \to 2^Y \setminus \{\emptyset\}$ a set-valued mapping. Then the following statements are equivalent:*

(a) T is upper semi-continuous;
(b) For each open set G in Y, $T^+(G) = \{x \in X : T(x) \subset G\}$ is open in X;
(c) For each closed set F in Y, $T^{-1}(F) = \cup_{y \in F} T^{-1}(y)$ is closed in X, where $T^{-1}(y) = \{x \in X : y \in T(x)\}$;
(d) For each $x \in X$ and every net $\{x_\delta : \delta \in D\}$ in X converging to x, and each open set G in Y with $T(x) \subset G$, $T(x_\delta) \subset G$ eventually, i.e., $T(x_\delta) \subset G$ for all $\delta_0 \geq \delta$ for some $\delta_0 \in D$.

Proof. First let (a) hold. Let G be an open set in Y and $x_0 \in T^+(G)$. By upper semicontinuity of T at x_0, there exists an open neighborhood $U(x_0)$ of x_0 such that $T(U(x_0)) \subset G$. Hence $U(x_0) \subset T^+(G)$ and hence $T^+(G)$ is an open set. Thus (a) implies (b).

Now let $T^+(G)$ be open for every open set G in Y. Let $x_0 \in X$ and G be an open set containing $T(x_0)$. $T^+(G)$ is an open neighborhood of x_0 and $T(T^+(G)) \subset G$. Hence T is upper semicontinuous at x_0. Since x_0 is arbitrary, (b) implies (a). That $(b) \iff (c)$ is evident from Lemma 2.18.

Now we prove that (b) implies (d). Let $\{x_\delta : \delta \in D\}$ be a net converging to $x \in X$ and G an open subset of Y with $T(x) \subset G$. Then by (b), $T^+(G)$ is open and $x \in T^+(G)$. Since $x_\delta \to x$, $x_\delta \in T^+(G)$ eventually. Hence $T(x_\delta) \subset G$ eventually.

Finally, we prove that (d) implies (b). Let H be an open set in Y. If possible, let $T^+(H)$ be not open. Then there is a point $x_0 \in X$ such that $x_0 \in T^+(H)$ is not an interior point of $T^+(H)$. Let $D_0 = \mathcal{N}(x_0)$ be the system of all open neighborhoods of x_0. Then D_0 ordered partially by inclusion is a directed set. We choose $x_\delta \in D_0$ such that $x_\delta \notin T^+(H)$. This is possible as x_0 is not an interior point of $T^+(H)$. Evidently $\{x_\delta : \delta \in D_0\}$ is a net converging to x_0 and $T(x_0) \subset H$. Hence by (d), $T(x_\delta) \subset H$ eventually, which contradicts the fact that $x_\delta \notin T^+(H)$ for all $\delta \in D_0$. □

Theorem 2.31 *Let X and Y be topological spaces and $T : X \to 2^Y \setminus \{\emptyset\}$ a set-valued mapping. Then the following statements are equivalent:*

(a) T is lower semi-continuous;
(b) For each open set G in Y, $T_+(G)$ is open in X;
(c) For each closed set F in Y, $T^+(F)$ is closed in X;

(*d*) *For each $x \in X$ and each net $\{x_\delta : \delta \in D\}$ in X converging to x, and each open set G in Y with $T(x) \cap G \neq \emptyset$, $T(x_\delta) \cap G \neq \emptyset$, i.e., $T(x_\delta) \cap G \neq \emptyset$ for all $\delta \geq \delta_0$ for some $\delta_0 \in D$.*

Proof. Let (*a*) hold. Let G be an open set in Y and $x_0 \in T_+(G)$. By the lower semi-continuity of T at x_0, there exists an open neighborhood $U(x_0)$ of x_0 such that $T(x) \cap G \neq \emptyset$ for each $x \in U(x_0)$, i.e., $U(x_0) \subset T_+(G)$. Hence $T_+(G)$ is open in X. Now by virtue of Lemma 2.19, $\cup_{y \in G} T^{-1}(y)$ is open. Thus (*a*) implies (*b*). We now suppose that (*b*) holds. Let $x_0 \in X$ and G be an open set in Y such that $T(x_0) \cap G \neq \emptyset$. Then $x_0 \in T_+(G)$. Hence by virtue of Lemma 2.19 and (*b*), $T_+(G) = \cup_{y \in G} T^{-1}(y)$ is an open neighborhood of x_0 in X. It follows that T is is lower semi-continuous at x_0. Thus (*b*) implies (*a*).

Now that (*b*) $\iff$ (*c*) follows from the Lemma 2.18. Finally, by giving similar argument as given in Theorem 2.30 we can prove that (*b*) implies (*d*). □

Theorem 2.32 (*a*) *Let X and Y be topological spaces with Y a T_3 space and $T : X \to 2^Y \setminus \{\emptyset\}$ a set-valued upper semi-continuous mapping with closed values. Then the graph $T = G(T) = \{(x, y) \in X \times Y : y \in T(x)\}$ is closed.*

(*b*) *Let X and Y be topological spaces with Y compact and $T : X \to 2^Y \setminus \{\emptyset\}$ a set-valued mapping with closed graph (i.e., $G(T)$ is closed). Then T is upper semi-continuous.*

Proof. (*a*) Let $\{(x_\delta, y_\delta) : \delta \in D\}$ be a net in $G(T)$ converging to (x, u). If possible, let $(x, u) \notin G(T)$, i.e., $u \notin T(x)$. Since $T(x)$ is closed and Y is T_3, there exist open sets G_1 containing u and G_2 containing $T(x)$ with $G_1 \cap G_2 = \emptyset$. Now since T is upper semi-continuous, by Theorem 2.30 (*d*), $T(x_\delta) \subset G_2$ eventually. But since $y_\delta \in T(x_\delta)$ for each $\delta \in D$, $y_\delta \in G_2$ ventually. This contradicts the fact that $y_\delta \to u$ as $u \in G_1$ and $G_1 \cap G_2 = \emptyset$.

(*b*) If possible, let T be not upper semi-continuous at a point $x \in X$. Let $\{x_\delta : \delta \in D\}$ be a net converging to x. Then there must exist, by Theorem 2.30 (*d*), at least one open set G in Y with $f(x) \subset G$ such that $T(x_\delta) \not\subset G$ eventually. We can choose a subnet $\{x_{\delta'} : \delta' \in D'\}$ of the net $\{x_\delta : \delta \in D\}$ such that $T(x_{\delta'}) \not\subset G$ for each $\delta' \in D'$. For this we can select $u_{\delta'}$ from each $T(x_{\delta'})$ such that $u_{\delta'} \notin G'$. Now since G' is compact, $\{u_{\delta'} : \delta' \in D'\}$ has a subnet $\{u_{\delta''} : \delta'' \in D''\}$ converging to a point $u \in G'$. Clearly, $\{(x_{\delta''}, u_{\delta''} : \delta'' \in D''\}$ is a net in $G(T)$ which converges to $(x, u) \notin G(T)$ as $u \notin T(x) \subset G$. □

Theorem 2.33 *Let X and Y be topological spaces, $T : X \to K(Y)$ a set-valued upper semi-continuous mapping and K a compact subset of X. Then $T(K) = \cup_{x \in K} T(x)$ is a compact subset of Y.*

Proof. Let $\{G_\alpha : \alpha \in I\}$ be an open covering of $T(K)$. Then for $\alpha \in I$, there exists an open set H_α in Y such that $G_\alpha = T(K) \cap H_\alpha$. For each $x \in K$, $T(x)$ being compact is covered by a finite number of H_α, say $H_{\alpha_1}, H_{\alpha_2}, \ldots, H_{\alpha_n}$ with $\bigcup_{i=1}^n H_{\alpha_i} \supset T(x)$.

We set $H_x = \bigcup_{i=1}^{n} H_{\alpha_i}$. Then $\{T^+(H_x) : x \in K\}$ is an open covering of K. Since K is compact, there is a finite subcovering $T^+(H_{x_1}), T^+(H_{x_2}), \ldots, T^+(H_{x_n})$ of K. It follows that $\{H_{x_i} : i = 1, 2, \ldots, n\}$ cover $T(K)$ and $\bigcup_{i=1}^{n} G_{x_i} = \bigcup_{i=1}^{n}(H_{x_i} \cap (K)) = T(K)$. Hence $\{G_{x_i} : i = 1, 2, \ldots, n\}$ is a subcover $T(K)$. □

Theorem 2.34 *Let X and Y be topological spaces, $T : X \to 2^Y \setminus \{\emptyset\}$ a set-valued upper semi-continuous (or lower semi-continuous) mapping, K is a connected subset of Y and $T(x)$ is a connected subset of Y for each $x \in K$. Then $T(K)$ is a connected subset of Y.*

Proof. If possible, we suppose that $T(K)$ is not connected. Then there are two disjoint non-empty open subsets G_1 and G_2 of $T(K)$ such that $T(K) = G_1 \cap G_2$. Hence there exist two non- empty open subsets H_1 and H_2 of Y such that $G_1 = H_1 \cap T(K)$ and $G_2 = H_2 \cap T(K)$. Thus $T(K) \subset H_1 \cup H_2$. By upper semi-continuity of T, $T^+(H_1)$ and $T^+(H_2)$ are open sets of X. Let $x \in K$, then $T(x) \subset H_1 \cup H_2$. But since $T(x)$ is connected, it follows that $T(x)$ is contained in either H_1 or in H_2. Thus $K \subset T^+(H_1) \cup T^+(H_2)$. Obviously, $T^+(H_1) \cap T^+(H_2) = \emptyset$ and $K \cap T^+(H_1) \neq \emptyset$ and $K \cap T^+(H_2) \neq \emptyset$. Thus K is not connected, which is a contradiction. Hence $T(K)$ must be connected. □

For the proof in the case of lower semi-continuity, we refer to Klein and Thompson (1984, p. 90).

Theorem 2.35 *Let X, Y and Z be topological spaces, and $T_1 : X \to 2^Y \setminus \{\emptyset\}$ and $T_2 : Y \to 2^Z \setminus \{\emptyset\}$ are set-valued upper semi-continuous mappings. Then the set-valued mapping $T : X \to 2^Z \setminus \{\emptyset\}$ defined by $T = T_2 \circ T_1 = T_2(T_1(x))$ is upper semi-continuous.*

Proof. Let G be an open subset of G. Then $T^+(G) = (T_2 \circ T_1)^+(G) = \{x \in X : (T_2 \circ T_1)(x) \subset G\} = \{x \in X : T_1(x) \subset T_2^+(G)\} = T_1^+[T_2^+(G)]$ is an open subset of X. Hence T is upper semi-continuous. □

2.7.1 *End Points*

Definition 2.6 For set-valued mapping $T : X \to 2^X \setminus \{\emptyset\}$, a point $x_0 \in X$ us said to be en end point of T if $T(x_0) = \{x_0\}$. Let X be a topological space. Then an upper semi-continuous set-valued mapping $T : X \to 2^X \setminus \{\emptyset\}$ with closed values is said to be a topological contraction if, for each non-empty closed subset A of X with $T(A) = A$, A is a singleton set, i.e., A is an end point of T.

Theorem 2.36 *Let X be a compact Hausdorff topological space and $T : X \to 2^X \setminus \{\emptyset\}$ a set-valued topological contraction. Then T has a unique end point $x_0 \in X$ such that $\{x_0\} = \bigcap_{n=0}^{\infty} T^n(X)$, where $T^0(X) = X$ and $T^n(X) = T(T^{n-1}(X))$ for $n = 1, 2, \ldots$.*

Proof. For each $n = 0, 1, 2, \ldots$, let $F_n = T^n(X)$. Since T is upper semi-continuous with closed (and hence compact) values, by Theorem 2.33 F_n is compact for each