Set Valued Mappings

Def: Let A be a non-empty subset of a metric space X. For each $x \in X$, we define

$$d(x, A) = \inf\{d(x, a) : a \in A\}. \quad \text{—}(1)$$

Now, let $CB(X)$ denote the set of non-empty closed bounded subsets of X and $K(X)$ denote the set of non-empty compact subsets of X.
Clearly, $K(X) \subset CB(X)$.

For $A, B \in CB(X)$, we define

$$\delta(A, B) = \sup\{d(x, B) : x \in A\} \quad \text{—}(2)$$

$$H(A, B) = \max\{\delta(A, B), \delta(B, A)\} \quad \text{—}(3)$$

$$= \max\{\sup_{a \in A} d(a, B), \sup_{b \in B} d(b, A)\}.$$

e.g. If $X = \mathbb{R}$, $A = [1, 2]$, $B = [2, 3]$.
Then

$$\delta(A, B) = \sup_{a \in A} d(a, B) = 1$$

&

$$\delta(B, A) = \sup_{b \in B} d(b, A) = 1.$$

Hence ~~H(A)~~ $H(A, B) = 1$.

Note: The distance δ is not symmetric.

Now, we discuss some properties of δ & H.

Proposition:

Let (X,d) be a metric space and $A,B,C \in CB(X)$. Then

a) $\delta(A,B) = 0 \iff A \subset B$

b) $B \subset C \Rightarrow \delta(A,C) \leq \delta(A,B)$

c) $\delta(A \cup B, C) = \max\{\delta(A,C), \delta(B,C)\}$

d) $\delta(A,B) \leq \delta(A,C) + \delta(C,B)$.

Proof: (a) ~~Let~~ By definition of δ, we have

$$\delta(A,B) = 0$$

$$\iff \sup_{x \in A} d(x,B) = 0$$

$$\iff \sup_{x \in A}\{\inf_{b \in B} d(x,b)\} = 0$$

$$\iff d(x,b) = 0$$

$$\iff d(x,B) = 0 \quad \forall\ x \in A.$$

Since B is closed in X, so

$$d(x,B) = 0 \iff x \in B.$$

But $x \in A$. So $A \subset B$

i.e. $\delta(A,B) = 0 \iff A \subset B$.

b) Suppose that $B \subset C$. Then

$d(x,c) \le d(x,b) \quad \forall x \in X, b \in B$

$c \in C$.

$\Rightarrow \inf d(x,c) \le \inf d(x,b)$

$\Rightarrow \quad d(x,C) \le d(x,B) \quad \forall x \in X$.

If we restrict x to A, then

$d(x,C) \le d(x,B) \quad \forall x \in A$

$\Rightarrow \sup d(x,C) \le \sup d(x,B) \quad \forall x \in A$

$\Rightarrow \quad \delta(A,C) \le \delta(A,B)$.

c) By definition

$\delta(A \cup B, C) = \sup_{x \in A \cup B} d(x,C) \quad - (1)$

Since $x \in A \cup B$ implies

$x \in A$ or $x \in B$,

so (1) become

$\delta(A \cup B, C) = \max\left\{ \sup_{x \in A} d(x,C), \sup_{x \in B} d(x,C) \right\}$

$= \max\{\delta(A,C), \delta(B,C)\}$.

d) Let $a \in A, b \in B$ and $c \in C$. Then by triangular inequality

$$d(a,b) \leq d(a,c) + d(c,b) \quad - (1)$$

this implies that

$$d(a,B) \leq d(a,c) + d(c,B) \quad - (2)$$

Since $c \in C$ is arbitrary, so (2) become

$$d(a,B) \leq d(a,C) + \delta(C,B) \quad - (3)$$

Now, $a \in A$ is also arbitrary, so

$$\delta(A,B) \leq \delta(A,C) + \delta(C,B) \quad - (4).$$

Now we show that H given in (3) (page 1) is a metric on $CB(X)$.

<u>Proposition:</u> Let (X,d) be a metric space. Then H is a metric on $CB(X)$.

Proof: H1) By definition of H, we have

$$H(A,B) \geq 0.$$

H2) $$H(A,B) = 0$$

$$\Leftrightarrow \max\{\delta(A,B), \delta(B,A)\} = 0$$

$\Leftrightarrow \delta(A,B)=0$ & $\delta(B,A)=0$

$\Leftrightarrow A\subset B$ & $B\subset A$

$\Leftrightarrow A=B$.

Hence $H(A,B)=0 \Leftrightarrow A=B$.

H3) $H(A,B) = \max\{\delta(A,B), \delta(B,A)\}$

$= \max\{\delta(B,A), \delta(A,B)\}$

$= H(B,A)$.

H4) By definition of H, we have

$H(A,B) = \max\{\delta(A,B), \delta(B,A)\}$

$\leq \max\{\delta(A,C)+\delta(C,B), \delta(B,C)+\delta(C,A)\}$

$\leq \max\{\delta(A,C), \delta(C,A)\} + \max\{\delta(B,C), \delta(C,B)\}$

$= H(A,C) + H(C,B)$

$\Rightarrow H(A,B) \leq H(A,C) + H(C,B)$.

Hence H is a metric on $CB(X)$, and is called Hausdorff metric.

Note: The metr H ~~dep~~: depends on metric d. It is ~~each~~ easy to show that the completeness of (X,d) implies completeness of $(CB(X), H)$.

Moreover, in a similar way it can be shown that $(K(X), H)$ is also a metric space.

Remark: For $A, B \in CB(X)$ and $a \in A$, and for $\epsilon > 0$, there must exist a point $b \in B$ such that

$$d(a,b) \leq H(A,B) + \epsilon.$$

Proposition: Let X be a metric space. Then

$$H(A \cup B, C \cup D) \leq \max\{H(A,C), H(B,D)\}$$

for all $A, B, C, D \in CB(X)$.

Proof:

Since

$$\delta(A \cup B, C \cup D) = \max\{\delta(A, C \cup D), \delta(B, C \cup D)\}$$