

FRACTURE MECHANICS

WHAT IS FRACTURE MECHANICS

Fracture mechanics is mechanics

of solids containing planes of displacement discontinuities (cracks) with special attention to their growth

Fracture mechanics is a failure theory that

1. determines material failure by *energy* criteria, possibly in conjunction with strength (or yield) criteria
2. considers failure to be *propagating* throughout the structure rather than simultaneous throughout the entire failure zone or surface.

Linear elastic fracture mechanics (LEFM)

is the basic theory of fracture, that deals with sharp cracks in elastic bodies.

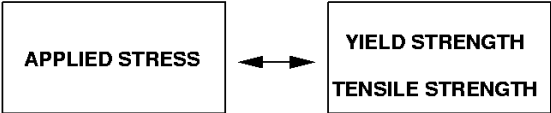
It is applicable to any materials as long as the material is elastic except in a vanishingly small region at the crack tip (assumption of small scale yielding), brittle or quasibrittle fracture, stable or unstable crack growth

Elastic-plastic fracture mechanics

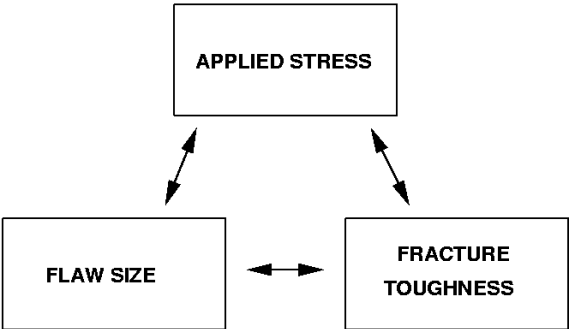
is the theory of ductile fracture, usually characterized by stable crack growth (ductile metals) the fracture process is accompanied by formation of large plastic zone at the crack tip

COMPARISON OF THE FRACTURE MECHANICS APPROACH TO THE DESIGN WITH THE TRADITIONAL STRENGTH OF MATERIALS APPROACH

APPROACH BASED ON STRENGTH OF MATERIALS



APPROACH BASED ON STRENGTH OF MATERIALS



GOVERNING EQUATIONS OF LINEAR ELASTICITY

In this study we shall consider only statics. Individual particles of the body will be identified by their coordinates x_i ($i = 1, 2, 3$) in the undeformed configuration.

Displacement field

$$u_i = u_i(x_1, x_2, x_3) = u_i(x_j).$$

Strain field

$$\epsilon_{ij} = \frac{1}{2}(u_{i,j} + u_{j,i}). \quad (1)$$

Equations of equilibrium

$$\begin{aligned} \sigma_{ij,j} + X_i &= 0 \\ \sigma_{ij} &= \sigma_{ji}. \end{aligned} \quad (2)$$

Surface tractions

$$p_i = \sigma_{ij}n_j. \quad (3)$$

Constitutive equations

$$\begin{aligned} \sigma_{ij} &= L_{ijkl}(\epsilon_{kl} - \epsilon_{ij}^0) \\ L_{ijkl} &= L_{jikl} = L_{ijlk}. \end{aligned} \quad (4)$$

The fourth-order tensor L_{ijkl} is known as *the stiffness tensor*. Suppose that a strain energy density function $U(\epsilon_{ij})$ per unit volume exists such that

$$\sigma_{ij} = \frac{\partial U}{\partial \epsilon_{ij}}. \quad (5)$$

Eqs. (2) and (4) readily provide

$$L_{ijkl} = L_{klij}$$

and that

$$U = \frac{1}{2} L_{ijkl} \epsilon_{ij} \epsilon_{kl}. \quad (6)$$

Providing the strain energy U has a minimum in the stress-free state then L_{ijkl} is positive definite

$$L_{ijkl} \epsilon_{ij} \epsilon_{kl} > 0 \quad (7)$$

for all non-zero symmetric tensors ϵ_{ij} . It is now possible to invert Eq. (4) to get

$$\epsilon_{ij} = M_{ijkl} \sigma_{kl} + \sigma_{ij}^0 \quad (8)$$

M_{ijkl} is known as *the compliance tensor*. Note that

$$M_{ijkl} = M_{jikl} = M_{ijlk} = M_{klij}$$

and also

$$L_{ijrs} M_{rskl} = \frac{1}{2} (\delta_{ik} \delta_{jl} + \delta_{il} \delta_{jk}) = I_{ijkl}, \quad (9)$$

where δ_{ij} is the Kronecker delta and I_{ijkl} represents the fourth-order identity tensor.

Note: we used standard Cartesian tensor notation in which repeated suffixes are summed over the range 1, 2, 3.

AVERAGES

In preparation for evaluation of the overall moduli we first review some basic formulae for the determination of average stresses and strains. To that end, we assume that the displacements fields are continuous, and the strain fields are compatible; also, the stress fields and tractions are continuous and in equilibrium

Consider first an arbitrary homogeneous medium of volume V with the boundary S . In general, the volume average of a quantity is just the ordinary volume average given by

$$\langle f \rangle = \frac{1}{V} \int_V f dV. \quad (10)$$

Let $\epsilon(\mathbf{x})$ and $\sigma(\mathbf{x})$ be certain fields in V . Their volume averages are defined as

$$\langle \epsilon \rangle = \frac{1}{V} \int_V \epsilon(\mathbf{x}) dV \quad \langle \sigma \rangle = \frac{1}{V} \int_V \sigma(\mathbf{x}) dV \quad (11)$$

After applying the divergence theorem we arrive at

$$\langle \epsilon_{ij} \rangle = \frac{1}{2V} \int_S (u_i n_j + u_j n_i) dS \quad (12)$$

$$\langle \sigma_{ij} \rangle = \frac{1}{2V} \int_S (p_i x_j + p_j x_i) dS \quad (13)$$

Next, consider a heterogeneous elastic medium which consists of a homogeneous matrix V_2 and homogeneous inclusion V_1 . Evaluation of the above volume averages requires an

application of a generalized (but still standard) divergence theorem. Let f be continuous in V and continuously differentiable in the interior of V_1 and V_2 . We may now apply the divergence theorem separately to V_1 and V_2 to conclude that

$$\int_V \frac{\partial f}{\partial x_i} dV + \int_{\Sigma} [f] m_j dS = \int_S f n_i dS \quad (14)$$

where $[f]$ denotes the jump in the value of f as we travel across Σ from V_1 to V_2 . Now, assume that perfect bonding exists. When setting $f = u_i$ in Eq. (14) we immediately recover Eq. (12). Since tractions are continuous across Σ

$$[\sigma_{ij}] m_j = 0$$

setting $f = \sigma_{ik} x_j$ yields Eq. (13). We may now conclude that Eqs. (12) and (13) apply to any heterogeneous material, generally anisotropic, consisting of a homogeneous matrix and an arbitrary number of homogeneous inclusions.

EXAMPLES 1.1

Consider an arbitrary composite material with outer boundary S

1. Suppose that the composite is loaded by displacements u_i on S , which are compatible with the uniform strain E_{ij} , i.e. $u_i = E_{ij}x_j$ (affine displacements). Show that

$$\langle \epsilon_{ij} \rangle = E_{ij}.$$

2. Suppose that the composite is loaded by prescribed tractions p_i on S , which are compatible with the uniform stress Σ_{ij} , i.e. $p_i = \Sigma_{ij}n_j$. Show that

$$\langle \sigma_{ij} \rangle = \Sigma_{ij}.$$

3. Let σ_{ij} be a self-equilibrated stress field ($\sigma_{ij,j}=0$) and u_i is a displacement field associated with strain $\epsilon_{ij} = \frac{1}{2}(u_{i,j} + u_{j,i})$. Show that if either $u_i = 0$ or $\sigma_{ij}n_j = 0$ on the boundary then

$$\int_V \sigma_{ij}\epsilon_{ij} dV = 0.$$

4. For the boundary conditions of Exs. (1) and (2) show that (Hill's lemma)

$$\langle \sigma_{ij}\epsilon_{ij} \rangle = \langle \sigma_{ij} \rangle \langle \epsilon_{ij} \rangle \quad \text{and} \quad \langle U \rangle = \frac{1}{2} \langle \sigma_{ij} \rangle \langle \epsilon_{ij} \rangle .$$

MINIMUM ENERGY PRINCIPLES

We now give a brief review of the classical energy principles as they have been extensively used in assessing the bounds on the overall elastic properties of composites.

First, consider an arbitrary anisotropic elastic medium Ω with prescribed displacements u_i along its boundary. Let $\epsilon_{ij}, \sigma_{ij}, U$ be the associated strain, stress, and strain energy density, respectively. The purpose of this investigation is to show that the energy density U^* associated with any kinematically admissible displacement field u_i^* is greater than the energy function U associated with the true solution. Let

$$\epsilon_{ij}^* = \frac{1}{2}(u_{i,j}^* + u_{j,i}^*) \quad \sigma_{ij}^* = L_{ijkl}\epsilon_{ij}^*$$

$$U^* = \frac{1}{2}\sigma_{ij}^*\epsilon_{ij}^*.$$

In the next step, calculate the energy of the difference state with displacements $(u_i^* - u_i)$, which is positive

$$\frac{1}{2}(\sigma_{ij}^* - \sigma_{ij})(\epsilon_{ij}^* - \epsilon_{ij}) \geq 0$$

Therefore,

$$\frac{1}{2} \int_{\Omega} (\sigma_{ij}^*\epsilon_{ij}^* - \sigma_{ij}\epsilon_{ij}) d\Omega \geq \frac{1}{2} \int_{\Omega} (\sigma_{ij}^*\epsilon_{ij} + \sigma_{ij}\epsilon_{ij}^* - 2\sigma_{ij}\epsilon_{ij}) d\Omega$$

Applying Betti's theorem $\sigma_{ij}^*\epsilon_{ij} = \sigma_{ij}\epsilon_{ij}^*$ yields

$$\frac{1}{2} \int_{\Omega} (\sigma_{ij}^*\epsilon_{ij}^* - \sigma_{ij}\epsilon_{ij}) d\Omega \geq \int_{\Omega} \sigma_{ij}(\epsilon_{ij}^* - \epsilon_{ij}) d\Omega = 0$$

and finally

$$\int_{\Omega} U^* d\Omega \geq \int_{\Omega} U d\Omega \quad (15)$$

we recover a special case of the theorem of minimum potential energy.

Next, consider the second boundary value problem with prescribed tractions along the boundary of the anisotropic solid. Once again, let u_i be the required solution and $\epsilon_{ij}, \sigma_{ij}, W$ be the corresponding strain, stress, and stress (complementary) energy density function, respectively. Suppose that τ_{ij} is any statically admissible stress field and define the associated field η_{ij}

$$\eta_{ij} = M_{ijkl}\tau_{kl}.$$

Again, using the trick of computing the positive energy associated with difference state

$$\frac{1}{2}(\tau_{ij} - \sigma_{ij})(\eta_{ij} - \epsilon_{ij}) \geq 0$$

yields

$$\begin{aligned} \frac{1}{2} \int_{\Omega} (\tau_{ij}\eta_{ij} - \sigma_{ij}\epsilon_{ij}) d\Omega &\geq \frac{1}{2} \int_{\Omega} (\tau_{ij}\epsilon_{ij} + \sigma_{ij}\eta_{ij} - 2\sigma_{ij}\epsilon_{ij}) d\Omega \\ \int_{\Omega} (\tau_{ij} - \sigma_{ij})\epsilon_{ij} d\Omega &= 0. \end{aligned}$$

It now follows that

$$\int_{\Omega} M_{ijkl}\tau_{ij}\tau_{kl} d\Omega \geq \int_{\Omega} M_{ijkl}\sigma_{ij}\sigma_{kl} d\Omega, \quad (16)$$

which is the special case of the theorem of minimum complementary energy.

EXAMPLES 1.2

Consider an arbitrary heterogeneous body with outer boundary S .

1. Suppose that the medium is loaded by prescribed displacements u_i on S . Then, if the material is stiffened in any way (keeping the boundary fixed) show that strain energy increases (Hill's stiffening theorem).
2. Show that if the stiffness tensor L_{ijkl} is increased by a positive amount, then the corresponding compliance tensor M_{ijkl} decreases by the positive definite amount.

VARIATIONAL PRINCIPLES

Consider an arbitrary anisotropic elastic body Ω loaded by prescribed displacements u_i along a portion of its boundary Γ_u and prescribed tractions p_i on Γ_p . The minimum of total potential energy $\Pi = E_i + E_e$ is then given by

$$\begin{aligned}\delta\Pi &= \delta(E_i + E_e) \\ &= \int_{\Omega} \delta\epsilon_{ij}\sigma_{ij} d\Omega - \int_{\Omega} \delta u_i X_i d\Omega - \int_{\Gamma_p} \delta u_i \bar{p}_i d\Gamma = 0.\end{aligned}\quad (17)$$

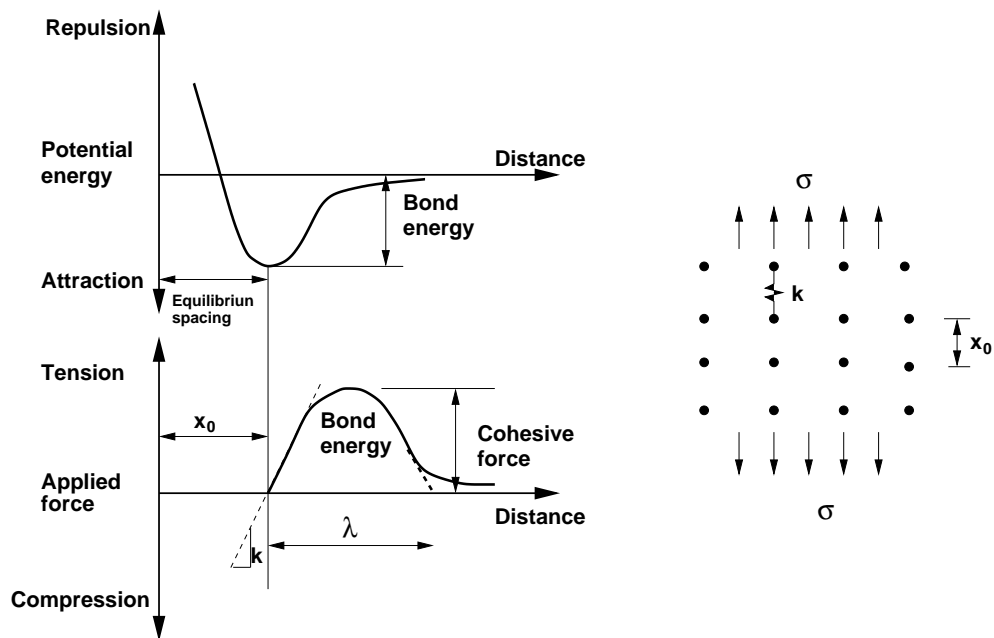
Eq. (17) represents the *Lagrange variational principle* of the minimum of total potential energy. Principle of the minimum of complementary energy follows from the *Castiglian variational principle* and assumes the form

$$\begin{aligned}\delta\Pi^* &= \delta(E_i^* + E_e^*) \\ &= \int_{\Omega} \delta\sigma_{ij}\epsilon_{ij} d\Omega - \int_{\Gamma_u} \delta p_i \bar{u}_i d\Gamma = 0.\end{aligned}\quad (18)$$

Note that applying the Lagrange variational principle provides the Cauchy equations of equilibrium and static (traction) boundary conditions. However, when invoking the Castiglian variational principle we arrive at the geometrical equations and kinematic (displacement) boundary conditions.

AN ATOMISTIC VIEW OF FRACTURE

It comes out from the assumption that a material fractures when sufficient stress and work are applied on the atomic level to break the bonds that hold atoms together. The bond strength is supplied by the attractive forces between atoms.



AN ATOMISTIC VIEW OF FRACTURE - Continue

The bond energy is provided by

$$E_b = \int_{x_0}^{\infty} P dx \quad (19)$$

where x_0 is the equilibrium spacing and P is the applied force. Assume that the cohesive strength at the atomic level can be estimated by idealizing the interatomic force-displacement relationship as one half the period of a sine wave:

$$P = P_c \sin\left(\frac{\pi(x - x_0)}{\lambda}\right) \quad (20)$$

with the origin defined at x_0 . For small displacements we get

$$P = P_c \left(\frac{\pi(x - x_0)}{\lambda}\right) \quad (21)$$

and the bond stiffness is given by

$$k = \frac{P_c \pi}{\lambda}. \quad (22)$$

Multiplying both sides of Eq. (22) by the number of bonds per unit area and dividing by the gage length x_0 gives

$$\sigma_c = \frac{E \lambda}{\pi x_0} \quad \sigma_c = \frac{E}{\pi} \quad (23)$$

where E is the elastic modulus and σ_c is the cohesive strength.

AN ATOMISTIC VIEW OF FRACTURE - Continue

Introduce a surface energy γ_s resulting from non-equilibrium configuration of atoms on an arbitrary surface as

$$2\gamma_s = \int_{x_0}^{x_0+\lambda} \sigma(x)dx \Rightarrow \gamma_s = \frac{1}{2} \int_0^\lambda \sigma_c \sin\left(\frac{\pi x}{\lambda}\right) dx = \sigma_c \frac{\lambda}{\pi} \quad (24)$$

Note that the surface energy equals one half of the fracture energy since two surfaces are created when material fractures. Finally, substituting for λ from Eq. (23) into Eq. (24) and solving for σ_c gives

$$\sigma_c = \sqrt{\frac{E\gamma_s}{x_0}}. \quad (25)$$

Example

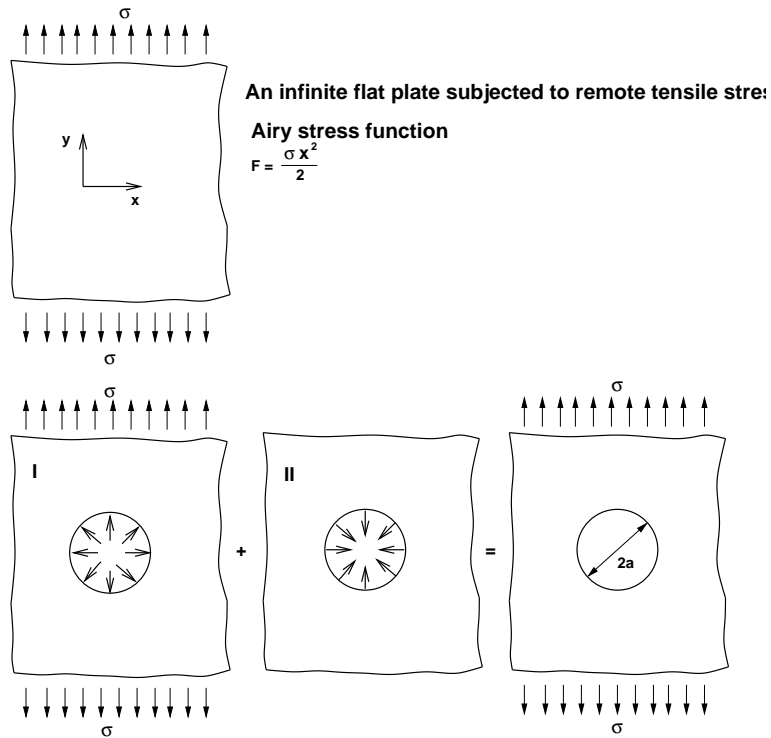
$$\gamma_s = 1 - 10\text{J/m}^2, \quad E = 10^{11} - 10^{12}\text{N/m}^2, \quad x_0 = 2 * 10^{-10}\text{m} \Rightarrow \sigma_c = E/5.$$

Recall, that the theoretical cohesive strength is approximately E/π . But practical and experimental observations suggest that the true fracture strength is typically three to four orders of magnitude below the theoretical value. This discrepancy, as pointed out already by Leonardo da Vinci, Griffith, and others, is due to flaws (defects) in these materials. As shown by the previous derivation, fracture cannot occur unless the stress at the atomic level exceeds the cohesive strength of material. *Thus flaws must lower the global strength by magnifying the strength locally* \rightarrow *concept of stress concentration.*

Typical flaws include: defects, cracks, secondary phases, etc.

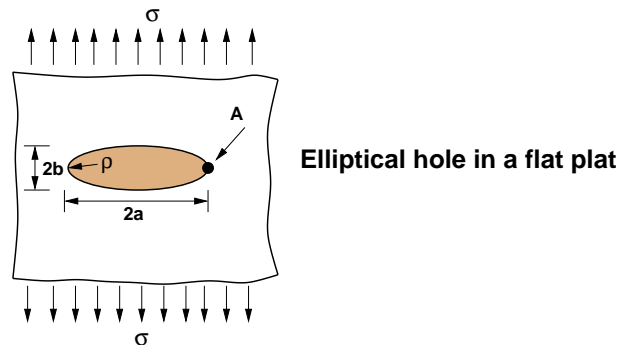
STRESS IN AN INFINITE PLATE WITH AS A CIRCULAR HOLE

This problem can be solved by introducing the Airy stress function in polar coordinates.



STRESS IN AN INFINITE PLATE WITH AN ELLIPTICAL HOLE

The first quantitative evidence for the stress concentration factor was provided by Inglis, who analyzed elliptical holes in flat infinite plates.



Inglis (1913) found that the maximum net section stress at point A (see the figure above) is provided by

$$\sigma_A = \sigma \left(1 + \frac{2a}{b} \right) \quad (26)$$

where σ is the nominal (remote) stress. Note that if $a = b$ (circle) then $\sigma_A = 3\sigma$. When defining the radius of curvature $\rho = b^2/a$ the maximum local stress σ_A attains the form

$$\sigma_A = \sigma \left(1 + 2\sqrt{\frac{a}{\rho}} \right) \quad \text{and if } a \gg \rho \quad \sigma_A = 2\sigma\sqrt{\frac{a}{\rho}}. \quad (27)$$

The above criterion suffers from the major drawback. In particular, if $\rho \rightarrow 0$ then $\sigma_A \rightarrow \infty$. This is not realistic, because no material can withstand infinite stress.

E.g., in ductile materials (metals), the infinite stress is avoided by yielding and non-linear deformation \rightarrow blunting of the crack tip. Moreover, accepting an infinite stress immediately suggests that localized yielding will occur at the crack tips for any nonzero value of the remote stress σ . The commonly employed failure criterion such as Von Mises predicts yielding for any load level and is therefore inadequate for judging the crack stability \rightarrow some local criterion based on fracture mechanics is needed.

STRESS IN AN INFINITE PLATE WITH AN ELLIPTICAL HOLE - Continue

An infinitely sharp crack in a continuum is a mathematical abstraction that is not relevant to real materials, which are made of atoms. In the absence of plastic deformation, the minimum radius a crack tip can have is on the order of the atomic radius. Thus setting $\rho = x_0$ yields

$$\sigma_A = \sigma_A = 2\sigma \sqrt{\frac{a}{x_0}}. \quad (28)$$

Assuming that fracture occurs when $\sigma_A = \sigma_c$ results in the expression for the remote stress at failure as

$$\sigma_f = \sqrt{\frac{E\gamma_s}{4a}}. \quad (29)$$

Note that Eq. (29) must be viewed as a rough estimate of failure stress, because the continuum assumption of Inglis analysis breaks at the atomic level.

Example

$$\gamma_s = 1-10\text{J/m}^2, \quad E = 10^{11}-10^{12}\text{N/m}^2, \quad x_0 = 2*10^{-10}\text{m} \quad a = 5000x_0 \Rightarrow \sigma_f = E/700.$$

GRIFFITH ENERGY CRITERION

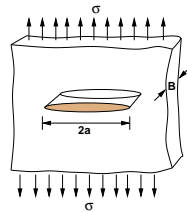
The paradox of a sharp crack motivated Griffith to develop a fracture theory based on energy rather than local stress. He observed that to introduce a crack into an elastically stressed body one would have to balance the decrease in potential energy (due to the release of stored elastic energy and the work done by external loads) and the increase in surface energy resulting from the presence of the crack which creates new surfaces. Recall, that *surface energy* arises from the non-equilibrium configuration of atoms at any surface of a solid. Likewise he reasoned that an existing crack would grow by some increment if the necessary surface energy was supplied to the system.

According to the First law of thermodynamics, when a system goes from a non-equilibrium state to equilibrium, there will be a net decrease in energy. In 1920 Griffith applied this idea to the formation of crack. To that end, suppose that the crack is formed by the sudden annihilation of the tractions acting on its surface. At the same instant, the strain and thus the potential energy possess their original values. But in general, this new state is not in equilibrium. It is not a state of equilibrium, then, by the theorem of minimum of potential energy, the potential energy must be reduced by the attainment of equilibrium. If it is a state of equilibrium the energy does not change. Therefore, a crack can form (or an existing crack can grow) only if such a process causes the total energy to decrease or remain constant. Thus the critical condition for the fracture can be defined as the point at which crack growth occurs under equilibrium conditions. In mathematical terms the above statement reads

$$\frac{dE}{dA} = \frac{d\Pi}{dA} + \frac{dW_s}{dA} = 0. \quad (30)$$

GRIFFITH ENERGY CRITERION - Continue

Let's follow Griffith's treatment.



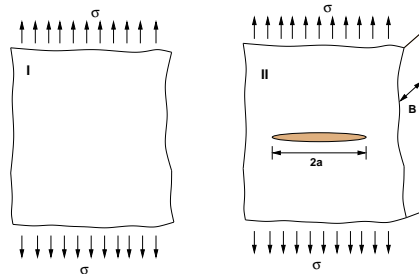
Griffith crack

Through-thickness crack in an infinite plate subject to a remote tensile stress

Griffith wrote an expression for the change in total energy that would result from the introduction of an Inglis crack into an infinitely large, elastically stressed body as a sum of the decrease in potential energy and the increase in surface energy

$$E - E_0 = -\frac{\pi\sigma^2 a^2 B}{E} + 4aB\gamma_s \quad (31)$$

where the first term on the right hand side represents the decrease in potential energy and the second term is the increase in surface energy.



GRIFFITH ENERGY CRITERION - Continue

Introducing Eq. (31) into Eq. (30) gives

$$2\gamma_s = \frac{\pi\sigma^2 a}{E} \quad (32)$$

Thus Eq. (32) provides the remote fracture stress at failure in the form

$$\sigma_f = \sqrt{\frac{2E\gamma_s}{\pi a}}. \quad (33)$$

Note that the second derivative of Π

$$\frac{d^2\Pi}{da^2} = -\frac{2\pi\sigma^2 B}{E} \quad (34)$$

is always negative. Therefore, any crack growth will be unstable (as the crack length changes from the equilibrium length, the energy will always decrease) and the crack will continue to run.

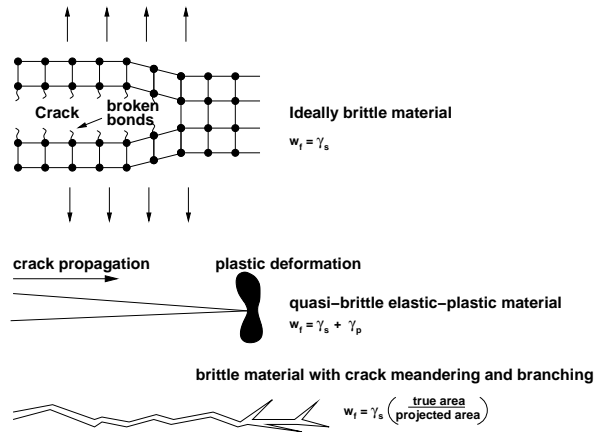
Note that the Griffith model, Eq. (31), applies only to linear elastic material behavior. Thus *the global behavior of the structure must be linear*. Any nonlinear effects such as plasticity must be confined to a small region near the crack tip.

GRIFFITH ENERGY CRITERION - Continue

Note, that the Griffith criterion applies to ideally brittle materials containing sharp cracks as it assumes that the work required to create new surfaces is proportional to the surface energy only. Eq. (33), however, can be generalized for any type of energy dissipation by introducing the fracture energy w_f

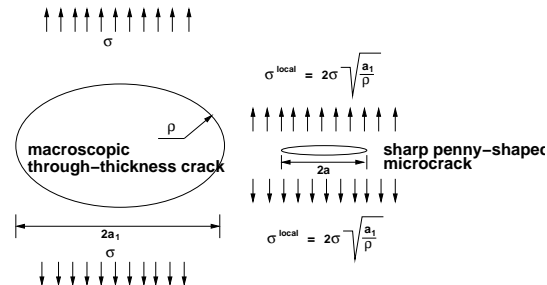
$$\sigma_f = \sqrt{\frac{2Ew_f}{\pi a}}. \quad (35)$$

where w_f could include plastic, viscoelastic, viscoplastic and other effects, depending on material. For the linear elastic solid with the plastic zone confined to a small region near the crack tip the fracture energy is constant. In many ductile materials, however, the fracture energy increases with with the crack growth. In such a case, the energy required for a unit advance of the crack is called the crack growth resistance R .

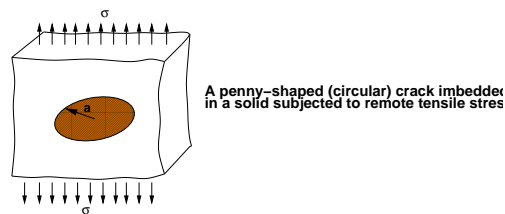


GRIFFITH ENERGY CRITERION - Continue

Example



A flat plate made from a brittle material contains a macroscopic through-thickness crack with half length a_1 and notch tip radius ρ . A sharp penny-shaped microcrack with radius a_2 is located near the crack tip of the larger flaw. Estimate the minimum size of the microcrack to cause failure in the plate when the Griffith equation is satisfied by the global stress and a_1 .



Note that σ_f for a penny-shaped crack is given by

$$\sigma_f = \sqrt{\frac{\pi E \gamma_s}{2(1 - \nu^2)a}}. \quad (36)$$

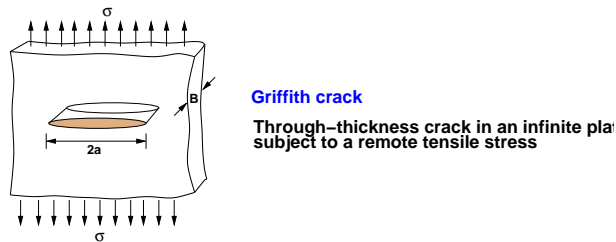
ENERGY RELEASE RATE

In 1956 Irwin proposed an energy approach equivalent to Griffith model but more suitable for solving engineering problems. He introduced an *energy release rate* G as a measure of the energy available for an increment of crack extension

$$G = -\frac{d\Pi}{dA} \quad \text{or} \quad G = -\frac{1}{B} \frac{d\Pi}{da} \quad (37)$$

where B is the thickness of a plane structure. Note that the term *rate* does not refer to a derivative with respect to time. G is the rate of change in potential energy with respect to crack area. G , as it follows from the derivative of a potential, is also called the *crack extension force* or the *crack driving force*.

Consider again the Griffith crack.



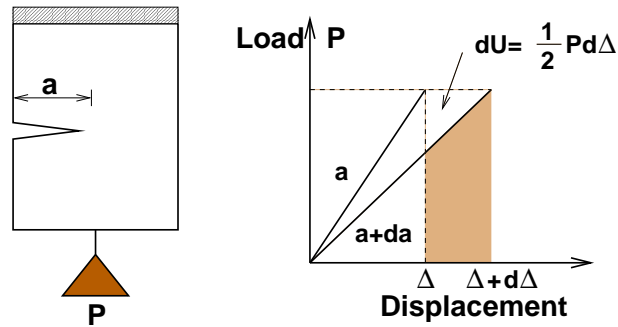
Recall Eq. (31) and write

$$-\frac{d\Pi}{dA} = G = \frac{\pi\sigma^2 a}{E} \quad [\text{N/m}]. \quad (38)$$

Thus the crack extension occurs when G reaches a critical value $G_f = 2w_f$, where G_f is a measure of the *fracture toughness*. Accepting the hypotheses of elastic fracture renders G_f constant.

ENERGY RELEASE RATE - load control test vs. displacement control test

First consider a crack plate that is dead loaded. Since the load is fixed at P the structure is said to be load controlled.



The potential energy of an elastic body is given by

$$\Pi = U - A \quad (39)$$

where U is the strain energy stored in the body and W represents the work done by external forces. For the present loading conditions we have

$$U = E_i = \int_0^{\Delta} P d\Delta = \frac{1}{2} P \Delta \quad (40)$$

$$A = -E_e = P \Delta \quad (41)$$

The energy release rate (fracture energy) is thus provided by

$$G = \frac{1}{B} \left(\frac{dU}{da} \right)_P = \frac{P}{2B} \left(\frac{d\Delta}{da} \right)_P \quad (42)$$

Since the compliance C of the structure assumes the form

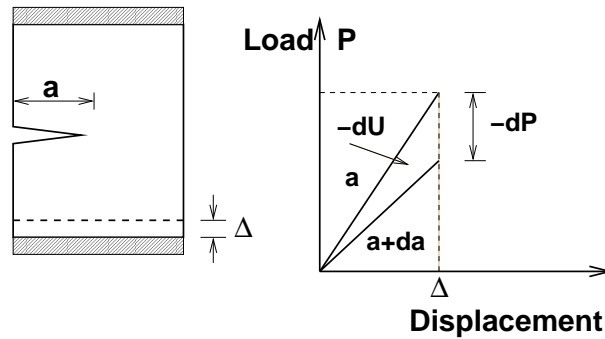
$$C = \frac{\Delta}{P} \quad \text{and} \quad \frac{d\Delta}{da} = P \frac{dC}{da}, \quad (43)$$

we finally arrive at

$$G = \frac{P^2}{2B} \frac{dC}{da}. \quad (44)$$

Note that under the load controlled conditions the energy required for the crack extension is supplied by the applied load.

As a next step consider a loading case in which the displacement is fixed. The structure is displacement controlled.



When the structure is displacement controlled the external work supplied to the structure is zero, and therefore $A = 0$ and $\Pi = U$. The energy release rate is then given by

$$G = -\frac{1}{B} \left(\frac{dU}{da} \right)_{\Delta} = -\frac{\Delta}{2B} \left(\frac{dP}{da} \right)_{\Delta}. \quad (45)$$

With the help of Eq. (43) we get

$$\Delta = PC \quad \text{and} \quad \frac{dP}{da} = -\frac{\Delta}{C^2} \frac{dC}{da} \quad (46)$$

and finally

$$G = \frac{P^2}{2B} \frac{dC}{da}. \quad (47)$$

Note that under the displacement controlled conditions the energy required for the crack extension is supplied by the strain energy.

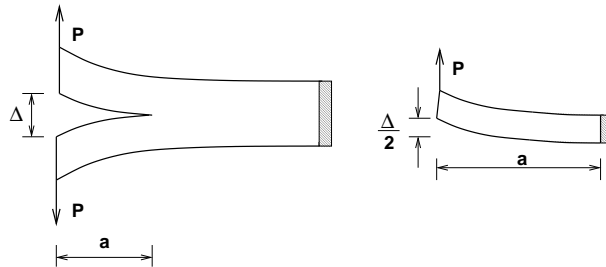
After comparing Eqs. (44) and (47) we see that the energy release rate, is the same as defined in Eq. (37), for both load and displacement control and also

$$\left(\frac{dU}{da} \right)_P = - \left(\frac{dU}{da} \right)_\Delta \quad (48)$$

ENERGY RELEASE RATE - continue

EXAMPLE:

determine the energy release rate for a double cantilever beam (DCB) specimen



Solution: Suppose that the crack is sufficiently deep. This assumption allows application of the beam theory for the derivation of crack opening at the point of the applied load.

Recall

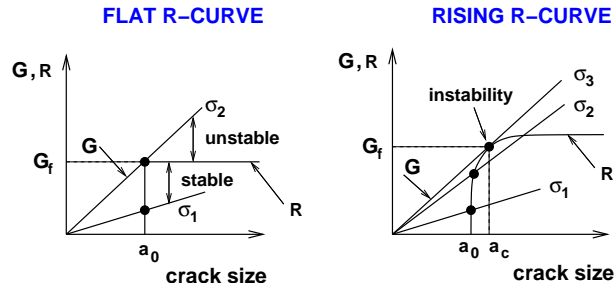
$$\begin{aligned}\frac{\Delta}{2} &= \frac{Pa^3}{3EI} \\ I &= \frac{1}{12}Bh^3 \\ \Delta &= CP \\ C &= \frac{2a^3}{3EI}\end{aligned}$$

Substituting C into Eq. (47) gives

$$G = \frac{P^2}{2B} \frac{dC}{da} = \frac{12P^2a^2}{B^2h^3E} \quad (49)$$

INSTABILITY AND THE R-CURVE

According to definition crack extension occurs when $G = 2w_f = R$, where R is called the material resistance to crack extension. Depending on how G and R vary with the crack size the crack growth may be stable or unstable as shown in the figure below, which corresponds to a response of the Griffith crack.



A plot of R versus crack extension is called a *resistance or R curve*. The corresponding plot of G versus crack extension is the *driving force*.

Condition for the stable crack growth

$$\begin{aligned} G &= R \\ \frac{dG}{dR} &\leq \frac{dR}{da} \end{aligned} \tag{50}$$

Condition for the unstable crack growth

$$\frac{dG}{dR} > \frac{dR}{da} \tag{51}$$

INSTABILITY AND THE R-CURVE - continue

Some final comments

When the resisting curve is flat, one can define a critical value of energy release rate, G_f , unambiguously. A material with a rising R curve, however, cannot be uniquely characterized with a single toughness value. According to Eq. (51) a flaw structure fails when the driving force curve is tangent with R curve, but this point of tangency depends on the shape of the driving force, which depends on configuration of the structure.

The R curve for an ideally brittle material is flat because the surface energy is an invariant property. However, when nonlinear material behavior accompanies fracture, the R curve can take on a variety of shapes.

Materials with rising R curves can be characterized by the value of G at initiation of crack growth. This value, however, characterizes only the onset of crack growth and provides no information on the shape of the R curve.

Ideally, the R curve, should only be a property of the material and not depend on the size or shape of the crack body. Much of fracture mechanics assumes that the fracture toughness is material property.

ENERGY RELEASE RATE - THE GENERAL ENERGY BALANCE

Consider a body with a preexisting crack loaded up to a certain level at which the crack advances an elemental length δa in its own plane. The required energy δW^F is given by

$$\delta W^F = RB\delta a, \quad (52)$$

where B is the plate thickness and R represents the crack growth resistance. When R is a material property not dependent on the crack history, flat R curve, notation $R = G_c$ is often adopted. Hereafter, consider a quasistatic process and suppose that the only energy-consuming process is fracture. Thus the available energy for fracture, or (elemental energy release δW^R), reads

$$\delta W^R = GB\delta a = \delta W - \delta U, \quad (53)$$

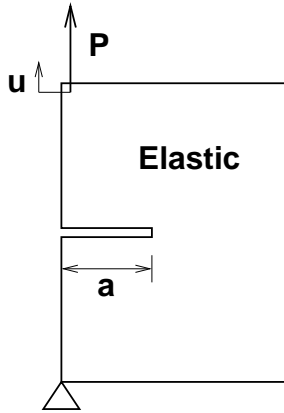
where δW represents, in the infinitesimal process, the total energy supplied to the structure (external work), and δU corresponds to the elastic energy stored in the structure. The specific available energy G (energy release rate) is a measure of energy available for an increment of crack extension. The energy balance at the onset of fracture requires

$$G\delta a = R\delta a. \quad (54)$$

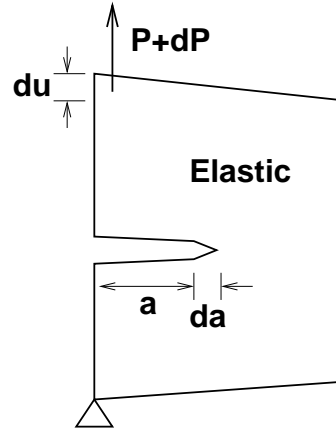
Note that G is a state function which depends on the instantaneous geometry and boundary conditions but not how they were attained in the fracture process. G is path independent.

ENERGY RELEASE RATE - continue

INITIAL SOLUTION



CO-PLANAR CRACK GROWTH UPON FURTHER LOADING



P – applied load
 u – point load displacement

ENERGY RELEASE RATE FROM THE PRINCIPLE OF VIRTUAL DISPLACEMENTS

The primary unknowns (independent variables) in the principle of virtual work are equilibrium displacements which follow from the solution of the elastic problem. The associated equilibrium forces, $P = P(u, a)$, can be determined by elastic equilibrium of the structure.

Consider an elastic body with a crack length a subjected to virtual displacement δu . First, suppose that there is no crack growth. Hence

$$\delta W - \delta[U]_a = 0. \quad (55)$$

Next, consider a general process at which both u and a may vary. Then Eq. (53) attains the form

$$GB\delta a = P(u, a)\delta u - \left(\left[\frac{\partial U(u, a)}{\partial u} \right]_a \delta u + \left[\frac{\partial U(u, a)}{\partial a} \right]_u \delta a \right). \quad (56)$$

where $P(u, a)$ is the load and u is the point load displacement. When considering equilibrium variation at $\delta a = 0$ we arrive at the second Castigliano's theorem

$$P(u, a) = \left[\frac{\partial U(u, a)}{\partial u} \right]_a. \quad (57)$$

Eq. (56) together with Eq. (57) yield

$$G = G(u, a) = - \left[\frac{\partial U(u, a)}{\partial a} \right]_u. \quad (58)$$

ENERGY RELEASE RATE FROM THE PRINCIPLE OF VIRTUAL FORCES

The primary unknowns (independent variables) in the principle of virtual forces are equilibrium forces (stresses).

Starting with the complementary energy U^* given by

$$U^* = Pu - U, \quad (59)$$

denoting $u = u(P, a)$, $U^*(P, a)$ and considering an equilibrium process in which both P and a may vary, and using Eq. (53) we get

$$GB\delta a = -u(P, a)\delta P + \left(\left[\frac{\partial U^*(P, a)}{\partial P} \right]_a \delta P + \left[\frac{\partial U^*(P, a)}{\partial a} \right]_u \delta a \right). \quad (60)$$

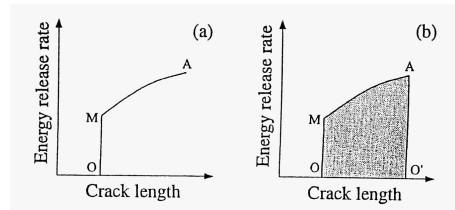
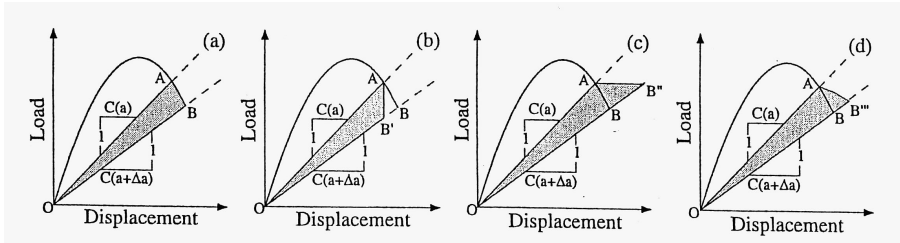
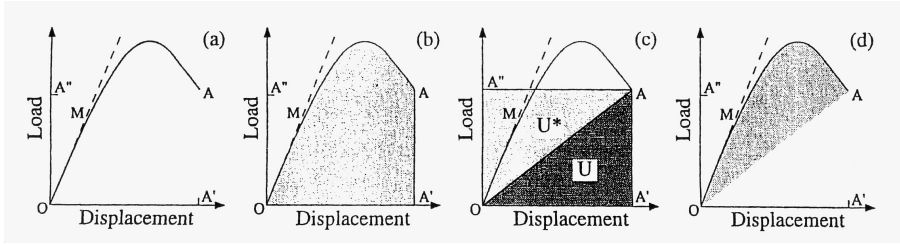
When considering equilibrium variation at $\delta a = 0$ we arrive at the first Castigliano's theorem

$$u(P, a) = \left[\frac{\partial U^*(P, a)}{\partial P} \right]_a. \quad (61)$$

Finally, Eqs. (60) and (61) give

$$G = G(P, a) = \left[\frac{\partial U^*(P, a)}{\partial a} \right]_P. \quad (62)$$

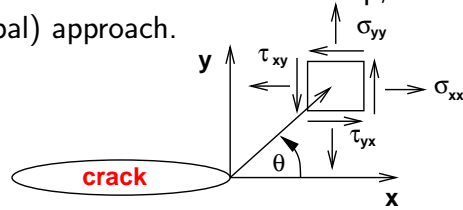
GRAPHICAL REPRESENTATION OF FRACTURE PROCESS



$$GB\Delta a = \text{area}(OAB'') = \frac{1}{2}P(\overline{AB''}) = \frac{1}{2}P[PC(a + \Delta a) - PC(a)] = \frac{1}{2}P^2C'(a)\Delta a$$

GENERAL NEAR-TIP FIELDS. STRESS INTENSITY FACTORS

For certain cracked configurations subjected to external forces, it is possible to derive closed-form solutions for the stresses in the body, assuming linear elastic material behavior. The early works on this subject are due to Westergaard and Irwin. Irwin, in particular, proved that so-called local approach, in which the essentials of LEFM are formulated in terms of stresses close to the crack tip, is essentially equivalent to the Griffith energetic (or global) approach.

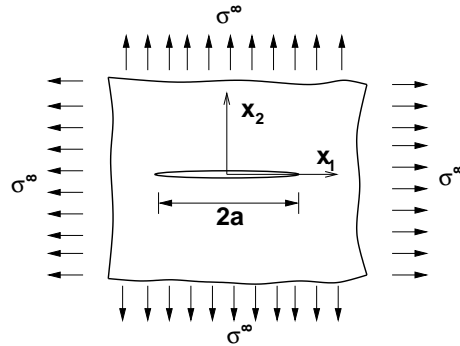


When defining a polar coordinate system (r, θ) with the origin at the crack tip the stress field in any linear elastic cracked body can be written as

$$\sigma_{ij} = \left(\frac{k}{\sqrt{r}} \right) f_{ij}(\theta) + \text{other terms} \quad (63)$$

where k is a constant and f_{ij} is dimensionless function of θ . As evident from Eq. (63) the stress near the crack tip varies with $1/\sqrt{r}$, regardless of the configuration of the cracked body. Note that when $r \rightarrow 0$ the stress approaches to ∞ . In other words, when a body contains a crack, a strong concentration develops around a crack tip. However, for linear elastic material this stress concentration has the same distribution close to the crack tip regardless of the size shape and specific boundary conditions of the body. Only the intensity of the stress concentration varies. For the same intensity, the stresses around the crack tip are identical.

CENTER CRACKED INFINITE PANEL



For the center cracked infinite panel loaded by remote normal stress σ^∞ in all directions Griffith showed that the normal stress σ_{yy} along the uncracked part of the crack plane ($y = 0, x^2 - a^2 > 0$) is given by

$$\sigma_{yy} = \sigma^\infty \frac{|x|}{\sqrt{x^2 - a^2}}. \quad (64)$$

Introducing the polar coordinate, setting $\theta = 0, x - a = r, x = r + a, x + a = r + 2a, x^2 - a^2 = (x + a)(x - a)$, and expanding the term $(1 + r/a)/\sqrt{(1 + r/2a)}$ via Taylor series we get

$$\sigma_{yy} = \frac{\sigma^\infty \sqrt{a}}{\sqrt{2r}} \left[1 + \frac{3r}{4a} - \frac{5r^2}{32a^2} \dots \right]. \quad (65)$$

The term in the brackets tends to 1 when $r \ll a$. Recall Eq. (63) to see that in this particular case we have

$$k = \frac{\sigma^\infty \sqrt{a}}{\sqrt{2}}. \quad (66)$$

STRESS INTENSITY FACTOR - THREE TYPES OF LOADING

There are three types of loading that a crack can experience.

1. Mode I: principal load is applied to the crack plane, tends to open the crack
2. Mode II: in-plane shear loading, tends to slide one crack face with respect to the other in its own plane
3. Mode III: out-of-plane shear loading, tends to slide one crack face with respect to the other out of plane

A crack body can be loaded in any one of these modes, or combination of them. Nevertheless, each mode produces the $1/\sqrt{r}$ singularity at the crack tip, but the proportionality constant k and function f_{ij} in Eq. (63) depend on a specific mode.

It is customary to define k in terms of the *stress intensity factor* K and write

$$K = k\sqrt{2\pi}. \quad (67)$$

Eq. (63) thus receives the form

$$\lim_{r \rightarrow 0} \sigma_{ij}^{(I,II,III)} = \frac{K_{(I,II,III)}}{\sqrt{2\pi r}} f_{ij}^{(I,II,III)}(\theta) \quad K[\text{Nm}^{-1/2}] \quad (68)$$

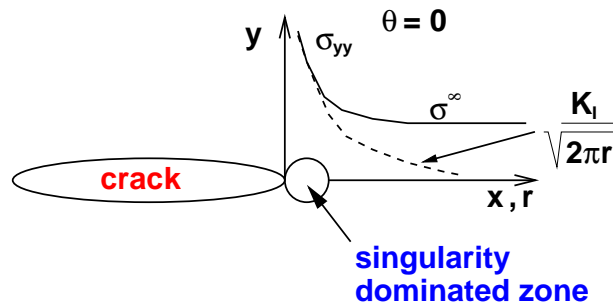
where I, II, III refer to individual loading modes. Note that individual contributions to a given stress component are additive

$$\sigma_{ij}^{\text{total}} = \sigma_{ij}^{(I)} + \sigma_{ij}^{(II)} + \sigma_{ij}^{(III)}.$$

However

$$K_{\text{total}} \neq K_I + K_{II} + K_{III}.$$

STRESS CONCENTRATION FACTOR - THREE TYPES OF LOADING



When $\theta = 0$, the shear stress is zero and crack plane is a principal plane for pure Mode I loading. Then the stresses, in the close vicinity of the crack tip, assume the form

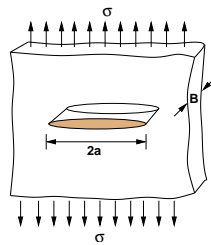
$$\sigma_{xx}^{(I)} = \sigma_{yy}^{(I)} = \frac{K_I}{\sqrt{2\pi r}}. \quad (69)$$

Note that Eq. (69) is only valid near the crack tip, where the $1/\sqrt{r}$ singularity dominates the stress field. Stresses far from the crack tip are governed by the remote boundary conditions. Therefore, the singularity dominated zone is defined as a region where Eq. (69), or more general Eq. (68), describes the crack tip stresses. Thus the stresses near the crack tip increase in proportion to K (K defines the amplitude of the crack tip singularity).

As intimated in the introductory part, for linear elastic material the stress concentration has the same distribution close to the crack tip regardless of the size shape and specific boundary conditions of the body. Thus the stress intensity factor K completely defines the crack tip conditions (single parameters description of the crack tip conditions).

RELATIONSHIP BETWEEN K AND G

1. Energy release rate G : quantifies the net change in potential energy due to increment of crack extension, global parameter
2. Stress intensity factor K : characterizes the stresses, strains and displacement fields near the crack tip, local parameter



Griffith crack

Through-thickness crack in an infinite plate subject to a remote tensile stress

In the limit of LEFM parameters G and K are uniquely related. As an example, consider again a through crack in an infinite plate subject to a uniform stress. In this particular case we have

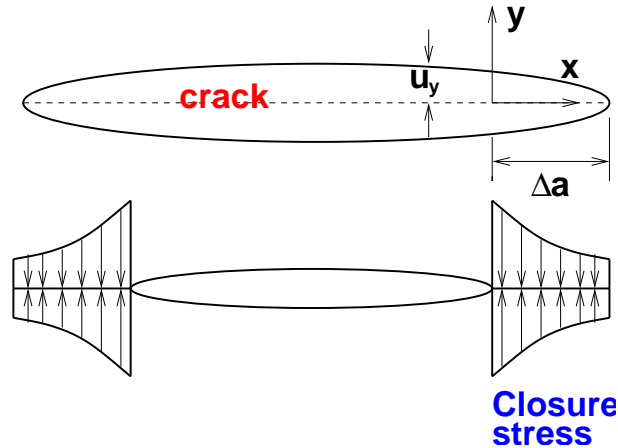
$$G = \frac{\pi a \sigma^2}{E}$$
$$K_I = \sigma \sqrt{\pi a}$$

and therefore

$$G = \frac{K_I^2}{E'}, \quad (70)$$

where $E' = E$ for plane stress and $E' = \frac{E}{1-\nu^2}$ for plane strain.

RELATIONSHIP BETWEEN K AND G - CRACK CLOSURE ANALYSIS



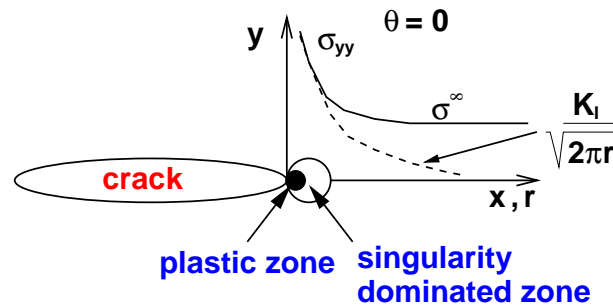
To arrive at a general relationship between G and K we follow Irwin and assume that the energy release rate G associated with the advancement of crack Δa can be linked to the work required to close the crack in this region. According to definition

$$G = \lim_{\Delta a \rightarrow 0} \left(\frac{\Delta U}{\Delta a} \right)_P \quad (71)$$

where ΔU is the work of crack closure.

CRACK TIP PLASTICITY

Recall that the LEFM applies to sharp cracks. The assumption of sharp cracks, however, leads to the prediction of infinite stresses at the crack tip. On the other hand, stresses in real materials are finite because the crack tip radius is finite (recall an atomic view on fracture). In addition, inelastic deformation, e.g., plasticity in metals, crazing in polymers or damage in concrete, results in further reduction of crack tip stresses → modification of the LEFM to account for the crack tip yielding.

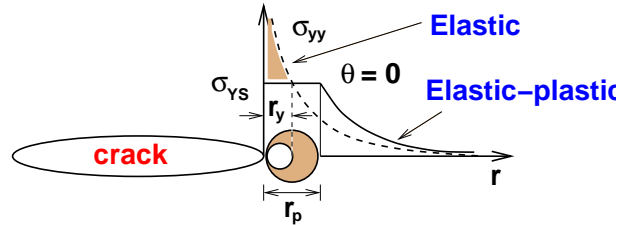


If the *plastic zone* at the crack tip is sufficiently small (confined within the singularity dominated zone), there are two simple approaches available that provide corrections to the LEFM:

1. the Irwin approach
2. the strip yield model

Note: although the term *plastic zone* usually applies to metals it will be used here to represent the inelastic crack tip behavior in more general sense.

THE IRWIN APPROACH



1. First-order estimate of the plastic zone size: consider the crack plane ($\theta = 0$) and suppose that the boundary between elastic plastic behavior occurs when the stress σ_{yy} given by Eq. (69) satisfies the yield criterion ($\sigma_{yy} = \sigma_{Ys}$ for plane stress). Thus substituting the yield stress σ_{yy} into Eq. (69) and solving for r gives a first order estimate of plastic zone size in the form

$$r_y = \frac{1}{2\pi} \left(\frac{K_I}{\sigma_{Ys}} \right)^2 \quad (72)$$

Flaw: the analysis is based on purely elastic crack tip analysis

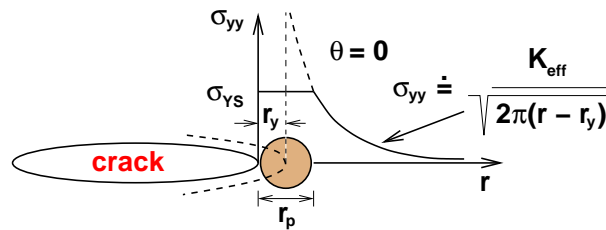
2. Second-order estimate of the plastic zone size: note that when yielding occurs the stresses must redistribute ahead of the crack tip to satisfy equilibrium. To that end, consider a simple force balance to get

$$\sigma_{Ys} r_p = \int_0^{r_y} \sigma_{yy} dr = \int_0^{r_y} \frac{K_I}{\sqrt{2\pi r}} dr = \frac{K_I}{\sqrt{2\pi}} \left[\frac{1}{2} \right]_0^{r_y} \quad (73)$$

Substituting for r_y from Eq. (72) into Eq. (73) gives a second order estimate of the plastic zone as

$$r_p = \frac{1}{\pi} \left(\frac{K_I}{\sigma_{YS}} \right)^2 \quad (74)$$

which is twice as large as r_y , the first order estimate. Note that the redistributed stress in the elastic region is higher than predicted by Eq. (69), which implies a higher effective stress intensity factor (K_{eff}). Irwin found that a good approximation of K_{eff} can be obtained by placing the tip of the effective crack in the center of the plastic zone.

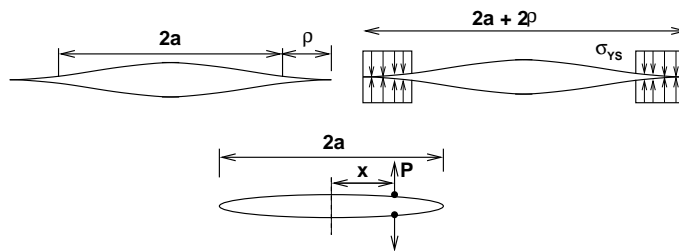


Determination of the effective crack size a_{eff}

1. set $a_{eff} = a + r_y$
2. write $K_{eff} = C(a_{eff})\sigma\sqrt{\pi a_{eff}}$ where $C(a_{eff})$ is the geometry correction factor
3. iterate within the first two steps to solve for K_{eff}

THE STRIP YIELD MODEL - suitable for polymers

The strip yield model was first proposed by Dugdale and Barenblatt. They assumed a long slender plastic zone at the crack tip in nonhardening material in plane stress. In further discussion we limit our attention to a through crack in infinite plate. This model is a classical application of the principle of superposition as it approximates the elastic-plastic behavior by superimposing two elastic solutions: a through crack under remote tension and a through crack with closure stresses at the tip.



The idea is as follows. Since the stresses at the strip yield zone are finite, there cannot be a singularity at the crack tip (the stress intensity factor at the tip of plastic zone must be equal to zero). Thus the plastic zone length ρ is found from the condition that the stress intensity factors from the remote tension and closure stress cancel one another.

To proceed, consider first a through crack in an infinite plate loaded by a normal force P applied at a distance x from the center line of the crack. The stress intensities for

the two crack tips are then give by

$$K_I(+a) = \frac{P}{\sqrt{\pi a}} \sqrt{\frac{a+x}{a-x}} \quad K_I(-a) = \frac{P}{\sqrt{\pi a}} \sqrt{\frac{a-x}{a+x}} \quad (75)$$

In the next step, we identify the force P at a point with the closure stress σ_{YS} through the

$$P = -\sigma_{YS} dx$$

and replace a with $a + \rho$ to arrive at the stress intensity factor from closure stress

$$K_{closure} = -\frac{\sigma_{YS}}{\sqrt{\pi(a+\rho)}} \int_a^{a+\rho} \left[\sqrt{\frac{a+\rho+x}{a+\rho-x}} + \sqrt{\frac{a+\rho-x}{a+\rho+x}} \right] dx \quad (76)$$

Solving this integral yields

$$K_{closure} = -2\sigma_{YS} \sqrt{\frac{a+\rho}{\pi}} \cos^{-1} \left(\frac{a}{a+\rho} \right) \quad (77)$$

The stress intensity from the remote tensile stress is given by

$$K_{\sigma} = \sigma \sqrt{\pi(a+\rho)} \quad (78)$$

Finally, equating Eqs. (77) and (78) gives

$$\frac{a}{a+\rho} = \cos \left(\frac{\pi \sigma}{2\sigma_{YS}} \right) \quad (79)$$

Expanding the right hand side of Eq. (79) via Taylor series provides

$$\frac{a}{a + \rho} = 1 - \frac{1}{2!} \left(\frac{\pi\sigma}{2\sigma_{YS}} \right)^2 + \frac{1}{4!} \left(\frac{\pi\sigma}{2\sigma_{YS}} \right)^4 - \frac{1}{6!} \left(\frac{\pi\sigma}{2\sigma_{YS}} \right)^6 + \dots \quad (80)$$

Neglecting all but the first two terms and solving for the plastic zone size gives

$$\rho = \frac{\pi^2 \sigma^2 a}{2\sigma_{YS}} = \frac{\pi}{8} \left(\frac{K_I}{\sigma_{YS}} \right)^2 \quad (81)$$

Recall the Irwin approach which gives the size of plastic zone r_p as

$$r_p = \frac{1}{\pi} \left(\frac{K_I}{\sigma_{YS}} \right)^2$$

and notice that $1/\pi = 0.318$ and $\pi/8 = 0.392$. Therefore, the Irwin approach and the strip yield model predict similar plastic zone sizes. The K_{eff} follows from the strip yield model after replacing a by $a + \rho$. This yields for the through crack in the infinite plate under remote stress

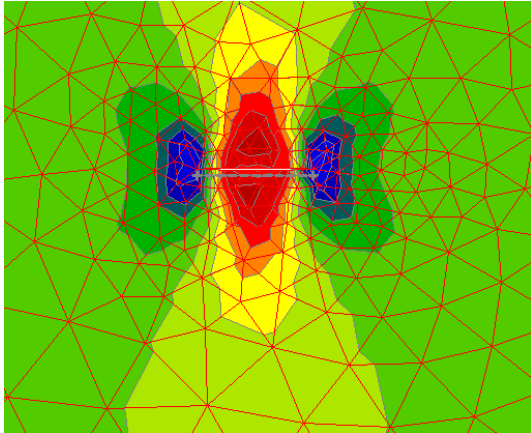
$$K_{eff} = \sigma \sqrt{\pi a \sec \left(\frac{\pi\sigma}{2\sigma_{YS}} \right)} \quad (82)$$

The actual a_{eff} is somewhat less than $a + \rho$. More realistic estimate of K_{eff} was derived by Burdekin and Stone

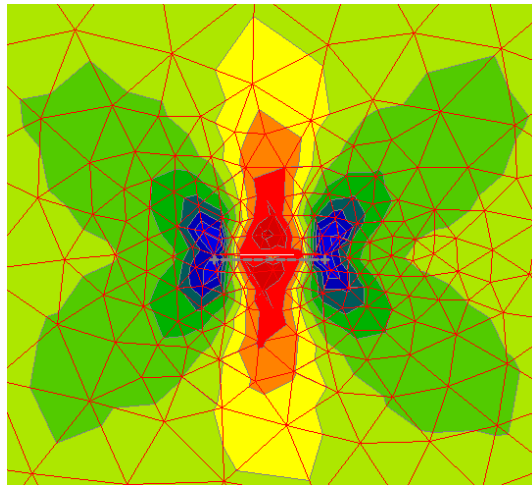
$$K_{eff} = \sigma_{YS} \sqrt{\pi a} \left[\frac{8}{\pi^2} \ln \sec \left(\frac{\pi\sigma}{2\sigma_{YS}} \right) \right]^2 \quad (83)$$

PLANE STRESS VS. PLANE STRAIN

PLANE STRESS



PLANE STRAIN



K AS A FRACTURE CRITERION - LOCAL FRACTURE CRITERION FOR MODE I (K_{Ic})

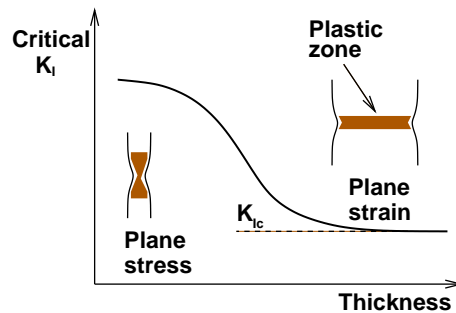
Here we limit our attention to a pure mode I failure. In the limit of the LEFM the stress state of the material in the singularity dominated zone (excluding a very small plastic zone at the crack tip) is uniquely determined by K_I . When assuming the material fails at some combination of local stresses and strains, then the crack extension must occur at a critical K_{Ic} value called fracture toughness. This value is a material constant independent of the size and geometry of the crack body and may be determined by performing a fracture test. If certain conditions are met the K_I value that provokes failure is set to K_{Ic} . Since the energy release rate is uniquely related to stress intensity, G also provides a single-parameter description of the crack tip conditions, and G_c , or G_f is an alternative measure of fracture.

1. Effect of loading mode: the critical stress intensity factor for a given mode is a material constant, but K_c varies with the loading mode

$$K_{Ic} \neq K_{IIc} \neq K_{IIIc}$$

Under combination of loading modes, an initially straight crack kinks upon fracture and the fracture criteria must give not only the loading combination that produces the fracture, but also the kink direction. Mixed-mode fracture is still a subject of an ongoing research. Nevertheless, the vast majority of practical applications consider only the Mode I fracture.

2. Effect of specimen dimensions: the critical stress intensity factor is a material constant only when certain conditions are met. Recall, e.g., that a lower degree of stress triaxiality usually results in higher toughness. Also, the through thickness constraint may affect the shape of the R curve. In particular, the R curve for a material in *plane strain* may be relatively flat (single valued toughness), while the *plane stress* R curve usually rises with crack growth.



3. Limits to the validity of LEFM: according to the American Society for Testing and Materials (ASTM) standard for K_{Ic} the following specimen size requirements must be met to obtain a valid K_{Ic} results in *metals*:

$$a, B, (W - a) \geq 2.5 \left(\frac{K_I}{\sigma_{YS}} \right)^2$$

where a, B, W are the crack size, thickness, and width of the specimen, respectively. The thickness requirement ensures nearly plane strain conditions and the requirement on in-plane dimensions ensures that the nominal behavior is predominantly linear elastic.

SIZE EFFECT - QUANTIFICATION OF FRACTURE MECHANICS SF

In the classical theories based on plasticity or limit analysis (or other theories in which the material failure criterion is expressed in terms of critical stresses or strains), the strength of geometrically similar structures is independent of the structure size. However, the failure behavior of structures made of brittle or quasibrittle material (e.g., concrete) is usually size dependent. Such failures are said to exhibit size effect. The size effect on the structural strength is represented by the deviation of the actual load capacity of a structure from the load capacity predicted by any theory based on critical stresses or strains.

Fracture mechanics size effect, resulting from the release of stored energy of the structure into fracture front, is the most important source of size effect. The size effect is, for design engineers, the most compelling reason for adopting fracture mechanics (Z.P. Bažant).

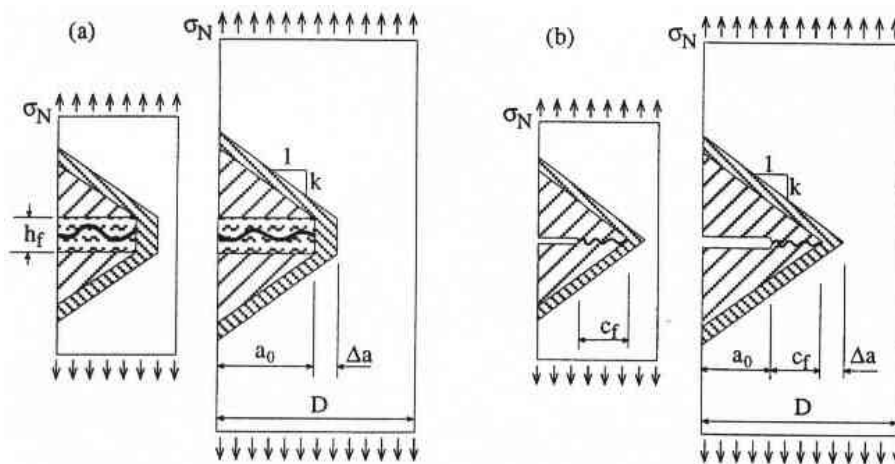
Since the size effect is understood as the dependence of the structure strength on the structure size, it is rigorously defined through a comparison of structure strength of geometrically similar structures of different sizes. The structure strength is commonly defined as the nominal stress σ_{N_u} (load divided by a typical cross-sectional area) at the peak load

$$\sigma_N = c_N \frac{P}{bD} \quad \text{for 2D similarity} \quad \sigma_N = c_N \frac{P}{D^2} \quad \text{for 3D similarity} \quad (84)$$

where P is the applied load, b is the thickness in 2D structure, D is the characteristic dimension of the structure or specimen and c_N is a certain coefficient, which may be set to 1.

EXPLANATION OF FRACTURE MECHANICS SIZE EFFECT (Bažant)

First, consider a uniformly stressed panel and suppose that fracture propagates via the formation of a crack band of thickness h_f . The load required to propagate the band follows from energy balance equation, i.e., energy available is equal to the fracture energy (the energy required for band extension). To that end, assume that due to presence of crack band the strain energy in the band and cross-hatched area drops from $\sigma_N^2/2E$ to zero (this region is called the stress relief zone). Next, consider a geometrically similar panel.



It is usually the case, that the larger the panel, the larger the crack band and consequently the larger the cross-hatched area \implies in a larger structure, more energy is released in a strip by the same extension of the crack band. It is usually assumed that the edges of the specimen are fixed during the crack advance (displacement control), and so the external work is zero. The condition balancing the total energy released from the stress relief zone and the fracture energy needed to advance the crack by Δa reads

$$b(h_f \Delta a + 2ka_0 \Delta a) \frac{\sigma_N^2}{2E} = G_f b \Delta a \quad (85)$$

Following Bažant we further denote

$$Bf' = \sqrt{\frac{G_f E}{h_f}} = \text{const} \quad \text{and} \quad D_0 = \frac{h_f D}{2ka_0} = \text{const}$$

where f' is the tensile strength and $D/a_0 = \text{const}$ due to geometrical similarity. Combining the above expressions together with Eq. (85) gives the Bažant size effect equation in the form

$$\sigma_{N_u} = \frac{Bf'}{\sqrt{1 + D/D_0}} \quad (86)$$

Note that both Bf' and D_0 depend on the fracture properties of the material and on the geometry of the structure, but not on the structure size. Also note that Eq. (86) is approximate, valid only within a range of about 1:20 for most structures.

SIZE EFFECT IN PLASTICITY

Remember that the size effect is defined by comparing geometrically similar structures of different sizes. The goal here is to investigate the effect of the size on the nominal strength σ_{N_u} written as

$$\sigma_{N_u} = c_N \frac{P_u}{bD} \quad (87)$$

Consider a reference structure of size D and geometrically similar one of size $D' = \lambda D$, where λ is the scaling factor and write stresses at an arbitrary point of coordinates (x_1, x_2) in terms of nominal stress σ_N as $\sigma_{ij}(\sigma_N, x_1, x_2)$. Next define a set homologous points of coordinates $(x'_1 = \lambda x_1, x'_2 = \lambda x_2)$. The similitude (podobnost) laws state that

$$\sigma'_{ij}(\sigma_N, x'_1, x'_2) = \sigma_{ij}(\sigma_N, x_1, x_2) \quad \text{with } x'_1 = \lambda x_1, x'_2 = \lambda x_2 \quad (88)$$

$$\epsilon'_{ij}(\sigma_N, x'_1, x'_2) = \epsilon_{ij}(\sigma_N, x_1, x_2) \quad \text{with } x'_1 = \lambda x_1, x'_2 = \lambda x_2 \quad (89)$$

$$u'(\sigma_N, x'_1, x'_2) = \lambda u(\sigma_N, x_1, x_2) \quad \text{with } x'_1 = \lambda x_1, x'_2 = \lambda x_2 \quad (90)$$

Eqs. (88), (89) and (90) thus imply that the stress and strain maxima also occur at homologous points. Therefore, if failure is assumed to occur when the stress or strain or in case of plasticity a certain function $\Psi(\sigma, \epsilon)$ reaches a critical value, $\Psi(\sigma, \epsilon) = \Psi_c$, then the two similar structures will fail at the same nominal stress ($\sigma'_{N_u} = \sigma_{N_u}$). In such a case we say that there is no size effect.

SIZE EFFECT IN LEFM

Let D be a characteristic length (arm depth in DCB specimen) and all the remaining dimensions being proportional (length-to-depth ratio for the DCB). To proceed it is desirable to express G and K_I in terms of the variables P or σ_N , D and $\alpha = a/D$. In particular, we write

$$K_I = \frac{P}{b\sqrt{D}}\hat{k}(\alpha) = \sigma_N\sqrt{D}k(\alpha) \quad \text{and} \quad G = \frac{P^2}{b^2DE'}\hat{g}(\alpha) = \frac{\sigma_N^2}{E'}Dg(\alpha) \quad (91)$$

where $\hat{k}(\alpha)$, $k(\alpha)$, $\hat{g}(\alpha)$, $g(\alpha)$ are dimensionless constants, $\hat{g}(\alpha) = \hat{k}^2(\alpha)$, $g(\alpha) = k^2(\alpha)$, $k(\alpha) = \hat{k}(\alpha)/c_N$, $g(\alpha) = \hat{g}(\alpha)/c_N^2$, and α is the relative crack depth.

Consider now a family of geometrically similar plane cracked structures loaded in mode I. Let a_0 and $\alpha_0 = a_0/D$ are the initial crack and initial relative crack length, respectively. Suppose that $k(\alpha)$ increases with α (positive geometries), then σ_N decreases with the crack advance and the peak load coincides with the onset of crack growth ($K_I = K_{Ic}$) and

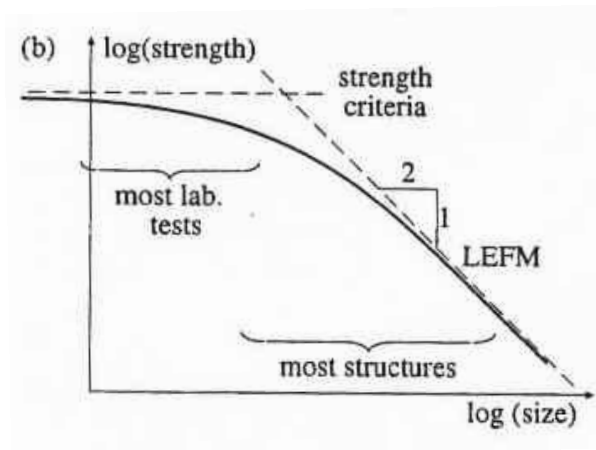
$$\sigma_{N_i} = \sigma_{N_u} = \frac{K_{Ic}}{\sqrt{D}k(\alpha_0)} \quad \sigma_N = \frac{K_{Ic}}{\sqrt{D}k(\alpha)} \quad (92)$$

to keep $K_I = K_{Ic}$ during the crack growth. Evidently, since α_0 is constant for geometrically similar structures, the nominal strength is always proportional to the square root of the size.

Therefore, for similar precracked structures, the nominal strengths are related

$$\begin{aligned}
 D &= \lambda D_1 \\
 \sigma_{N_u} \sqrt{D} &= \sigma_{N_{u1}} \sqrt{D_1} \\
 \sigma_{N_u} \sqrt{\lambda D_1} &= \sigma_{N_{u1}} \sqrt{D_1} \\
 \sigma_{N_u} &= \lambda^{-\frac{1}{2}} \sigma_{N_{u1}}
 \end{aligned} \tag{93}$$

It follows, from above, that geometrically similar structures following LEFM exhibit the inverse square root size effect.



NONLINEAR FRACTURE MECHANICS (ELASTIC-PLASTIC FM)

Recall that LEFM is only valid as long as nonlinear material behavior is confined to a small region surrounding the crack tip. There are many materials, however, for which the applicability of LEFM is impossible or at least suspicious. Therefore, an alternative fracture mechanics model is required.

Elastic-plastic fracture mechanics applies to materials that exhibit time-independent, nonlinear behavior (plastic deformation). There are two parameters characterizing the nonlinear behavior at the crack tip:

1. CTOD - crack tip opening displacement
2. J counter integral

Critical values of CTOD and J give nearly size-independent measures of fracture toughness, even for relatively large amount of crack tip plasticity. Note that there are still limits to the applicability of J and CTOD, but these limit are much less restrictive than the validity requirements of LEFM.

CTOD - CRACK TIP OPENING DISPLACEMENT

Wells observed that

- There is an important class of structural steels that are too tough to be characterized by LEFM
- Blunting of initially sharp cracks prior to fracture. The degree of crack blunting increases in proportion to the toughness of the material → he proposed the crack tip opening displacement as a measure of fracture toughness.

Relationship between G and K_I and CTOD in the limit of LEFM (small scale yielding)

The Irwin approach

$$\begin{aligned}u_y &= \frac{\kappa + 1}{2\mu} K_I \sqrt{\frac{r_y}{2\pi}} \\r_y &= \frac{1}{2\pi} \left(\frac{K_I}{\sigma_{YS}} \right)^2 \\ \delta &= 2u_y = \frac{4K_I^2}{\pi\sigma_{YS}E} = \frac{4G}{\pi\sigma_{YS}}\end{aligned}$$

The strip yield model

$$\begin{aligned}\delta &= \frac{1 - \nu^2}{E} \frac{K_I^2}{\sigma_{YS}} \text{ plain strain} = \frac{G}{\sigma_{YS}} \\ \delta &= \frac{1}{E} \frac{K_I^2}{\sigma_{YS}} \text{ plain stress} = \frac{G}{\sigma_{YS}}\end{aligned}$$

Definition of CTOD

There are a number of alternative definitions of CTOD. The two most common are:

1. the displacement at the original crack tip
2. so called 90⁰ degree intercept

The above two definitions are equivalent if the crack blunts in semicircle.

Laboratory measurements of CTOD

Most laboratory measurements of CTOD have been made on edge-cracked specimens loaded in three-point bending. When inferring the CTOD from this experiment it is assumed that the specimen halves are rigid and rotate about a hinge point. Then, measuring the crack mouth opening (V), as is usually the case, enables to compute the CTOD from a similarity of triangles as

$$\frac{\delta}{r(W-a)} = \frac{V}{r(W-a)+a} \implies \delta = \frac{r(W-a)V}{r(W-a)+a} \quad (94)$$

where r is a rotational factor (dimensionless constant between 0 and 1). Note that the hinge model becomes inaccurate when displacements are primarily elastic. Therefore, the total displacement is usually separated into elastic and plastic components and the hinge model is applied only to plastic displacements to get

$$\delta = \delta_{el} + \delta_{pl} = \frac{K_I^2}{m\sigma_{YS}E'} + \frac{r_p(W-a)V_p}{r_p(W-a)+a} \quad (95)$$

The plastic rotational factor r_p is approximately 0.44 for typical materials (metals) and test specimens. Note that Eq.(95) reduces to small scale yielding for linear elastic condition and the hinge model dominates when $V = V_p$.

Stability criterion for crack growth

The crack is stable as long as

$$\text{CTOD} \leq \text{CTOD}_c$$

The determination of CTOD_c , however, is ambiguous. Usually, the value of CTOD_{in} at the onset of crack growth is measured. In the limit of LEFM this value can be used to infer the critical value of stress intensity factor K_{Ic} (structural steel with low fracture toughness).

Specimen size requirements

$$B \geq 25\text{CTOD}_{in}$$

K_{Ic} from CTOD

$$\text{if } \text{CTOD}_{in} = \frac{4}{\pi} \frac{K_{Ic}^2}{\sigma_{YS} E'} \quad \text{then } B \geq 25 \frac{4}{\pi} \frac{K_{Ic}^2}{\sigma_{YS} E'}$$

Recall that for LEFM we require

$$B \geq 2.5 \left(\frac{K_{Ic}}{\sigma_{YS}} \right)^2$$

It is therefore evident that when inferring K_{Ic} from CTOD we may use specimens of substantially smaller sizes.

J CONTOUR INTEGRAL

Rice presented a path-independent contour integral of analysis of cracks and showed that the value of this integral, called J , is equal to the energy release rate in a nonlinear elastic body that contains crack. Hutchinson and also Rice and Rosengren further showed that J uniquely characterizes crack tip stresses and strains in nonlinear material. Thus the J integral can be viewed as both an energy parameter and a stress intensity parameter.

J as nonlinear energy release rate

Recall Eq. (37) and write the nonlinear energy release rate in the form

$$J = -\frac{d\Pi}{dA} \quad \text{or} \quad J = -\frac{1}{B} \frac{d\Pi}{da} \quad (96)$$

Next, consider a cracked plate which exhibits a nonlinear load-displacement curve.

- Load control

$$\Pi = U - W = U - P\Delta = -U^* \quad (97)$$

where U^* is the complimentary strain energy given by

$$U^* = \int_0^P \Delta dP$$

Thus substituting Eq. (97) into Eq. (96) provides

$$J = \left(\frac{dU^*}{da} \right)_P = \left(\frac{d}{da} \int_0^P \Delta dP \right)_P = \int_0^P \left(\frac{d\Delta}{da} \right)_P dP \quad (98)$$

- Displacement control

$$\Pi = U \quad (99)$$

where the strain energy U is given by

$$U = \int_0^{\Delta} P d\Delta$$

After introducing Eq. (99) into Eq. (96) we get

$$J = - \left(\frac{dU}{da} \right)_{\Delta} = - \left(\frac{d}{da} \int_0^{\Delta} P d\Delta \right)_{\Delta} = - \int_0^{\Delta} \left(\frac{dP}{da} \right)_{\Delta} d\Delta \quad (100)$$

After inspecting Eqs. (98) and (100) we arrive at

$$\int_0^P \left(\frac{d\Delta}{da} \right)_P dP = - \int_0^{\Delta} \left(\frac{dP}{da} \right)_{\Delta} d\Delta \quad (101)$$

Recall that in the limit of LEFM

$$J = G = \frac{K^2}{E'} \quad (102)$$

J as a path-independent line integral

Consider an arbitrary counter-clockwise path (Γ) around the tip of a crack. The J integral is given by

$$J = \int_{\Gamma} \left(w \, dy - p_i \frac{\partial u_i}{\partial x} \, ds \right) \quad (103)$$

where w is the strain energy density defined as

$$w = \int_0^{\epsilon_{ij}} \sigma_{ij} \, d\epsilon_{ij} \quad (104)$$

and $p_i = \sigma_{ij} n_j$ are the surface tractions. Rice showed that the value of J is independent of the path of integration around the crack. Thus J is called a *path-independent* integral.

J as a stress intensity factor

Under the assumption of nonlinear elasticity Hutchinson and Rice & Rosengren independently showed that J characterizes crack tip conditions. They assumed a power law relationship between plastic strain and stress which in case of uniaxial deformation reads (Ramberg-Osgood law)

$$\frac{\epsilon}{\epsilon_0} = \frac{\sigma}{\sigma_0} + \alpha \left(\frac{\sigma}{\sigma_0} \right)^n \quad (105)$$

where σ_0 is a reference stress value usually equal to yield stress, $\epsilon_0 = \sigma_0/E$, α is a dimensionless constant, and n is the strain hardening exponent. H & R & R showed that

for J to remain path independent the quantity stress \times strain must vary with $1/r$ near the crack tip. When limiting our attention to the plastic zone (elastic deformations are small small compare to elastic ones) Eq. (105) reduces to a simple power law. Stresses and strains ahead of the crack tip then receive the following forms

$$\sigma_{ij} = k_1 \left(\frac{J}{r} \right)^{\frac{1}{n+1}} \quad (106)$$

$$\epsilon_{ij} = k_2 \left(\frac{J}{r} \right)^{\frac{n}{n+1}} \quad (107)$$

where k_1, k_2 are proportionality constants. Note that for linear elastic material $n = 1$ and indeed above equations predict a $1/\sqrt{r}$ singularity as expected.

It can be concluded that J integral defines the amplitude of the HRR singularity, just as the stress intensity factor characterizes the amplitude of the linear elastic singularity. J completely describes the conditions within the plastic zone. Thus a structure in small-scale yielding has two singularity dominated zones: one in the elastic region, where stress varies as $1/\sqrt{r}$ and one in the plastic zone where stress varies as $r^{-1/(n+1)}$.

DETERMINATION OF J FOR A STATIONARY CRACK

1. In elastic material $J = G$ and G is uniquely related to the stress intensity factor
2. Application of the line integral definition of J , Eq. (103) - not suitable for experimental measurements
3. Invoking the energy release rate definition of J , Eq. (96) - more suitable for experimental measurements

Laboratory measurements of J based on energy release rate definition

1. Determination of J by measuring on a series of specimens (Landes and Begly)
2. Determination of J by measuring on a single specimen - unloading compliance method for monitoring crack growth

A good insight to how determine J experimentally can be provided by an analysis of an edge cracked plate in bending assuming the plastic region spreads over the total ligament length. To that end consider a single edge notched bend (SENB) specimen.

Procedure: First split the angle ψ into elastic and inelastic parts as $\psi = \psi_{el} + \psi_p$. If $\psi_{el} \ll \psi_p$ then

$$M = \sigma_0 B (W - a)^2 g(\psi_p, \frac{\sigma_0}{E}, n)$$

$$\psi_p = \frac{v}{L}$$

$$P = \frac{B}{L} (W - a)^2 \sigma_0 h(\frac{v}{L}, \frac{\sigma_0}{E}, n)$$

When taking the derivative of F with respect to a we get

$$\frac{\partial P}{\partial a} = -\frac{\partial P}{\partial(W-a)} = 2\frac{B}{L}(W-a)\sigma_0 h\left(\frac{v}{L}, \frac{\sigma_0}{E}, n\right) = -\frac{2P}{W-a} \quad (108)$$

The energy release rate based definition of J gives

$$J = -\frac{1}{B} \left(\frac{\partial U}{\partial a} \right)_{\Delta=const} = -\frac{1}{B} \int_0^{\Delta} \left(\frac{\partial P}{\partial a} \right)_{\Delta=const} d\Delta \quad (109)$$

Introducing Eq. (108) into Eq. (109) gives

$$J = \frac{2}{B(W-a)} \int_0^{\Delta} P d\Delta = \frac{2U_p}{B(W-a)} \quad (110)$$

where factor 2 represents the ratio between J and plastic work U_p per a unit area of the ligament length assuming a unit thickness of the specimen. Let us denoting this factor in general by η . Then

$$\beta = \frac{JB(W-a)}{U_p}.$$

From practice we have: $\eta = 2$ for SENB specimen and $\eta = 2 + 0.522(1 - a/W)$ for CT specimens. When including elastic effects Eq. (110) receives the form

$$J = \frac{\eta_{el}U_{el}}{B(W-a)} + \frac{\eta_p U_p}{B(W-a)} = \frac{K_I^2}{E'} + \frac{\eta_p U_p}{B(W-a)}. \quad (111)$$

CRACK GROWTH RESISTANCE CURVES

Note that many materials with high toughness do not fail catastrophically at a particular value of J or $CTOD$. Rather the material displays a rising R curve, where J and $CTOD$ increase with crack growth. Therefore, the initial crack growth is usually stable preceded by a small amount of apparent crack growth due to crack blunting. The onset of stable crack growth can be characterized by J_{IC} (initiation toughness). However, the precise point at which the crack begins to grow is not well defined so that the definition of J_{IC} is somewhat arbitrary. More complete description about the fracture behavior of ductile materials is provided by the entire R -curve.

The relative stability of the crack growth is indicated by the slope of the R curve at a given amount of crack extension.

Tearing modulus - slope on J resistance (R) curve

$$T_R = \frac{E}{\sigma_0^2} \frac{dJ_R}{da} \quad (112)$$

Applied tearing modulus - slope on the driving force

$$T_{app} = \frac{E}{\sigma_0^2} \left(\frac{dJ}{da} \right)_{\Delta T} \quad (113)$$

Conditions for crack growth

1. *Stable crack growth*

$$J = J_R \quad (114)$$

$$T_{app} \leq T_R \quad (115)$$

2. *Unstable crack growth*

$$T_{app} > T_R \quad (116)$$

Recall that the point of instability in a material with rising R curve depends on the size and geometry of the cracked structure. A critical value of J at instability is not a material property if J increases with crack growth. However, it is usually assumed that the R curve, including the J_{IC} value, is a material property independent of configuration. This is a reasonable assumption within certain limitations.

DETERMINATION J FOR A GROWING CRACK

We require for the R curve to be a material property independent of configuration. However, there is a geometry dependence of the R curve influenced by the way in which J is calculated. There are various ways to compute J that include:

- deformation $J \rightarrow J_D$ - based on the pseudo energy release rate definition of J
- far-field $J \rightarrow J_f$ - based on the contour integral definition of J
- modified $J \rightarrow$ modified $J_D(J_M)$

Deformation J

Recall that the J integral is based on deformation plasticity (or nonlinear elasticity) assumption for the material behavior and is only valid for stationary cracks. Consider now a growing crack which has grown from its initial length a_0 to a length a_1 and a corresponding load-displacement curve. The deformation J , however, is related to the area under the load-displacement curve for a stationary crack, rather than the area under the actual load-displacement, where the crack length varies. To that end, consider a deformation path for a stationary crack fixed at a_1 (nonlinear elasticity assumed). The area under this curve is the strain energy in an elastic material which depends only on the current load and crack length and is not history dependent which is not true for the actual elastic-plastic material. This energy is given by

$$U_D = U_D(P, a) = \left(\int_0^{\Delta} P d\Delta \right)_{a=a_1}. \quad (117)$$

Thus the J integral for a nonlinear elastic material with a growing crack is given by

$$J_D = -\frac{1}{B} \left(\frac{\partial U_D}{\partial a} \right)_{\Delta} = \frac{\eta U_D}{B(W-a)} \quad (118)$$

or

$$J_D = \frac{K_I^2}{E'} + \frac{\eta U_{D(p)}}{B(W-a)} \quad (119)$$

The calculation of $U_{D(p)}$ is usually performed incrementally, since the load-displacement curve based on deformation theory depends on the crack length.

***J* CONTROLLED FRACTURE**

As in the LEFM there are situations when J and thus also $CTOD$ completely characterizes crack tip conditions. However, there are limits to the validity of fracture mechanics analyses based on J and $CTOD$. In particular, such analyses become suspicious when there is excessive plasticity at the crack tip or significant crack growth. Fracture toughness given in terms of J then depends on the size and geometry of the structure or test specimen.

Required conditions for J -controlled fracture

- Stationary cracks:

small scale yielding - both K and J uniquely characterize crack tip conditions. In particular, in the close vicinity to the crack tip there exists the K -dominated zone where stresses are proportional to $1/\sqrt{r}$. In the plastic zone the elastic singularity no longer applies. However, when assuming monotonic, quasistatic loading there exists a J -dominate zone in the plastic region where the HRR solution is approximately valid and the stresses vary as $r^{1/n+1}$. Finally, the finite strain occurs within approximately 2δ from the crack tip, where large deformation invalidates the HRR theory.

elastic-plastic conditions - J is still approximately valid but there is no K field

large scale yielding - the size of the finite strain region becomes significant and there is no longer region uniquely characterized by J that exhibit a size and geometry dependence. Single parameter fracture mechanics is invalid.

- *J*-controlled crack growth:

Recall that in elastic material the prior crack growth has no adverse effects since the local crack tip fields depend only on current conditions. However, in elastic-plastic material the prior loading history does influence stresses and strains in the plastic region.

Consider a crack growth under *J*-controlled conditions. When crack advances there exists a region behind the growing crack tip where the material unloads elastically. Also recall that the material in the unloading region violates the assumptions of deformation plasticity. Just ahead of the crack tip there is a region of finite plastic strains where single parameter fracture is also invalid since the loading is highly nonproportional. Therefore, for the crack growth to be *J*-controlled both regions must be embedded within a zone of *J* – dominance. Otherwise, the measured *R*-curve is no longer uniquely characterized by *J*.

In small scale yielding there is always a zone of *J* dominance as the crack tip conditions are defined by the elastic stress intensity, which depends only on current values of load and crack size. There are three distinct stages of crack growth resistance in small scale yielding:

1. *Stage 1* - initial stage associated with crack blunting is essentially stationary
2. *Stage 2* - crack begins to grow, stresses and strains are history dependent and thus influenced by original blunt crack tip during the early stages of crack growth

3. *Stage 3* - steady state conditions are reached when the crack growth well beyond the initial blunted tip. During steady-state crack growth a plastic zone of constant size sweeps through the material leaving a plastic wake. The R -curve is flat - J does not increase with crack extension. If a rising or falling R curve appears then the local material properties vary with crack extension. The steady-state limit is usually not observed in laboratory tests on ductile material - typically the ligament is fully plastic during the crack growth that violates small scale yielding assumptions. Enormous specimens would be required to observe steady state growth in tough materials

FRACTURE TOUGHNESS TESTING OF METALS