Moreover, in a similar way it can be shown that $(K(X), H)$ is also a metric space.

Remark: For $A, B \in CB(X)$ and $a \in A$, and for $\epsilon > 0$, there must exist a point $b \in B$ such that

$$d(a,b) \leq H(A,B) + \epsilon.$$

Proposition: Let X be a metric space. Then

$$H(A \cup B, C \cup D) \leq \max\{H(A,C), H(B,D)\}$$

for all $A, B, C, D \in CB(X)$.

Proof:

Since

$$\begin{aligned} \delta(A \cup B, C \cup D) &= \max\{\delta(A, C \cup D), \delta(B, C \cup D)\} \\ &\leq \max\{\delta(A,C), \delta(B,D)\} \\ &\leq \max\{H(A,C), H(B,D)\} \quad \text{--- (1)} \end{aligned}$$

Similarly,

$$\delta(C \cup D, A \cup B) \leq \max\{H(A,C), H(B,D)\} \quad \text{--- (2)}$$

By definition of H, we have

$$H(A\cup B, C\cup D) = \max\{\delta(A\cup B, C\cup D), \delta(C\cup D, A\cup B)\}$$
$$\leq \max\{H(A,C), H(B,D)\}$$
$\forall\ A, B, C, D \in CB(X)$.

Def: let (X,d) be a metric space and $(CB(X),H)$ be a Hausdorff metric space. A mapping $T: X \to CB(X)$ is said to be Lipschitzian if there exist a constant $\alpha > 0$ such that
$$H(Tx,Ty) \leq \alpha d(x,y)$$
$\forall\ x, y \in X$.
T is called contraction if $\alpha < 1$.

Def: let $T: X \to CB(X)$. A point $x \in X$ is called fixed point of T if $x \in Tx$. The set of fixed points of T is denoted by
$$Fix(T) = \{x \in X : x \in Tx\}$$

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Nadler's Fixed Point Theorem

let X be a Complete metric Space and $T: X \to CB(X)$ a Contraction mapping. Then T has a fixed point in X.

Proof:

Suppose that $x_0 \in X$ and $x_1 \in Tx_0$. Then (By remark) there exist $x_2 \in Tx_1$ Such that

$$d(x_1, x_2) \leq H(Tx_0, Tx_1) + k \quad — (1)$$

where $k \in (0,1)$.

Similarly, there exist $x_3 \in Tx_2$ Such that

$$d(x_2, x_3) \leq H(Tx_1, Tx_2) + k^2 \quad — (2)$$

In Similar way, there exist a Sequence (x_n) in X Such That $x_{n+1} \in Tx_n$ and

$$d(x_n, x_{n+1}) \leq H(Tx_{n-1}, Tx_n) + k^n \quad — (3)$$

$\forall \; n \in \mathbb{N}$.

Now

$$d(x_n, x_{n+1}) \leq H(Tx_{n-1}, Tx_n) + k^n$$
$$\leq k\,d(x_{n-1}, x_n) + k^n$$
$$\leq k\left[H(Tx_{n-2}, Tx_{n-1}) + k^{n-1}\right] + k^n$$
$$\leq k\left[k\,d(x_{n-2}, x_{n-1}) + k^{n-1}\right] + k^n$$

$$= k^2 d(x_{n-2}, x_{n-1}) + k^n + k^n$$
$$= k^2 d(x_{n-2}, x_{n-1}) + 2k^n$$
$$\leq k^2\left[H(Tx_{n-3}, Tx_{n-2}) + k^{n-2}\right] + 2k^n$$
$$\leq k^2\left[k\,d(x_{n-3}, x_{n-2}) + k^{n-2}\right] + 2k^n$$
$$= k^3 d(x_{n-3}, x_{n-2}) + 3k^n$$
$$\vdots$$
$$\leq k^n d(x_0, x_1) + n k^n$$

$$\Rightarrow d(x_n, x_{n+1}) \leq k^n d(x_0, x_1) + n k^n \quad (4)$$
$$\Rightarrow \sum_{n=1}^{\infty} d(x_n, x_{n+1}) \leq d(x_0, x_1) \sum_{n=1}^{\infty} k^n + \sum_{n=1}^{\infty} n k^n \quad (5)$$

Since $\sum_{n=1}^{\infty} k^n < \infty$ and $\sum_{n=1}^{\infty} n k^n < \infty$,

we have
$$\sum_{n=1}^{\infty} d(x_n, x_{n+1}) < \infty.$$

This shows that (x_n) is a Cauchy Sequence.

Since X is complete, there exist $u \in X$ such that

$$\lim_{n \to \infty} x_n = u.$$

i.e. $d(x_n, u) = 0$ as $n \to \infty$.

By continuity of T, we have

$$H(Tx_n, Tu) = 0 \quad \text{as } n \to \infty$$

Since $x_{n+1} \in Tx_n$, so

$$d(x_{n+1}, Tu) = 0 \quad \text{as } n \to \infty$$

$\Rightarrow d(u, Tu) = 0$

$\Rightarrow u \in Tu$.

Ex: Let $X = \{1, 2, 3, 4\}$

$d(x, y) = 0$ for $x = y$ and

~~d(1,2) = 1~~, d(1,

$d(x, y) = |x - y|$ for $x \neq y$.

Def $T : X \to 2^X$ by

$$T(x) = \begin{cases} \{1, 2\} & ; x \in \{1, 3, 4\} \\ \{3\} & ; x \in \{2\}. \end{cases}$$

Check whether T is a contraction or not.