Lecture # 01

In the last lecture we discuss about the Banach Contraction principle, which States that
" A Contraction mapping on a Complete metric Space has a unique fixed point."

In this lecture, we will discuss Some examples and applications of Banach Contraction principle.

Example 1: Consider $(\mathbb{R}, d)$ and define a Self map T on $\mathbb{R}$ by
$$T(x) = \frac{1}{3}x.$$
we first Show that T is a Contraction.

$$\begin{aligned} d(Tx, Ty) &= |Tx - Ty| \quad \because d \text{ is usual metric} \\ &= \left|\frac{1}{3}x - \frac{1}{3}y\right| \\ &= \frac{1}{3}|x - y| \\ &= \frac{1}{3}d(x, y) \end{aligned}$$

So T is a Contraction with

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Lipschitz Constant $\alpha = \frac{1}{3}$.
Thus by Banach Contraction principle T has a unique fixed point.

We now give an application of Banach Contraction principle.

Consider the following first order initial value problem

$$\left.\begin{aligned} y'(t) &= f(t, y(t)) \\ y(0) &= y_0 \end{aligned}\right\} \text{ --- } (2)$$

where $f : I \times R^n \to R^n$ and $I = [0, b]$.
Note that (2) is the System of first order equations because f Takes values in R^n.
Assume that f is Continuous.
The $y \in C^1[0, b] = C^1(I)$ (the Banach Space of functions u whose first derivative is Continuous.

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on I together with the norm
$\|u\|_1 = \max\{\sup_{t\in I}\|u(t)\|, \sup_{t\in I}\|u'(t)\|\}$)

is a solution of ② iff
$y \in C(I)$ (the Banach space of functions u, which are continuous on I together with the norm
$\|u\|_0 = \sup_{t\in I} |u(t)|$)

is the solution of the corresponding integral equation

$$y(t) = y_0 + \int_0^t f(s, y(s))\,ds \quad \text{— ③}$$

of ①.

To obtain ③, we take integral of ② from 0 to t,

and make use of initial value, i.e. $y(0) = y_0$.

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Now we define Picard operator $T: C(I) \to C(I)$ by

$$T(y(t)) = y_0 + \int_0^t f(s, y(s))\, ds. \quad \text{—} (4)$$

To show that y is a solution of (2), we show that $T(y) = y$, i.e. y is a fixed point of T.

i.e. The solutions of (2) are the fixed points of Picard operator T.

We now prove the existence and uniqueness of the solution of initial value problem (2).

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Theorem: let $f : I \times \mathbb{R}^n \to \mathbb{R}^n$ be continuous and Lipschitz in y, That is, there exists $\alpha \geq 0$ such that

$$\| f(t,y) - f(t,z) \| \leq \alpha \| y - z \|_\alpha$$

$\forall\, y, z \in \mathbb{R}^n$. Then there exists a unique solution of (2).

Proof

We define weighted norm

(5) — $\|y\|_\alpha = \| e^{-\alpha t} y(t) \|_0$ on $C(I)$

clearly, $C(I)$ is a Banach space with respect to (5), since it is equivalent to the maximum norm, that is,

$$e^{-\alpha b} \| y \|_0 \leq \|y\|_\alpha \leq \| y \|_0 .$$

We now shown that T given is (4) is a contraction.

For this, let $y, z \in C(I)$ and for $t \in I$

$$\left(Ty(t) - Tz(t)\right) = \cancel{y_0} + \int_0^t f(s, y(s))\,ds$$
$$- \cancel{y_0} - \int_0^t f(s, z(s))\,ds$$
$$= \int_0^t \left(f(s, y(s)) - f(s, z(s))\right) ds \quad \text{--- (6)}$$

Thus, for $t \in I$, multiplying $e^{-\alpha t}$ on both sides of (6) and Taking norm, we have

$$e^{-\alpha t} \left| (Ty - Tz)(t) \right| = e^{-\alpha t} \left| \int_0^t \left(f(s, y(s)) - f(s, z(s))\right) ds \right|$$
$$\leq e^{-\alpha t} \int_0^t \alpha \cdot e^{\alpha s} \cdot e^{-\alpha s} \left| y(s) - z(s) \right| ds$$
$$\leq e^{-\alpha t} \left(\int_0^t \alpha e^{\alpha s}\, ds \right) \| y - z \|_\alpha \quad \because \text{using (5)}$$
$$\leq e^{-\alpha t} \left(e^{\alpha t} - 1 \right) \| y - z \|_\alpha$$
$$= \left(1 - e^{-\alpha t}\right) \| y - z \|_\alpha$$
$$= \left(1 - e^{-\alpha b}\right) \| y - z \|_\alpha$$

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This implies

$$\|Ty - Tz\|_\alpha \le (1-e^{-\alpha b})\|y-z\|_\alpha .$$

Since $(1-e^{-\alpha b}) < 1$, so by Banach Contraction principle there is a unique $y \in C(I)$ with $Ty = y$, equivalently (2) has a unique solution $y \in C^1(I)$.

This complete the proof.

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