Let us assume that there exist constants $c > 0$ and $\eta > 0$ such that

$$\forall\, y_n \in B(f(x), \eta) \cap M_n, \quad B_Y \subset cf'(x)(B_X) - T_{M_n}(y_n)$$

Then x belongs to $\mathrm{Liminf}_{n\to\infty} f^{-1}(M_n)$ and a similar statement holds true for upper limits.

This result is contained in the following, more general,

Theorem 1.2.9 *Let X and Y be two Banach spaces. We consider a sequence of closed subsets $M_n \subset Y$ and a map $f : X \mapsto Y$ differentiable on a neighborhood $\mathcal{U}$ of $f^{-1}(\mathrm{Liminf}_{n\to\infty} M_n)$ (respectively $f^{-1}(\mathrm{Limsup}_{n\to\infty} M_n)$) such that the derivatives $f'(x)$ are uniformly bounded on $\mathcal{U}$. Let us assume that there exist constants $c > 0$ and $\eta > 0$ such that for every $x \in \mathcal{U}$,*

$$\forall\, y_n \in B(f(x), \eta) \cap M_n, \quad B_Y \subset cf'(x)(B_X) - T_{M_n}(y_n)$$

Then

$$\mathrm{Liminf}_{n\to\infty} f^{-1}(M_n) = f^{-1}(\mathrm{Liminf}_{n\to\infty} M_n)$$

respectively

$$\mathrm{Limsup}_{n\to\infty} f^{-1}(M_n) = f^{-1}(\mathrm{Limsup}_{n\to\infty} M_n)$$

We postpone the proof of the last theorem to Chapter 3.

1.3 Set-Valued Maps

Sequences of subsets can be regarded as set-valued maps defined on the set $\mathbf{N}$ of integers.

Naturally, we can replace $\mathbf{N}$ by a metric (or even, topological) space X, and sequences of subsets $n \rightsquigarrow K_n$ by set-valued maps $x \rightsquigarrow F(x)$ (which associates with a point x a subset $F(x)$) and adapt the definition of upper and lower limits to this case, called the *continuous case*.

Before proceeding further, we recall in this section the basic definitions dealing with set-valued maps, also called *multifunctions*, *multivalued functions*, *point to set maps* or *correspondences*.

characterization of Set-valued mapping through its graph.

Definition 1.3.1 *Let X and Y be metric spaces. A set-valued map F from X to Y is characterized by its graph Graph(F), the subset of the product space $X \times Y$ defined by*

$$\text{Graph}(F) := \{(x, y) \in X \times Y \mid y \in F(x)\}$$

We shall say that $F(x)$ is the image *or the* value *of F at x.*

A set-valued map is said to be nontrivial *if its graph is not empty, i.e., if there exists at least an element $x \in X$ such that $F(x)$ is not empty.*

We say that F is strict *if all images $F(x)$ are not empty. The* domain *of F is the subset of elements $x \in X$ such that $F(x)$ is not empty:*

$$*_1 \quad \text{Dom}(F) := \{x \in X \mid F(x) \neq \emptyset\}$$

The image *of F is the union of the images (or values) $F(x)$, when x ranges over X:*

$$\text{Im}(F) := \bigcup_{x \in X} F(x)$$

The inverse *F^{-1} of F is the set-valued map from Y to X defined by*

$$*_2 \quad x \in F^{-1}(y) \iff y \in F(x) \iff (x, y) \in \text{Graph}(F)$$

$*_3$ The domain of F is thus the image of F^{-1} and coincides with the projection of the graph onto the space X and, in a symmetric way, the image of F is equal to the domain of F^{-1} and to the projection of the graph of F onto the space Y.

If K is a subset of X, we denote by $F|_K$ the *restriction* of F to K, defined by

$$F|_K(x) := \begin{cases} F(x) & \text{if } x \in K \\ \emptyset & \text{if } x \notin K \end{cases}$$

Let $\mathcal{P}$ be a property of a subset (for instance, closed, convex, etc..) Since we shall emphasize the symmetric interpretation of a set-valued map as a graph (instead of a map from a set to another one), we shall say as a general rule that *a set-valued map satisfies property $\mathcal{P}$ if and only if its graph satisfies it.*

For instance, a set-valued map is said to be closed (respectively convex, closed convex, measurable, monotone, maximal monotone)

Def 1: Let X & Y be two metric spaces. A set valued map $F: X \to Y$ is a map that associates with any $x \in X$, a subset $F(x)$ of Y. The subsets $F(x)$ are called the images or the values of F.

~~The~~

$*_1$ If Dom$(F) = X$, then F is called strict map.

$*_2$ $F^{-1}(y) = \{x \in X : (x, y) \in G(F)\}$

$*_3$ If $R(F) = \text{Dom}(F^{-1})$ then Dom(F) is the projection of Graph(F) onto Y. ~~and~~

[illegible]

if and only if its graph is closed (respectively convex, closed convex, measurable, monotone, maximal monotone.)

If the images of a set-valued map F are closed, convex, bounded, compact, and so on, we say that F is *closed-valued*, *convex-valued*, *bounded-valued*, *compact-valued*, and so on.

When $\star$ denotes an operation on subsets, we use the same notation for the operation on set-valued maps, which is defined by

$$F_1 \star F_2 : x \rightsquigarrow F_1(x) \star F_2(x)$$

We define in that way $F_1 \cap F_2$, $F_1 \cup F_2$, $F_1 + F_2$ (in vector spaces), etc. Similarly, if α is a map from the subsets of Y to the subsets of Y, we define

$$\alpha(F) : x \rightsquigarrow \alpha(F(x))$$

For instance, we shall use $\overline{F}$, $\text{co}(F)$, etc., to denote the set-valued maps $x \rightsquigarrow \overline{F}(x)$, $x \rightsquigarrow \text{co}(F(x))$, etc.

We shall write

$$F \subset G \iff \text{Graph}(F) \subset \text{Graph}(G)$$

and say that G is an *extension of* F.

Let us mention the following elementary properties:

$$\begin{cases} i) & F(K_1 \cup K_2) & = & F(K_1) \cup F(K_2) \\ ii) & F(K_1 \cap K_2) & \subset & F(K_1) \cap F(K_2) \\ iii) & F(X \backslash K) & \supset & \text{Im}(F) \setminus F(K) \\ iv) & K_1 \subset K_2 & \Longrightarrow & F(K_1) \subset F(K_2) \end{cases}$$

There are two manners to define the inverse image by a set-valued map F of a subset M:

$$\begin{cases} i) & F^{-1}(M) := \{x \mid F(x) \cap M \neq \emptyset\} \\ ii) & F^{+1}(M) := \{x \mid F(x) \subset M\} \end{cases}$$

The subset $F^{-1}(M)$ is called the *inverse image* of M by F and $F^{+1}(M)$ is called the *core* of M by F.

They naturally coincide when F is single-valued.
We observe at once that

$$F^{+1}(Y \setminus M) = X \setminus F^{-1}(M) \;\&\; F^{-1}(Y \setminus M) = X \setminus F^{+1}(M)$$

One can conceive as well two dual ways for defining composition products of set-valued maps (which coincide when the maps are single-valued):

Definition 1.3.2 *Let X, Y, Z be metric spaces and*

$$G : X \rightsquigarrow Y \;\&\; H : Y \rightsquigarrow Z$$

be set-valued maps.

1 — The usual composition product (called simply the product*) $H \circ G : X \rightsquigarrow Z$ of H and G at x is defined by*

$$(H \circ G)(x) := \bigcup_{y \in G(x)} H(y)$$

2 — The square product *$H \square G : X \rightsquigarrow Z$ of H and G at x is defined by*

$$(H \square G)(x) := \bigcap_{y \in G(x)} H(y)$$

Let **1** denote the identity map from one set to itself. We deduce the following formulas

$$\begin{cases} \text{Graph}(H \circ G) & = (G \times \mathbf{1})^{-1}(\text{Graph}(H)) \\ & = (\mathbf{1} \times H)(\text{Graph}(G)) \\ \text{Graph}(H \square G) & = (G \times \mathbf{1})^{+1}(\text{Graph}(H)) \end{cases} \tag{1.3}$$

as well as formulas which state that the inverse of a product is the product of the inverses (in reverse order):

$$\begin{cases} i) \;\; (H \circ G)^{-1}(z) & = G^{-1}(H^{-1}(z)) = \left(G^{-1} \circ H^{-1}\right)(z) \\ ii) \;\; (H \square G)^{-1}(z) & = G^{+1}(H^{-1}(z)) \end{cases}$$

We also observe that

$$\begin{cases} i) \quad x \in (H \square G)^{-1}(z) & \iff G(x) \subset H^{-1}(z) \\ ii) \quad x \in (G^{-1} \square H^{-1})(z) & \iff H^{-1}(z) \subset G(x) \end{cases}$$

and thus, that

$$G(x) = H^{-1}(z) \iff x \in (G^{-1} \square H^{-1})(z) \cap (H \square G)^{-1}(z)$$

Let us also point out the following relations: When M is a subset of Z, then

$$\begin{cases} i) \quad (H \circ G)^{-1}(M) = G^{-1}(H^{-1}(M)) \\ ii) \quad (H \circ G)^{+1}(M) = G^{+1}(H^{+1}(M)) \end{cases}$$

1.4 Continuity of Set-Valued maps

The concepts of semi-continuous maps have been introduced in 1932 by G. Bouligand[6] and K. Kuratowski[7]. We begin with the upper semicontinuous set-valued maps:

1.4.1 Definitions

In this section, X, Y, Z denote metric spaces. We describe the concepts of semicontinuous set-valued maps introduced by Bouligand, Kuratowski and Wilson in the early thirties.

Definition 1.4.1 *A set-valued map $F : X \rightsquigarrow Y$ is called* upper semicontinuous *at $x \in \mathrm{Dom}(F)$ if and only if for any neighborhood $\mathcal{U}$ of $F(x)$,*

*

$$\exists\, \eta > 0 \text{ such that } \forall\, x' \in B_X(x, \eta), \ F(x') \subset \mathcal{U}.$$

* there exist a n.hood of B(x; η) of x.

[6] who wrote: "Peut-on rendre plus profond hommage à la mémoire de René Baire qu'en poursuivant les conséquences d'une idée dégagée par lui et dont l'importance se révèle chaque jour accrue, la semi-continuité? Elle échappa tout le XIX^e siècle aux adeptes de la théorie des fonctions, et à plus forte raison, aux purs géomètres, qui s'adonnaient à des occupations moins subtiles."

[7] who also wrote:
"D'après Monsieur Baire, une fontion est dite semi-continue supérieurement..... La notion de semi-continuité dont nous allons nous servir ici est tout à fait analogue à celle-ci, mais concerne le cas où la fonction $F(x)$ admet comme valeurs des sous-ensembles fermés...."

* A set-valued map $F : X \to Y$ is upper semi-continuous at $x_0 \in X$ if for any open N containing $F(x_0)$ ($F(x_0) \subset N$) there exist a n.hood M of x_0 such that $F(M) \subset N$.

def: A set-valued map F is called lower semicontinuous at $x^0 \in X$ if for any $y^0 \in F(x^0)$ and any n.hood $N(y^0)$ of y^0, there exist a n.hood $N(x^0)$ of x^0 such that $\forall x \in N(x^0)$, $F(x) \cap N(y^0) \neq \emptyset$.

It is said to be upper semicontinuous *if and only if it is upper semicontinuous at any point of* Dom(F).

When $F(x)$ is compact, F is upper semicontinuous at x if and only if

$$\forall \varepsilon > 0, \exists \eta > 0 \text{ such that } \forall x' \in B_X(x, \eta), \; F(x') \subset B_Y(F(x), \varepsilon)$$

We observe that this definition is a natural adaptation of the definition of a *continuous* single-valued map. Why then do we use the adjective *upper semicontinuous* instead of continuous? One of the reasons is that the celebrated characterization of continuous maps — *a map f is continuous at x if and only if it maps sequences converging to x to sequences converging to $f(x)$* — does not hold true any longer in the set-valued case.

Indeed, the set-valued version of this characterization leads to the following definition.

Definition 1.4.2 *A set-valued map $F : X \rightsquigarrow Y$ is called* lower semicontinuous *at $x \in$ Dom(F) if and only if for any $y \in F(x)$ and for any sequence of elements $x_n \in$ Dom(F) converging to x, there exists a sequence of elements $y_n \in F(x_n)$ converging to y.*

It is said to be lower semicontinuous *if it is lower semicontinuous at every point $x \in$ Dom(F).*

Actually, as in the single-valued case, this definition is equivalent to:
For any open subset $\mathcal{U} \subset Y$ such that $\mathcal{U} \cap F(x) \neq \emptyset$,

$$\exists \eta > 0 \text{ such that } \forall x' \in B_X(x, \eta), \; F(x') \cap \mathcal{U} \neq \emptyset$$

Unfortunately, there exist set-valued maps which enjoy one property without satisfying the other.

Examples — The set-valued map $F_1 : \mathbf{R} \rightsquigarrow \mathbf{R}$ defined by

$$F_1(x) := \begin{cases} [-1, +1] & \text{if} \quad x \neq 0 \\ \{0\} & \text{if} \quad x = 0 \end{cases}.$$

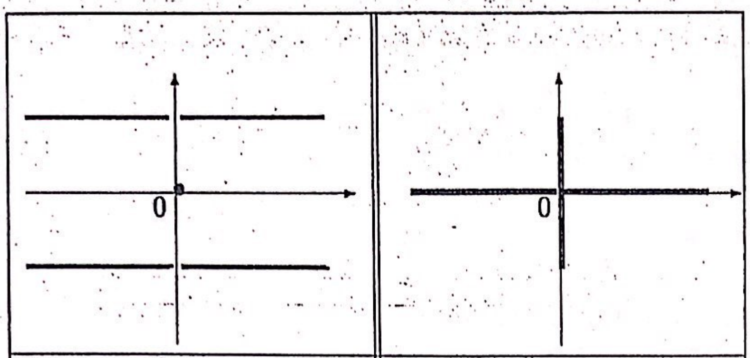
is lower semicontinuous at zero but not upper semicontinuous at zero.

The set-valued map $F_2 : \mathbf{R} \rightsquigarrow \mathbf{R}$ defined by

$$F_2(x) := \begin{cases} \{0\} & \text{if} \quad x \neq 0 \\ [-1, +1] & \text{if} \quad x = 0 \end{cases}$$

i.e. at $x = 0$, $f(x) = \{0\}$ and we can choose a n.hood $(-1, 1) = N$ of $f(x) = f(\{0\})$ but for any n.hood M (say $(-\frac{1}{2}, \frac{1}{2})$) of $x = 0$, $F(x) = [-1, 1] \notin N$. Hence f is not u.s.c at $x = 0$.

Figure 1.3: Semicontinuous and Noncontinuous Maps



lower s.c. & not upper s.c.	upper s.c. & not lower s.c.

is upper semicontinuous at zero but not lower semicontinuous at zero. □

We are therefore led to introduce still another

Definition 1.4.3 *We shall say that set-valued map F is* continuous *at x if it is* both *upper semicontinuous and lower semicontinuous at x, and that it is* continuous *if and only if it is continuous at every point of* $\mathrm{Dom}(F)$.

Remark — We can use the concepts of inverse images and cores to characterize upper and lower semicontinuous maps:

Proposition 1.4.4 *A set-valued map $F : X \rightsquigarrow Y$ is upper semicontinuous at x if the core of any neighborhood of $F(x)$ is a neighborhood of x and a set-valued map is lower semicontinuous at x if the inverse image of any open subset intersecting $F(x)$ is a neighborhood of x.*

Hence, F is upper semicontinuous if and only if the core of any open subset is open and it is lower semicontinuous if and only if the inverse image of any open subset is open.

If $\mathrm{Dom}(F)$ *is closed, then F is lower semicontinuous if and only if the core of any closed subset is closed and F is upper semicontinuous if and only if the inverse image of any closed subset is closed.*

We shall also need to adapt to the set-valued case the concept of Lipschitz applications.