3D Stress Components

The most general state of stress at a point may be represented by 6 components

From equilibrium principles:

$\tau_{xy} = \tau_{yx}$, $\tau_{xz} = \tau_{zx}$, $\tau_{zy} = \tau_{yz}$

Normal Stresses $\quad \sigma_x \quad \sigma_y \quad \sigma_z$

**Shear Stresses** $\quad \tau_{xy} \quad \tau_{yz} \quad \tau_{xz}$

Normal stress (σ) **:** the subscript identifies the face on which the stress acts. Tension is positive and compression is negative.
Shear stress (τ) **:** it has two subscripts. The first subscript denotes the face on which the stress acts. The second subscript denotes the direction on that face. A shear stress is positive if it acts on a positive face and positive direction or if it acts in a negative face and negative direction.
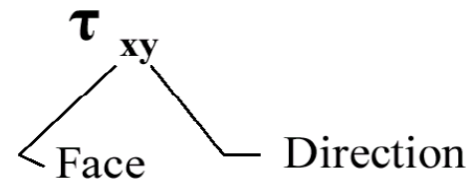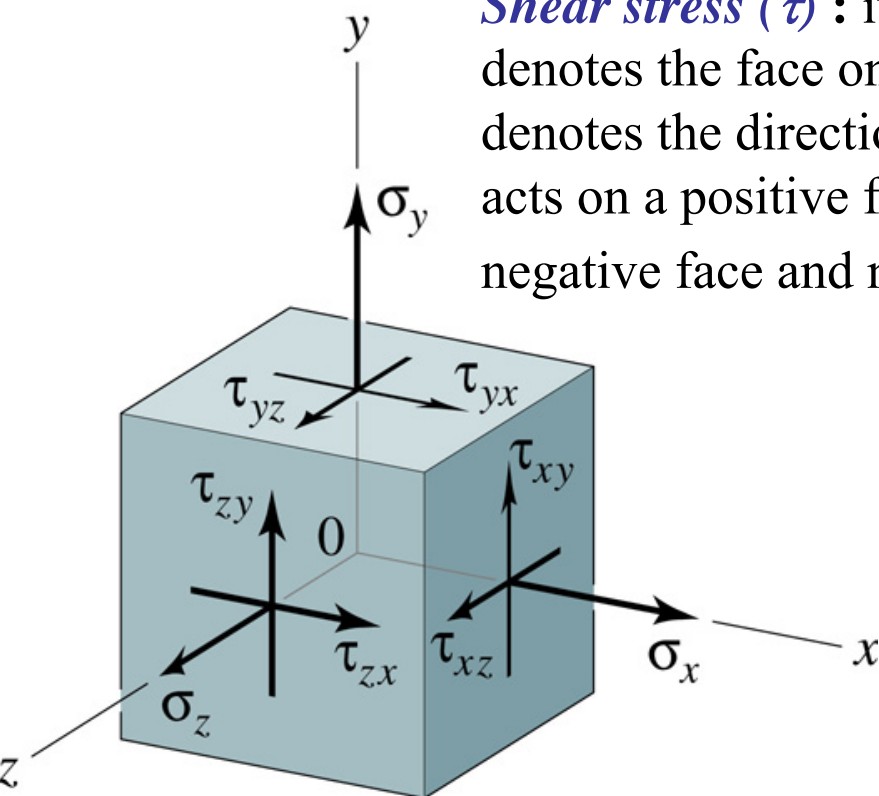
For static equilibrium $\tau_{xy} = \tau_{yx}$, $\tau_{xz} = \tau_{zx}$, $\tau_{zy} = \tau_{yz}$ resulting in six independent scalar quantities. These six scalars can be arranged in a 3x3 matrix, giving us a *stress tensor*.
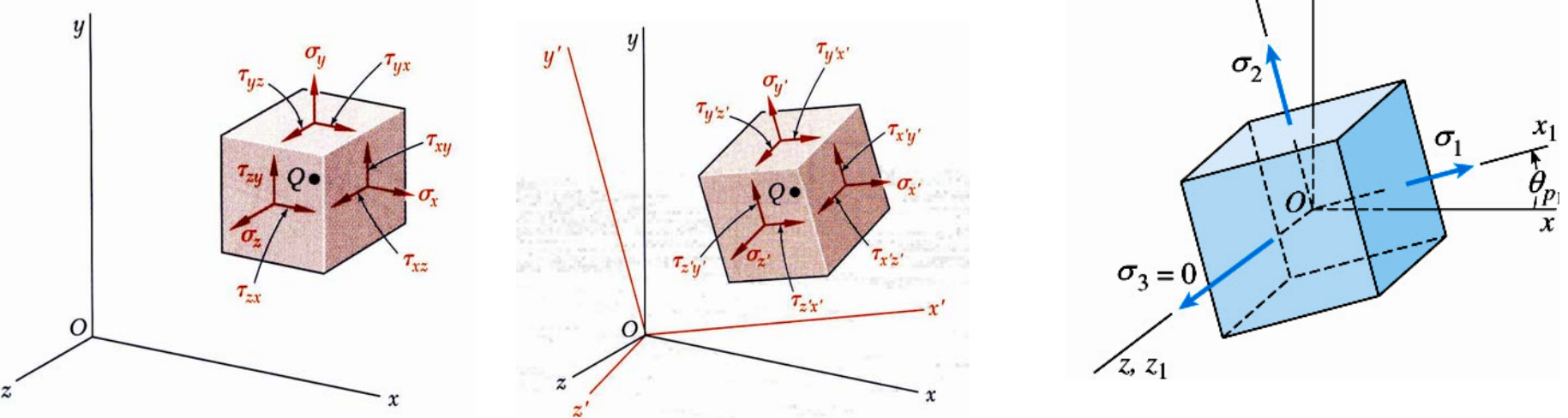
$$\boldsymbol{\sigma} = \sigma_{ij} = \begin{bmatrix} \sigma_x & \tau_{yx} & \tau_{zx} \\ \tau_{xy} & \sigma_y & \tau_{zy} \\ \tau_{xz} & \tau_{yz} & \sigma_z \end{bmatrix}$$

τ_{xy}

Face — Direction
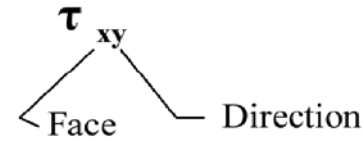
The sign convention for the stress elements is that a positive force on a positive face or a negative force on a negative face is positive. All others are negative.

The stress state is a second order tensor since it is a quantity associated with two directions (two subscripts direction of the surface normal and direction of the stress).

Same state of stress is represented by a different set of components if axes are rotated. There is a special set of components (when axes are rotated) where all the shear components are zero (principal stresses).

A property of a symmetric tensor is that there exists an orthogonal set of axes *1, 2* and *3 (called **principal axes**)* with respect to which the tensor elements are all zero except for those in the diagonal.

$$\sigma = \sigma_{ij} = \begin{bmatrix} \sigma_x & \tau_{yx} & \tau_{zx} \\ \tau_{xy} & \sigma_y & \tau_{zy} \\ \tau_{xz} & \tau_{yz} & \sigma_z \end{bmatrix} \qquad \xrightarrow{\text{\textit{Eigen values}}} \qquad \sigma' = \sigma'_{ij} = \begin{bmatrix} \sigma_1 & 0 & 0 \\ 0 & \sigma_2 & 0 \\ 0 & 0 & \sigma_3 \end{bmatrix}$$

In matrix notation the transformation is known as the ***Eigen-values***.

The principal stresses are the ***"new-axes"*** coordinate system. The angles between the ***"old-axes"*** and the ***"new-axes"*** are known as the ***Eigen-vectors***.

	"old" axes		
	x1	x2	x3
x'1	a11	a12	a13
"new" x'2	a21	a22	a23
x'3	a31	a32	a33

| principal stress	Cosine of angle between X and the principal stress	Cosine of angle between Y and the principal stress	Cosine of angle between Z and the principal stress
σ_1	k1	l1	m1
σ_2	k2	l2	m2
σ_3	k3	l3	m3

Plane Stress

State of stress in which two faces of the cubic element are free of stress. For the illustrated example, the state of stress is defined by
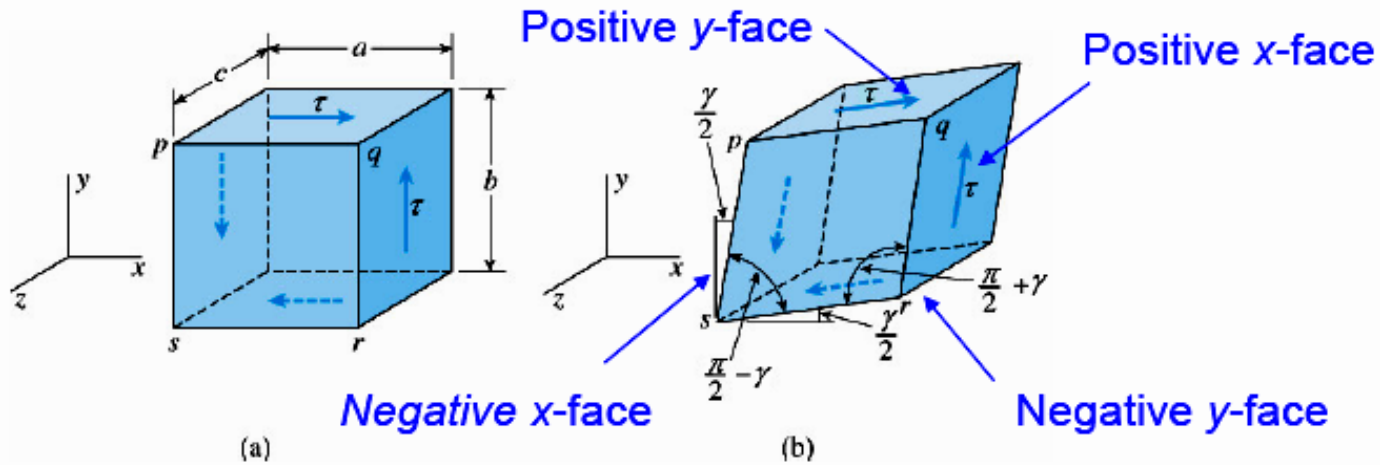
$$\sigma_x, \sigma_y, \tau_{xy} \quad \text{and} \quad \sigma_z = \tau_{zx} = \tau_{zy} = 0.$$
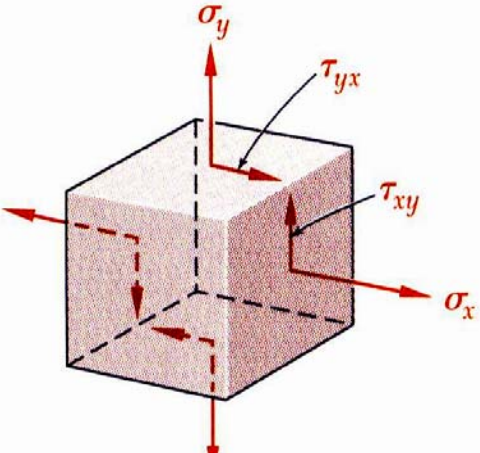
## *Sign Conventions for Shear Stress and Strain*
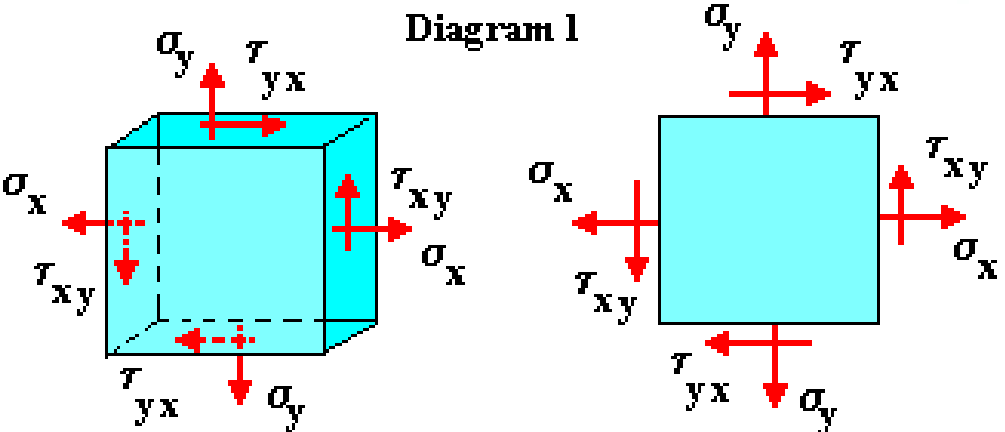
The Shear Stress will be considered positive when a pair of shear stress acting on opposite sides of the element produce a counterclockwise (ccw) torque (couple).

Positive y-face · Positive x-face · Negative x-face · Negative y-face

(a)   (b)

A shear strain in an element is positive when the angle between two positive faces (or two negative faces) is reduced, and is negative if the angle is increased.
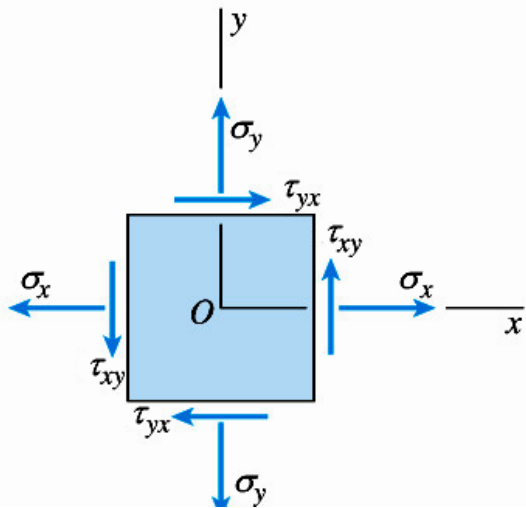
$$\begin{bmatrix} \sigma_{xx} & \tau_{yx} & 0 \\ \tau_{xy} & \sigma_{yy} & 0 \\ 0 & 0 & 0 \end{bmatrix}$$
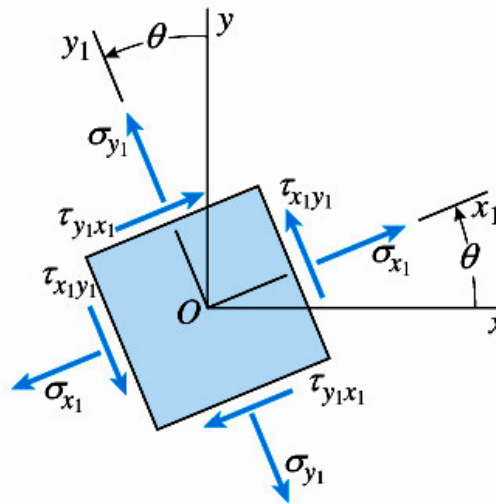
$$\begin{bmatrix} \sigma_{x1x1} & \tau_{y1x1} & 0 \\ \tau_{x1y1} & \sigma_{y1y1} & 0 \\ 0 & 0 & 0 \end{bmatrix}$$
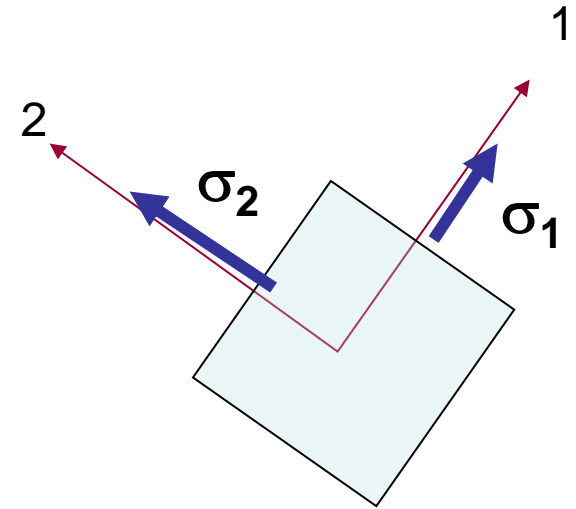
$$\begin{bmatrix} \sigma_1 & 0 & 0 \\ 0 & \sigma_2 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Stresses on Inclined Sections

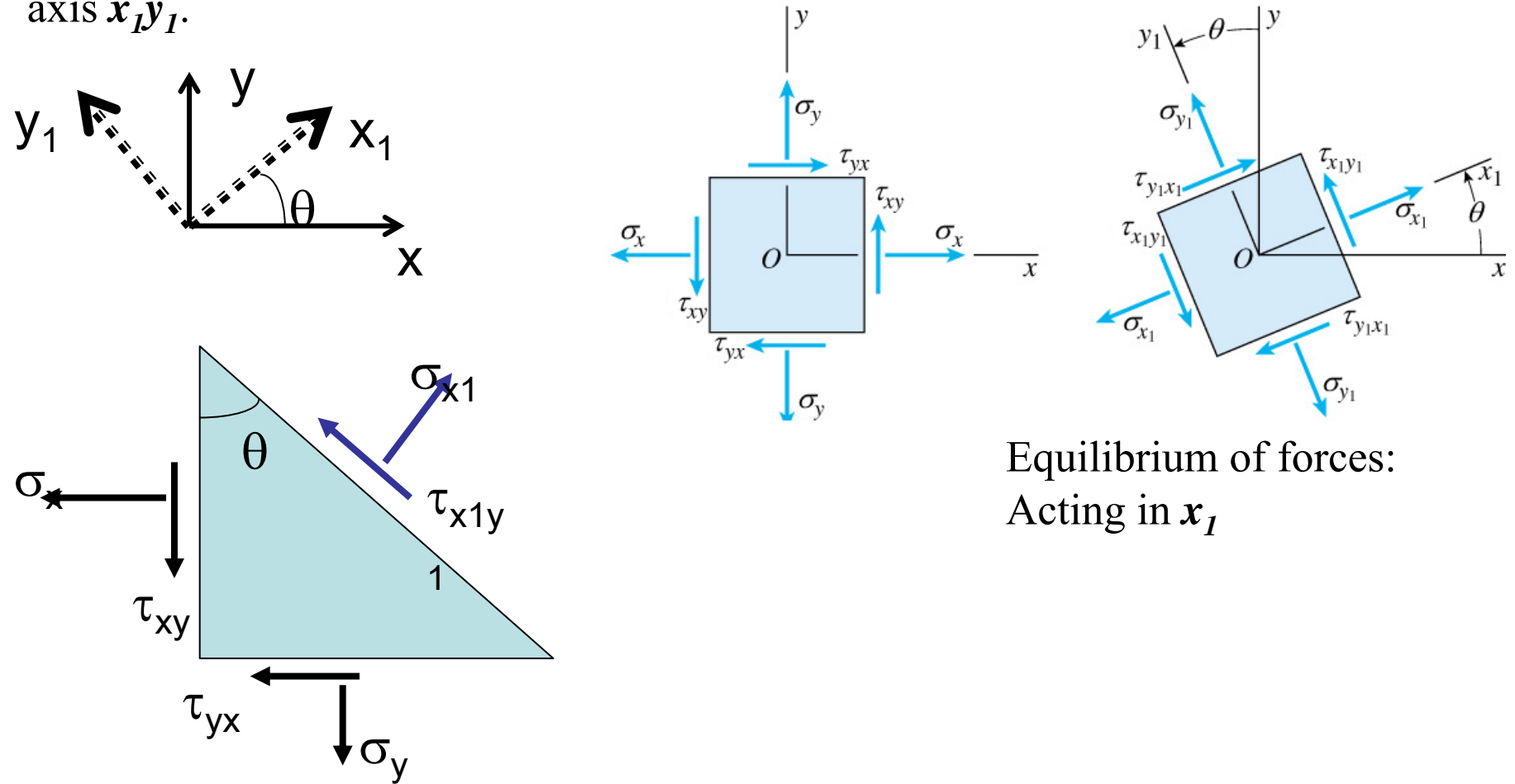Knowing the normal and shear stresses acting in the element denoted by the **xy** axes, we will calculate the normal and shear stresses acting in the element denoted by the axis **$x_1 y_1$**.

Equilibrium of forces:
Acting in **x_1**

$$\sigma_{X1} \cdot \frac{A_O}{\cos\theta} = \sigma_X \cos\theta \cdot A_O + \tau_{XY}\sin\theta \cdot A_O + \sigma_Y Sin\theta \cdot A_O \frac{Sin\theta}{Cos\theta} + \tau_{YX}\cos\theta \cdot A_O \frac{Sin\theta}{Cos\theta}$$
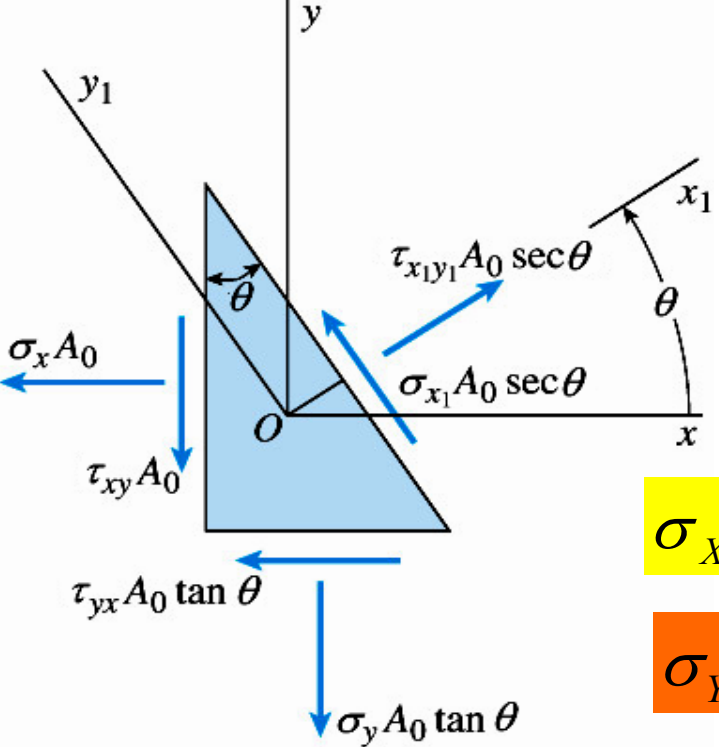
(b) Forces

Eliminating A_o, $sec\,\theta = 1/cos\,\theta$ and $\tau_{xy} = \tau_{yx}$

$$\sigma_{X1} = \sigma_X \cos^2 \theta + \sigma_Y \sin^2 \theta + 2\tau_{XY} \sin\theta\cos\theta$$

$$\sigma_{Y1} = \sigma_X \sin^2 \theta + \sigma_Y \cos^2 \theta - 2\tau_{XY} \sin\theta\cos\theta$$

Acting in y_1

$$\tau_{x1y1}A_o\sec\theta = -\sigma_x A_o\sin\theta + \tau_{xy}A_o\cos\theta + \sigma_y A_o\tan\theta\cos\theta - \tau_{yx}A_o\tan\theta\sin\theta$$

Eliminating A_o, $sec\,\theta = 1/cos\,\theta$ and $\tau_{xy} = \tau_{yx}$

$$\tau_{x1y1} = -\sigma_x \cdot \sin\theta \cdot \cos\theta + \sigma_y \cdot \sin\theta \cdot \cos\theta + \tau_{xy}\left(\cos^2 \theta - \sin^2 \theta\right)$$

Transformation Equations for Plane Stress

Using the following trigonometric identities:

Cos²θ = ½ (1+ cos 2θ)

Sin²θ = ½ (1- cos 2θ)

Sin θ cos θ = ½ sin 2θ

$$\sigma_{x1} = \frac{\sigma_x + \sigma_y}{2} + \frac{\sigma_x - \sigma_y}{2}\cos 2\theta + \tau_{xy}\sin 2\theta$$

$$\sigma_{y1} = \frac{\sigma_x + \sigma_y}{2} - \frac{\sigma_x - \sigma_y}{2}\cos 2\theta - \tau_{xy}\sin 2\theta$$

$$\tau_{x1y1} = -\frac{\sigma_x - \sigma_y}{2}\sin 2\theta + \tau_{xy}\cos 2\theta$$

These equations are known as *the transformation equations for plane stress.*

Special Cases

$$\sigma_{x1} = \frac{\sigma_x + \sigma_y}{2} + \frac{\sigma_x - \sigma_y}{2}\cos 2\theta + \tau_{xy}\sin 2\theta$$

$$\tau_{x1y1} = -\frac{\sigma_x - \sigma_y}{2}\sin 2\theta + \tau_{xy}\cos 2\theta$$

Case 1: Uniaxial stress

$$\sigma_y = 0 \quad \tau_{xy} = \tau_{yx} = 0$$

$$\sigma_{x1} = \sigma_x \cdot \left(\frac{1 + Cos2\theta}{2}\right)$$

$$\tau_{x1,y1} = -\sigma_x \cdot \left(\frac{Sin2\theta}{2}\right)$$

Case 3: Biaxial stress

$$\tau_{xy} = 0$$

$$\sigma_{x1} = \frac{\sigma_x + \sigma_y}{2} + \frac{\sigma_x - \sigma_y}{2} \cdot Cos2\theta$$

$$\tau_{x1,y1} = -\frac{\sigma_x - \sigma_y}{2} \cdot Sin2\theta$$

Case 2 : Pure Shear

$$\sigma_x = \sigma_y = 0$$

$$\sigma_{x1} = \tau_{xy} \cdot Sin2\theta$$

$$\tau_{x1,y1} = \tau_{xy} \cdot Cos2\theta$$
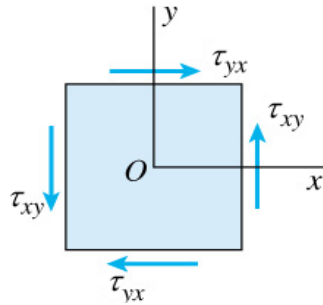
Example

An element in plane stress is subjected to stresses $\sigma_x=16000psi$, $\sigma_y=6000psi$, and $\tau_{xy}=\tau_{yx}=4000psi$ (as shown in figure below). Determine the stresses acting on an element inclined at an angle $\theta=45^o$ (*counterclockwise - ccw*).
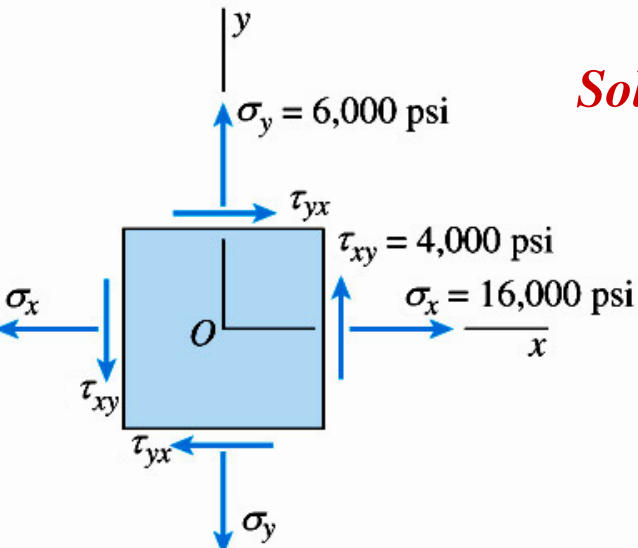
Solution: We will use the following transformation equations:

$$\sigma_{x1} = \frac{\sigma_x + \sigma_y}{2} + \frac{\sigma_x - \sigma_y}{2}\cos 2\theta + \tau_{xy}\sin 2\theta$$

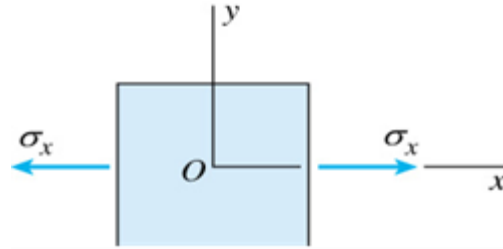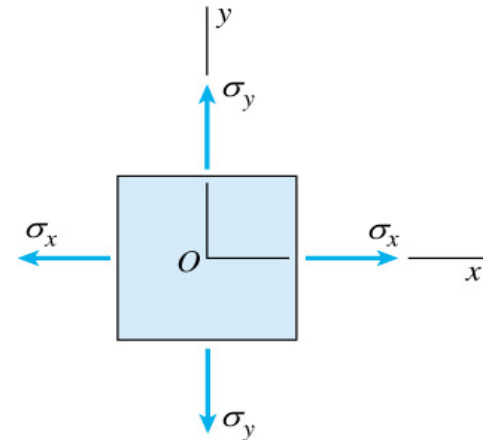$$\sigma_{y1} = \frac{\sigma_x + \sigma_y}{2} - \frac{\sigma_x - \sigma_y}{2}\cos 2\theta - \tau_{xy}\sin 2\theta$$

$$\tau_{x1y1} = -\frac{\sigma_x - \sigma_y}{2}\sin 2\theta + \tau_{xy}\cos 2\theta$$

Numerical substitution

$$\frac{\sigma_x + \sigma_y}{2} = \frac{(16000 + 6000)}{2} = 11000\,psi$$

For x-axis $\theta = +45^0 (ccw)$

$$\frac{\sigma_x - \sigma_y}{2} = \frac{(16000 - 6000)}{2} = 5000\,psi$$

$$Sin(2\theta) = Sin(90^0) = 1$$

$$Cos(2\theta) = Cos(90^0) = 0$$

$$\tau_{xy} = 4000\,psi$$

For y-axis $\theta = +45^0 + 90^0 (ccw)$

$$\sigma_{x1} = 11000 + 5000(0) + 4000(1) = 15000\,psi$$

$$Sin(2\theta) = Sin(270^0) = -1$$

$$\sigma_{y1} = 11000 - 5000(0) - 4000(1) = 7000\,psi$$

$$Cos(2\theta) = Cos(270^0) = 0$$

$$\tau_{x1y1} = -5000(1) + 4000(0) = -5000\,psi$$
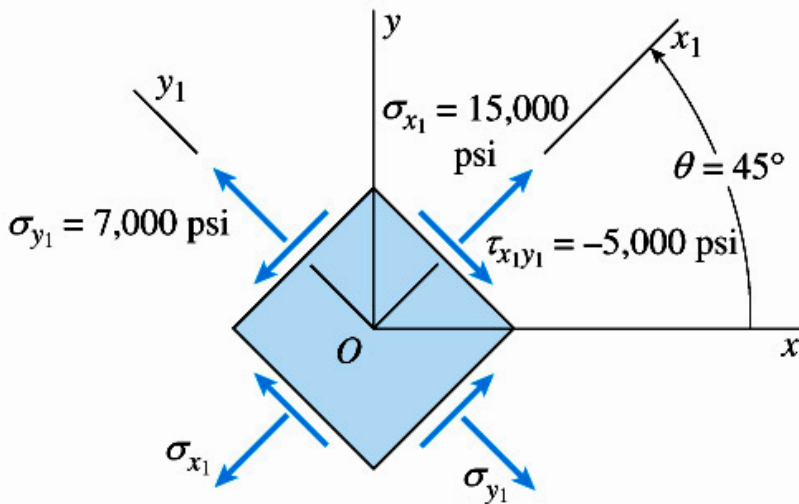
Note: $\quad \sigma_x + \sigma_y = \sigma_{x1} + \sigma_{y1}$

Example

A plane stress condition exists at a point on the surface of a loaded structure such as shown below. Determine the stresses acting on an element that is oriented at a clockwise (cw) angle of **15°** with respect to the original element, the principal stresses, the maximum shear stress and the angle of inclination for the principal stresses

Solution: We will use the following transformation equations:

$$\sigma_{x1} = \frac{\sigma_x + \sigma_y}{2} + \frac{\sigma_x - \sigma_y}{2}\cos 2\theta + \tau_{xy}\sin 2\theta$$

$$\tau_{x1y1} = -\frac{\sigma_x - \sigma_y}{2}\sin 2\theta + \tau_{xy}\cos 2\theta$$

For x-axis $\theta = -15^0 (cw)$

$Sin(2\theta) = Sin(-30^0) = -0.5$

$Cos(2\theta) = Cos(-30^0) = 0.866$

$$\frac{\sigma_x + \sigma_y}{2} = \frac{(-46+12)}{2} = -17 MPa$$

$$\frac{\sigma_x - \sigma_y}{2} = \frac{(-46-12)}{2} = -29 MPa$$

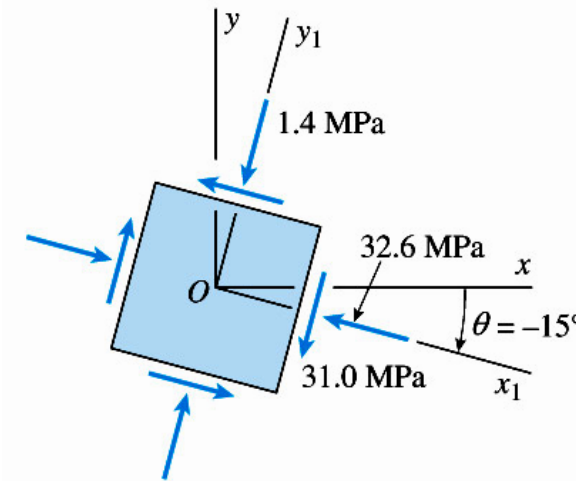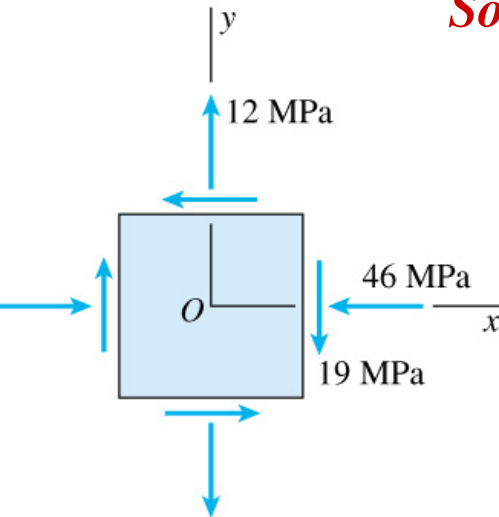$$\sigma_x + \sigma_y = \sigma_{x1} + \sigma_{y1}$$

$$\sigma_{y1} = -1.4 MPa$$

$$\tau_{xy} = -19 MPa$$

$$\sigma_{x1} = (-17) + (-29)(0.866) + (-19)(-0.5) = -32.6 MPa$$

$$\tau_{x1y1} = (-29)(-0.5) + (-19)(0.866) = -31 MPa$$

Example A rectangular plate of dimensions **3.0 in x 5.0 in** is formed by welding two triangular plates (see figure). The plate is subjected to a tensile stress of **600psi** in the long direction and a compressive stress of **250psi** in the short direction.

Determine the normal stress **sw** acting perpendicular to the line or the weld and the shear stress **tw** acting parallel to the weld. (Assume **sw** is positive when it acts in tension and **tw** is positive when it acts counterclockwise against the weld).

250 psi
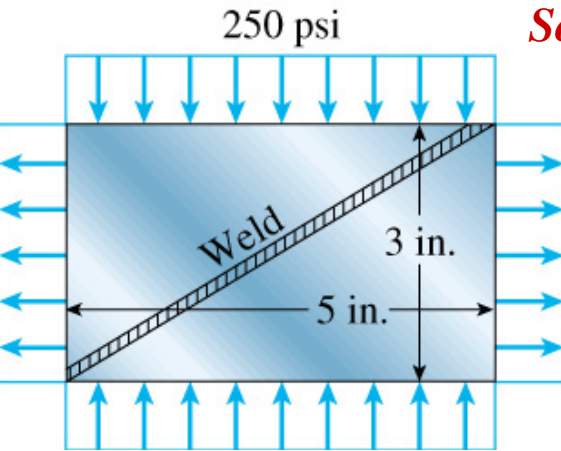
Solution:

$$\frac{\sigma_x + \sigma_y}{2} = 175\,psi \qquad \frac{\sigma_x - \sigma_y}{2} = 425\,psi \qquad \tau_{xy} = 0\,psi$$

$$\sigma_{x1} = 375\,psi \qquad \tau_{x1y1} = -375\,psi$$

Weld 3 in. 600 psi $\sigma_{y1} = -25\,psi$

5 in.

$$\sigma_x + \sigma_y = \sigma_{x1} + \sigma_{y1}$$

$$\sigma_{y1} = -25\,psi$$

25psi   -375psi 375psi

Θ = 30.96°

For x-axis $\tan\theta = \dfrac{3}{5} \Rightarrow$

$\theta = 30.96^0 \Rightarrow 2\theta = 61.92^0$

y

x
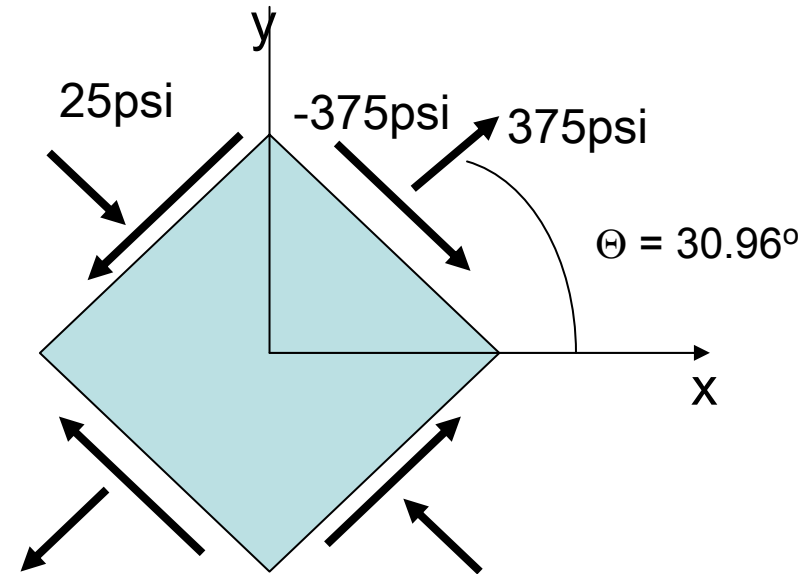
Stresses acting on the weld

σw

τw θ

25psi

375psi

θ = 30.96°   σw = -25psi and τw = 375psi

Principal Stresses and Maximum Shear Stresses

The sum of the normal stresses acting on perpendicular faces of plane stress elements is constant and independent of the angle θ.

$$\sigma_{X1} = \sigma_X \cos^2 \theta + \sigma_Y \sin^2 \theta + 2\tau_{XY} \sin \theta \cos \theta$$

$$\sigma_{Y1} = \sigma_X \sin^2 \theta + \sigma_Y \cos^2 \theta - 2\tau_{XY} \sin \theta \cos \theta$$

$$\sigma_{X1} + \sigma_{Y1} = \sigma_X + \sigma_Y$$

As we change the angle θ there will be *maximum and minimum normal and shear stresses* that are needed for design purposes.

The maximum and minimum normal stresses are known as the principal stresses. These stresses are found by taking the derivative of σ_{x1} with respect to θ and setting equal to zero.

$$\sigma_{x1} = \frac{\sigma_x + \sigma_y}{2} + \frac{\sigma_x - \sigma_y}{2} \cos 2\theta + \tau_{xy} \sin 2\theta$$

$$\frac{\delta\sigma_{x1}}{\delta\theta} = -(\sigma_x - \sigma_y)\sin 2\theta + 2\tau_{xy} \cos 2\theta = 0$$

$$\tan 2\theta_P = \frac{\tau_{xy}}{\left(\dfrac{\sigma_x - \sigma_y}{2}\right)}$$

The subscript *p* indicates that the angle θ_p defines the orientation of the principal planes. The angle θ_p has two values that differ by *90°*. They are known as the principal angles. For one of these angles σ_{x1} is a *maximum* principal stress and for the other a *minimum*. The principal stresses occur in mutually perpendicular planes.

$$R = \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2}$$

$$\frac{\sigma_x - \sigma_y}{2}$$

$$2\theta_p$$

$$\tau_{xy}$$
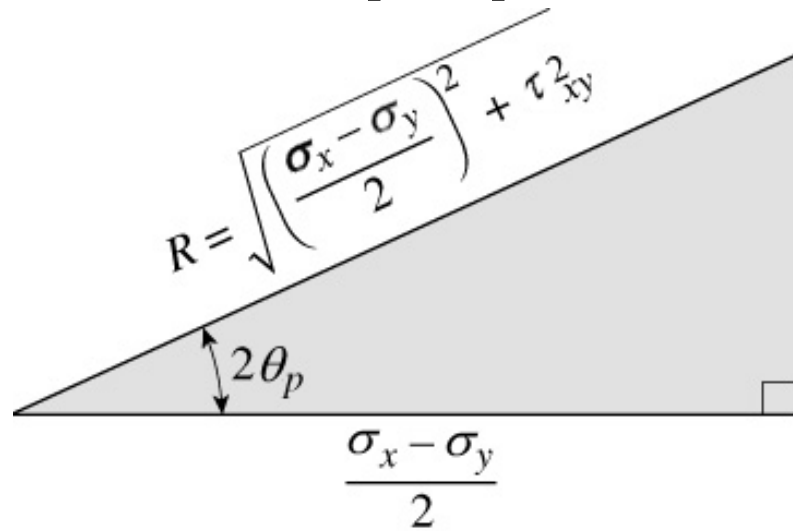
$$\tan 2\theta_P = \frac{\tau_{xy}}{\left(\sigma_x - \sigma_y\right)/2}$$

$$\sin 2\theta_P = \frac{\tau_{xy}}{R} \qquad \cos 2\theta_P = \frac{\left(\sigma_x - \sigma_y\right)}{2R}$$

$$\sigma_{x1} = \frac{\sigma_x + \sigma_y}{2} + \frac{\sigma_x - \sigma_y}{2}\cos 2\theta + \tau_{xy}\sin 2\theta$$

for the maximum stress $\theta = \theta_P$

$$\sigma_1 = \frac{\sigma_x + \sigma_y}{2} + \frac{\sigma_x - \sigma_y}{2}\left(\frac{\sigma_x - \sigma_y}{2R}\right) + \tau_{xy}\left(\frac{\tau_{xy}}{R}\right) = \frac{\sigma_x + \sigma_y}{2} + \left(\frac{\sigma_x - \sigma_y}{2}\right)^2\left(\frac{1}{R}\right) + \left(\tau_{xy}\right)^2\left(\frac{1}{R}\right)$$

$$\sigma_1 = \frac{\sigma_x + \sigma_y}{2} + \left(\frac{1}{R}\right)\left[\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \left(\tau_{xy}\right)^2\right]$$

But $\quad R = \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \left(\tau_{xy}\right)^2} \qquad \sigma_1 = \frac{\sigma_x + \sigma_y}{2} + \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \left(\tau_{xy}\right)^2}$

Principal Stresses

$$\sigma_1 = \left(\frac{\sigma_x + \sigma_y}{2}\right) + \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \left(\tau_{xy}\right)^2} = \sigma_{Average} + R$$

$$\sigma_2 = \left(\frac{\sigma_x + \sigma_y}{2}\right) - \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \left(\tau_{xy}\right)^2} = \sigma_{Average} - R$$

The plus sign gives the algebraically larger principal stress and the minus sign the algebraically smaller principal stress.

This are the *in-plane principal stresses*. The *third stress is zero* in plane stress conditions

Maximum Shear Stresses

The location of the angle for the maximum shear stress is obtained by taking the derivative of τ_{x1y1} with respect to θ and setting it equal to zero.

$$\tau_{x1y1} = -\frac{\sigma_x - \sigma_y}{2}\sin 2\theta + \tau_{xy}\cos 2\theta$$

$$\frac{\delta\tau_{x1y1}}{\delta\theta} = -(\sigma_x - \sigma_y)\cos 2\theta - 2\tau_{xy}\sin 2\theta = 0$$

$$\tan 2\theta_S = -\frac{(\sigma_x - \sigma_y)/2}{\tau_{xy}}$$

$$\cos 2\theta_{s1} = \frac{\tau_{xy}}{R} \quad \sin 2\theta_{s1} = -\frac{(\sigma_x - \sigma_y)}{2R} \quad \text{and} \quad \theta_{s1} = \theta_{P1} - 45^0$$

Therefore, **$2\theta_s - 2\theta_p = -90^o$ or $\theta_s = \theta_p +/- 45^o$**

The planes for maximum shear stress occurs at **45°** to the principal planes. The plane of the maximum positive shear stress τ_{max} is defined by the angle θ_{S1} for which the following equations apply:

$$\tan 2\theta_S = -\frac{(\sigma_x - \sigma_y)}{2\tau_{xy}} = -\frac{1}{\tan 2\theta_P} = -\cot 2\theta_P$$

$$\frac{\sin 2\theta_S}{\cos 2\theta_S} = -\frac{\cos 2\theta_P}{\sin 2\theta_P} = \frac{-\sin(90^o - 2\theta_P)}{\cos(90^o - 2\theta_P)} = \frac{\sin(2\theta_P - 90^o)}{\cos(2\theta_P - 90^o)}$$

The corresponding maximum shear is given by the equation

$$\tau_{MAX} = \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + (\tau_{xy})^2} = R$$

Another expression for the maximum shear stress

$$\tau_{MAX} = \frac{(\sigma_1 - \sigma_2)}{2}$$

The normal stresses associated with the maximum shear stress are equal to

$$\sigma_{AVER} = \frac{(\sigma_x + \sigma_y)}{2}$$

Equations of a Circle

General equation

$$\sigma_{x1} = \frac{\sigma_x + \sigma_y}{2} + \frac{\sigma_x - \sigma_y}{2}\cos 2\theta + \tau_{xy}\sin 2\theta$$

Consider

$$\sigma_{x1} - \sigma_{AVER} = \frac{\sigma_x - \sigma_y}{2}\cos 2\theta + \tau_{xy}\sin 2\theta$$

Equation (1)

$$\left(\sigma_{x1} - \sigma_{AVER}\right)^2 = \left[\frac{\sigma_x - \sigma_y}{2}\cos 2\theta + \tau_{xy}\sin 2\theta\right]^2$$

$$\tau_{x1y1} = -\frac{\sigma_x - \sigma_y}{2}\sin 2\theta + \tau_{xy}\cos 2\theta$$

Equation (2)

$$\left(\tau_{x1y1}\right)^2 = \left[-\frac{\sigma_x - \sigma_y}{2}\sin 2\theta + \tau_{xy}\cos 2\theta\right]^2$$

Equation (1) + Equation (2)

$$\left(\sigma_{x1} - \sigma_{AVER}\right)^2 + \left(\tau_{x1y1}\right)^2 = \left[\frac{\sigma_x - \sigma_y}{2}\cos 2\theta + \tau_{xy}\sin 2\theta\right]^2 + \left[-\frac{\sigma_x - \sigma_y}{2}\sin 2\theta + \tau_{xy}\cos 2\theta\right]^2$$

$$\left[\frac{\sigma_x - \sigma_y}{2}\cos 2\theta + \tau_{xy}\sin 2\theta\right]^2 = \left(\frac{\sigma_x - \sigma_y}{2}\right)^2\cos^2 2\theta + \left(\tau_{xy}\right)^2\sin^2 2\theta + 2\left(\frac{\sigma_x - \sigma_y}{2}\right)\left(\tau_{xy}\right)\sin 2\theta\cos 2\theta$$

$$\left[-\frac{\sigma_x - \sigma_y}{2}\sin 2\theta + \tau_{xy}\cos 2\theta\right]^2 = \left(-\frac{\sigma_x - \sigma_y}{2}\right)^2\sin^2 2\theta + \left(\tau_{xy}\right)^2\cos^2 2\theta - 2\left(\frac{\sigma_x - \sigma_y}{2}\right)\left(\tau_{xy}\right)\sin 2\theta\cos 2\theta$$
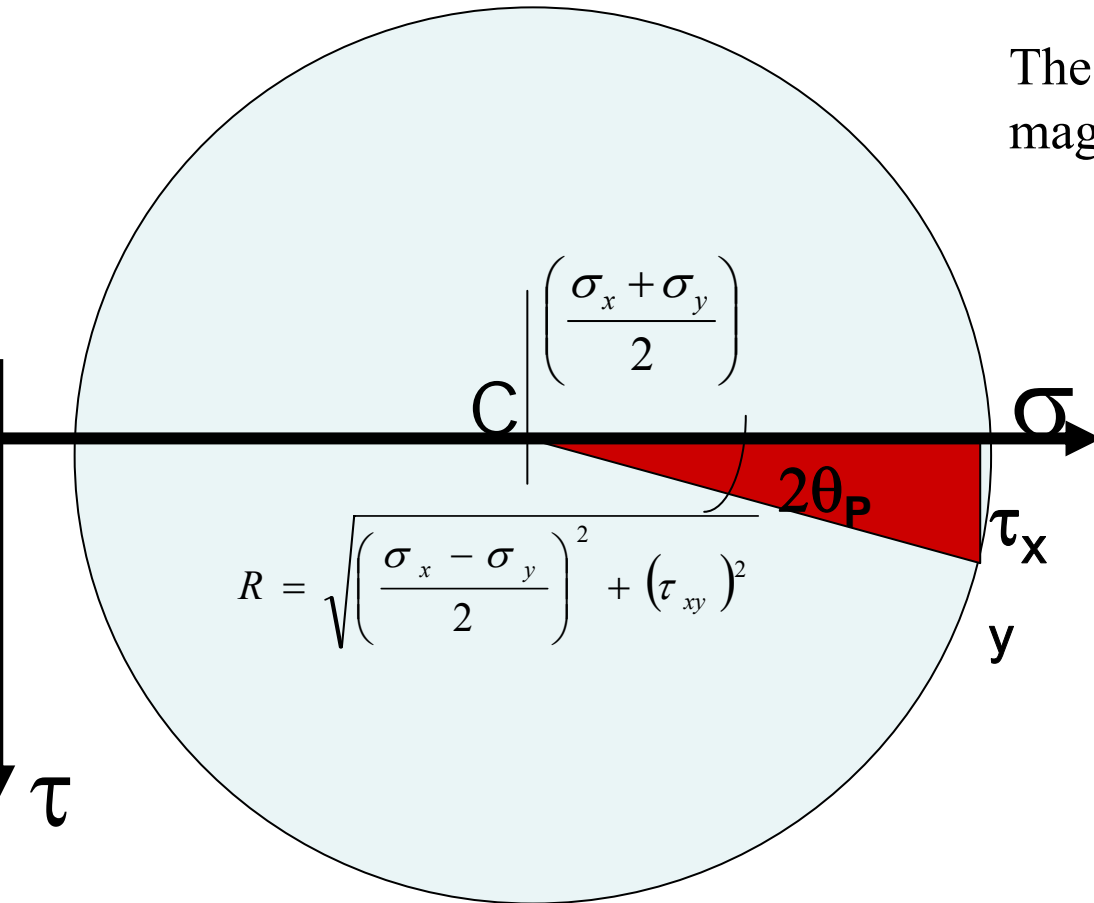
SUM
$$= \left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \left(\tau_{xy}\right)^2 = R^2$$

$$\left(\sigma_{x1} - \sigma_{AVER}\right)^2 + \left(\tau_{x1y1}\right)^2 = R^2$$

Mohr Circle

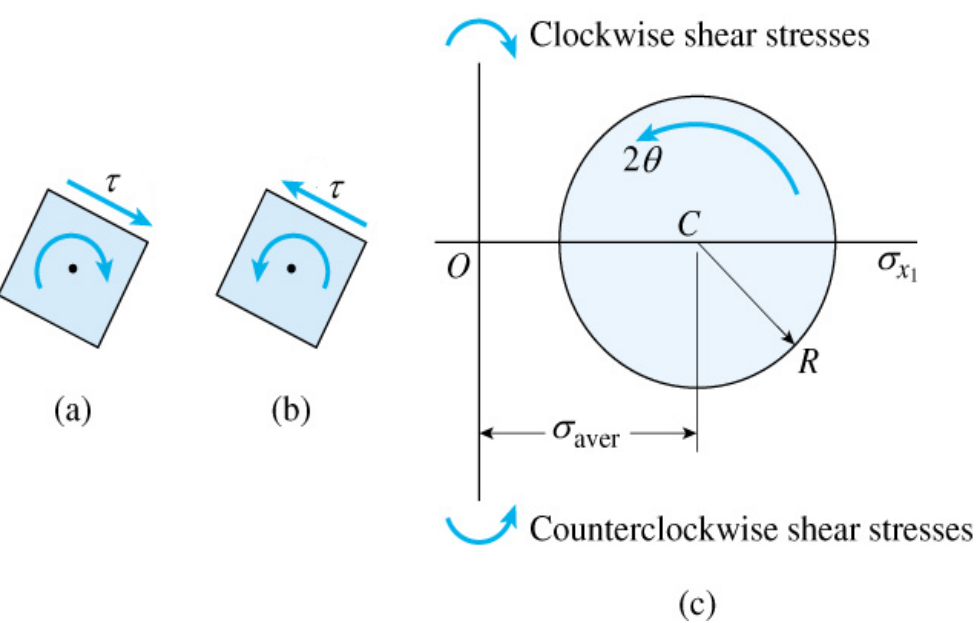The radius of the Mohr circle is the magnitude **R**.

$$R = \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \left(\tau_{xy}\right)^2}$$

The center of the Mohr circle is the magnitude

$$\sigma_{AVER} = \frac{\left(\sigma_x + \sigma_y\right)}{2}$$

State of Stresses

$$\left(\sigma_{x1} - \sigma_{AVER}\right)^2 + \left(\tau_{x1y1}\right)^2 = \left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \left(\tau_{xy}\right)^2$$

(a) (b) (c)

Clockwise shear stresses

Counterclockwise shear stresses

Alternative sign conversion for shear stresses:
(a) clockwise shear stress,
(b) counterclockwise shear stress, and
(c) axes for Mohr's circle.

Note that clockwise shear stresses are plotted upward and counterclockwise shear stresses are plotted downward.

Forms of Mohr's Circle

a) **We can plot the normal stress σ_{x1} positive to the right and the shear stress τ_{x1y1} positive downwards, i.e. the angle 2θ will be positive when counterclockwise or**

b) We can plot the normal stress σ_{x1} positive to the right and the shear stress τ_{x1y1} positive upwards, i.e. the angle 2θ will be positive when clockwise.

Both forms are mathematically correct. **We use (a)**

Two forms of Mohr's circle:

τ_{x1y1} is positive downward and the angle 2θ is positive counterclockwise, and

τ_{x1y1} is positive upward and the angle 2θ is positive clockwise. (*Note:* The first form is used here)

(a)

(a)

(b)

(b)
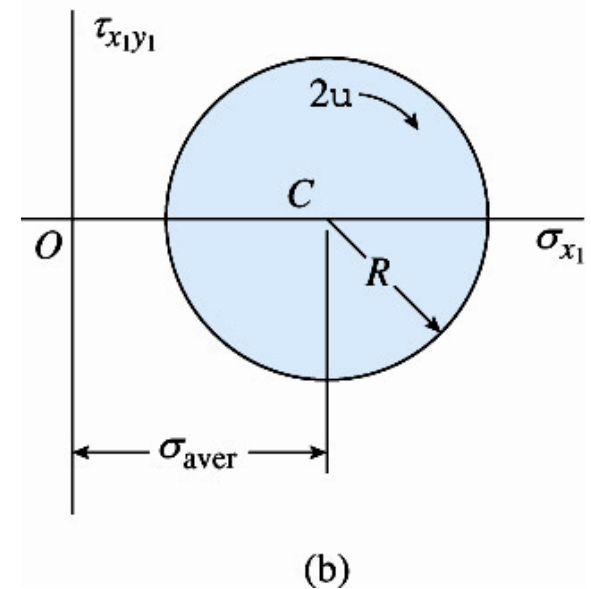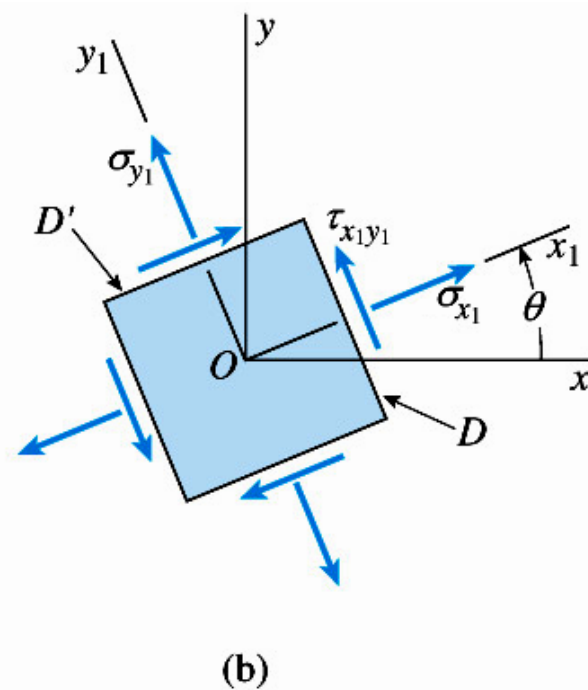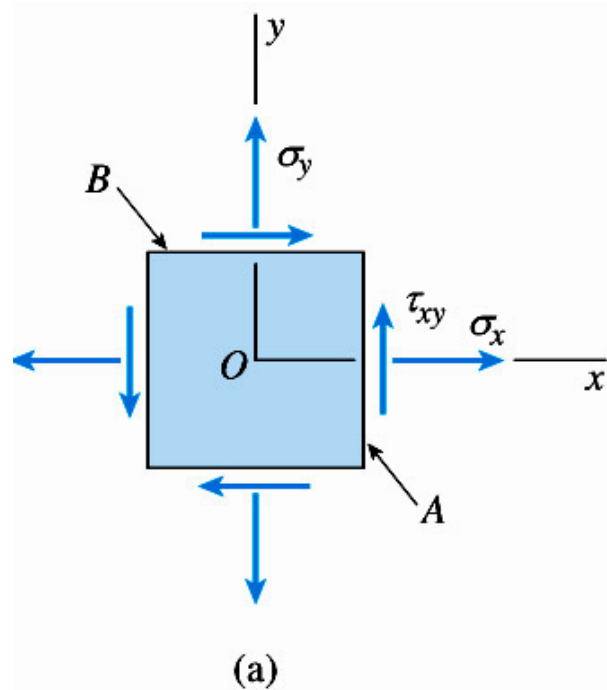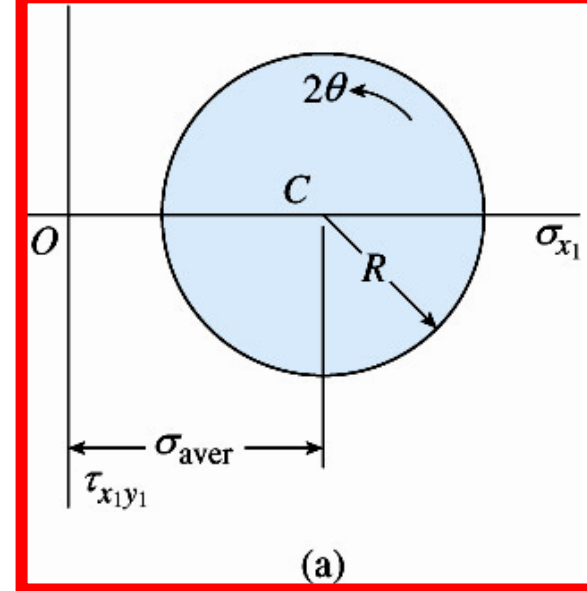
Construction of Mohr's circle for plane stress.

(c)

Example

At a point on the surface of a pressurized cylinder, the material is subjected to biaxial stresses $\sigma_x = 90MPa$ and $\sigma_y = 20MPa$ as shown in the element below.
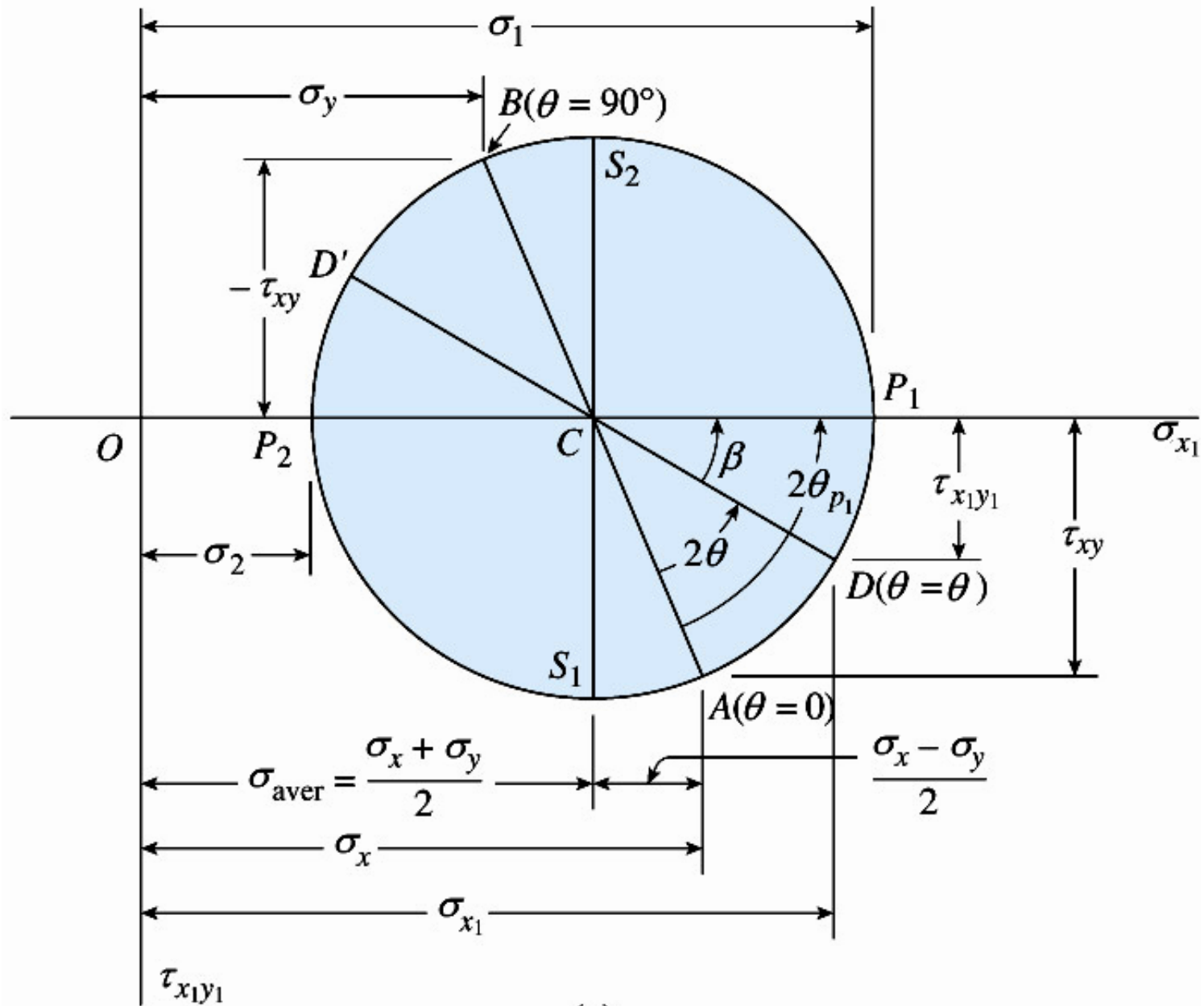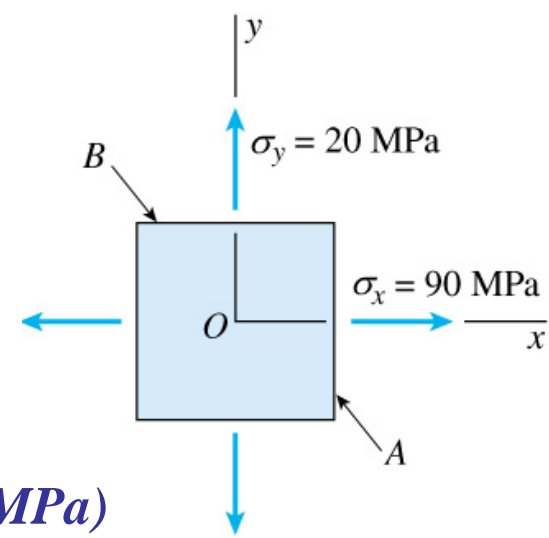
Using the Mohr circle, determine the stresses acting on an element inclined at an angle $\theta = 30°$ (Sketch a properly oriented element). $(\sigma_x = 90MPa, \quad \sigma_y = 20MPa \quad and \quad \tau_{xy} = 0MPa)$

Because the shear stress is zero, these are the principal stresses.

Construction of the Mohr's circle

The center of the circle is

$$\sigma_{Average} = \frac{(\sigma_x + \sigma_y)}{2} = \frac{(90 + 20)}{2} = 55MPa$$

$$R = \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + (\tau_{xy})^2} = \sqrt{\left(\frac{90 - 20}{2}\right)^2 + (0)^2} = 35MPa$$
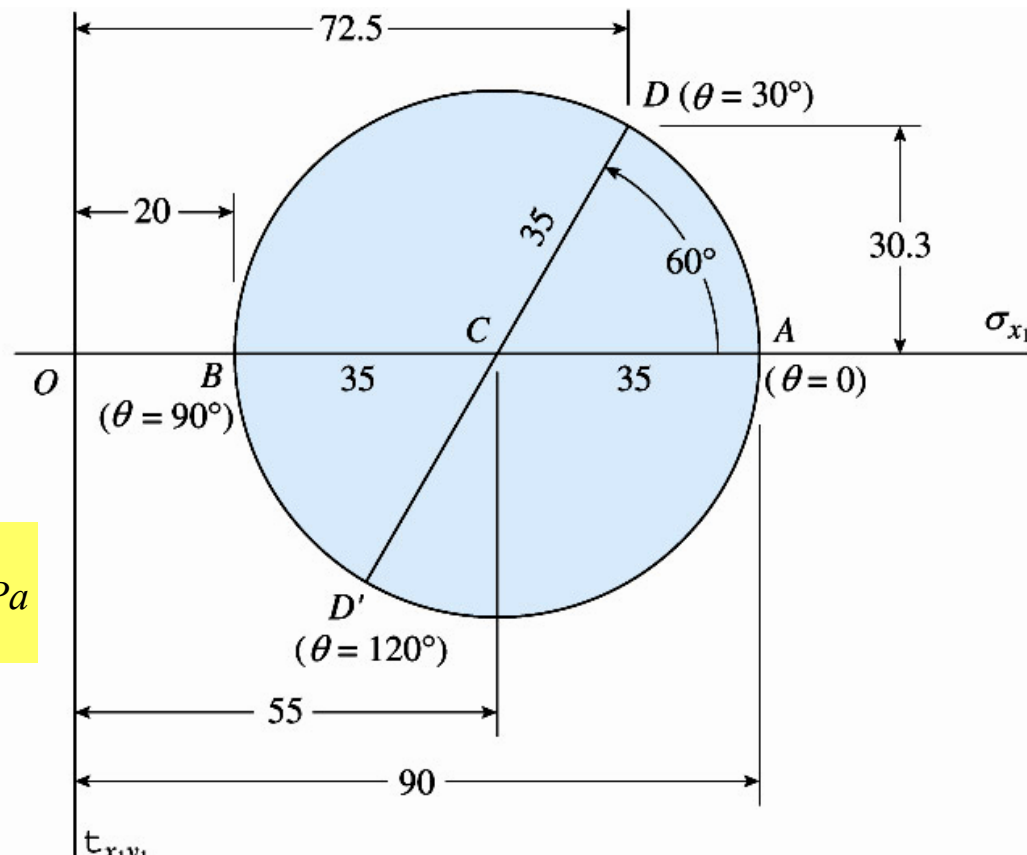
Stresses on an element inclined at θ = 30º

By inspection of the circle, the coordinates of point **D** are

$$\sigma_{x1} = \sigma_{Average} + R \cdot Cos60^0 = 55 + 35 \cdot Cos60 = 72.5 MPa$$

$$\tau_{x1y1} = -R \cdot Sin60^o = -35\left(Sin60^o\right) = -30.3 MPa$$

$$\sigma_{y1} = \sigma_{Average} - R \cdot Cos60^0 = 55 - 35 \cdot Cos60 = 37.5 MPa$$

Example An element in plane stress at the surface of a lar
machine is subjected to stresses $\sigma_x = $ **15000psi**,
5000psi and $\tau_{xy} = $ **4000psi**.

Using the Mohr's circle determine the following:
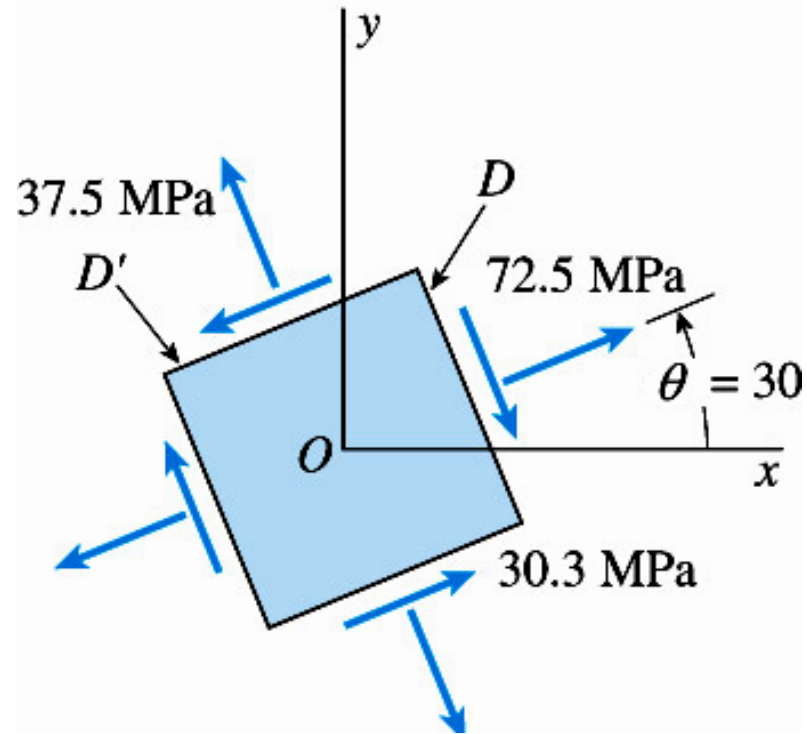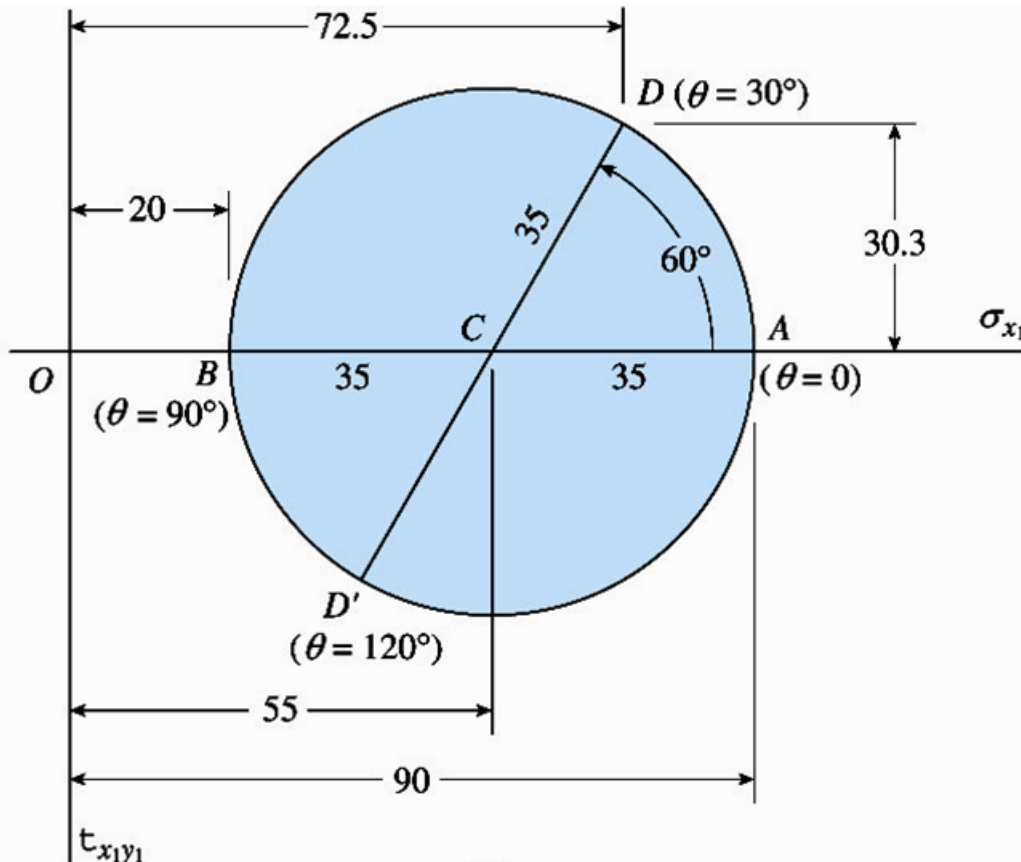a) The stresses acting on an element inclined at an
   angle $\theta = $ **40°**
b) The principal stresses and
c) The maximum shear stresses.
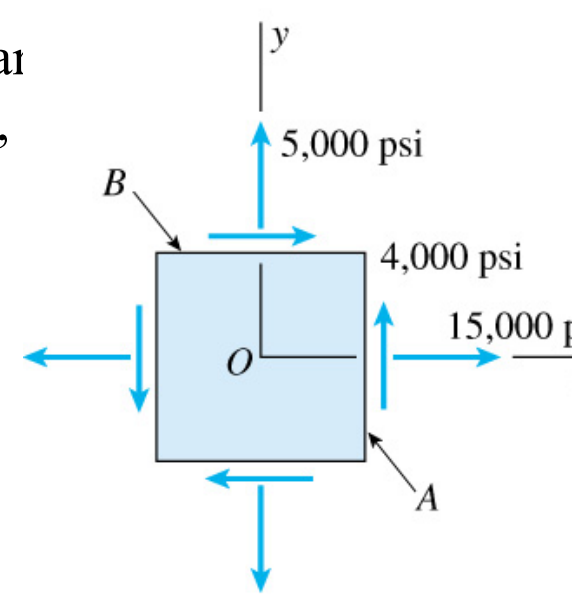
Construction of Mohr's circle:
Center of the circle (Point C):

$$\sigma_{Average} = \frac{(\sigma_x + \sigma_y)}{2} = \frac{(15000 + 5000)}{2} = 10000\,psi$$

Radius of the circle:

$$R = \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + (\tau_{xy})^2} = \sqrt{\left(\frac{15000 - 5000}{2}\right)^2 + (4000)^2} = 6403\,psi$$

Point A, representing the stresses on the **x** face of the element ($\theta = 0°$) has the
coordinates $\sigma_{x1} = 15000psi$ and $\tau_{x1y1} = 4000psi$

Point B, representing the stresses on the **y** face of the element ($\theta = 90°$) has the
coordinates $\sigma_{y1} = 5000psi$ and $\tau_{y1x1} = -4000psi$
The circle is now drawn through points A and B with center C and radius R

S_2 $(\theta_{s_2} = 64.3°)$

5,000

B $(\theta = 90°)$

D $(= 40°)$

4,000 4,000

6,403

6,403

80°

41.34°

P_1 $(\theta_{p_1} = 19.3°)$

5,000

P_2 $(\theta_{p_2} = 109.3°)$

6,403

38.66°

s_{x_1}

4,000

D'

A $(\theta = 0)$

S_1

$(\theta_{s_1} = -25.7°)$

10,000

5,000

15,000

$\tau_{x_1y_1}$
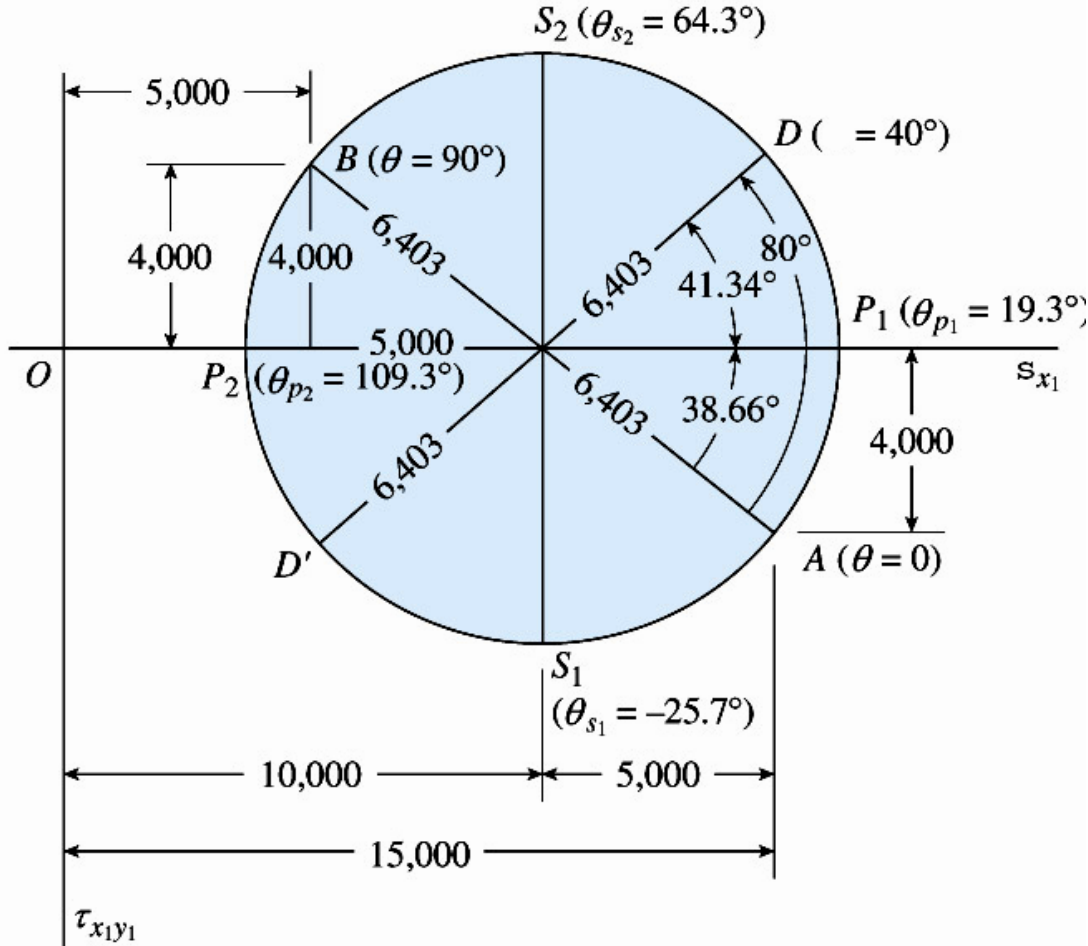
Stresses on an element inclined at θ = 40°

These are given by the coordinates of point D which is at an angle $2\theta = 80°$ from point A

By inspection the angle ACP_1 for the principal stresses (point P_1) is :

$$\tan\left(\overline{ACP_1}\right) = \frac{4000}{5000} \Rightarrow \overline{ACP_1} = 38.66°$$

Then, the angle P_1CD is $80° - 38.66° = 41.34°$

Knowing this angle, we can calculate the coordinates of point D (by inspection)

$$\sigma_{x1} = \sigma_{Average} + R \cdot Cos41.34° = 10000 + 6403 \cdot Cos41.34° = 14810\,psi$$

$$\tau_{x1y1} = -R \cdot Sin41.34° = -6403(Sin41.34°) = -4230\,psi$$

$$\sigma_{y1} = \sigma_{Average} - R \cdot Cos41.34° = 10000 - 6403 \cdot Cos41.34° = 5190\,psi$$

And of course, the sum of the normal stresses is

14810psi + 5190psi = 15000psi + 5000psi

Principal Stresses

The principal stresses are represented by points P_1 and P_2 on Mohr's circle.

The angle it was found to be $2\theta = 38.66°$ or $\theta = 19.3°$

$$\sigma_1 = \sigma_{Average} + R = 10000 + 6403 = 16403\,psi$$

$$\sigma_2 = \sigma_{Average} - R = 10000 - 6403 = 3597\,psi$$

Maximum Shear Stresses

These are represented by point S_1 and S_2 in Mohr's circle. Algebraically the maximum shear stress is given by the **radius** of the circle.

The angle ACS_1 from point A to point S_1 is $2\theta_{S1} = 51.34°$. This angle is negative because is measured clockwise on the circle. Then the corresponding θ_{S1} value is $-25.7°$.
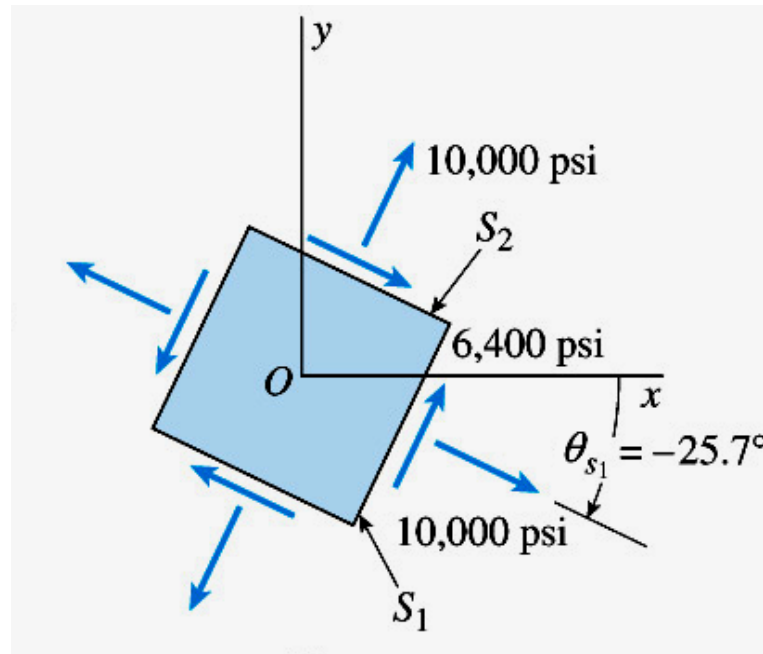
3-D stress state $\begin{bmatrix} 15000 & 4000 & 0 \\ 4000 & 5000 & 0 \\ 0 & 0 & 0 \end{bmatrix} Psi$ Transform to $\begin{bmatrix} 16403 & 0 & 0 \\ 0 & 3597 & 0 \\ 0 & 0 & 0 \end{bmatrix} Psi$

In matrix notation the transformation is known as the Eigen-values.

The principal stresses are the "new-axes" coordinate system. The angles between the "old-axes" and the "new-axes" are known as the Eigen-vectors.

principal stress	Cosine of angle between X and the principal stress	Cosine of angle between Y and the principal stress	Cosine of angle between Z and the principal stress
16403.1242	0.94362832	0.331006939	0
3596.876	-0.33101	0.943628	0
0	0	0	1

Example

At a point on the surface of a generator shaft the stresses are $\sigma_x = -50$MPa, $\sigma_y = 10$MPa and $\tau_{xy} = -40$MPa as shown in the figure. Using Mohr's circle determine the following:

(a) Stresses acting on an element inclined at an angle $\theta = 45°$,

(b) The principal stresses and

(c) The maximum shear stresses
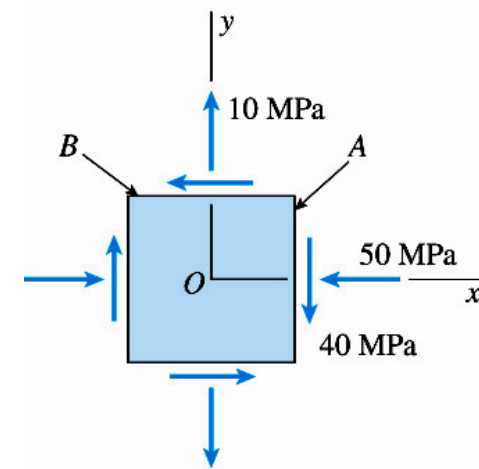
Construction of Mohr's circle
Center of the circle (Point C):

$$\sigma_{Average} = \frac{(\sigma_x + \sigma_y)}{2} = \frac{(-50) + (10)}{2} = -20 MPa$$

Radius of the circle:.

$$R = \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + (\tau_{xy})^2} = \sqrt{\left(\frac{(-50) - (10)}{2}\right)^2 + (-40)^2} = 50 MPa$$

Point A, representing the stresses on the *x* face of the element ($\theta = 0°$) has the coordinates $\sigma_{x1} = -50$MPa and $\tau_{x1y1} = -40$MPa

Point B, representing the stresses on the *y* face of the element ($\theta = 90°$) has the coordinates $\sigma_{y1} = 10$MPa and $\tau_{y1x1} = 40$MPa

The circle is now drawn through points A and B with center C and radius R.

Diagram labels: 50; S_2; $A\ (\theta=0)$; D'; 50; 50; 40; 90°; 53.13°; $P_1\ (\theta_{p_1}=116.6°)$; x_1; P_2; 36.87°; C; O; $(\theta_{p_2}=26.6°)$; 50; 50; 40; $D\ (\theta=45°)$; S_1; $B\ (\theta=90°)$; $(\theta_{s_1}=71.6°)$; 20; 10
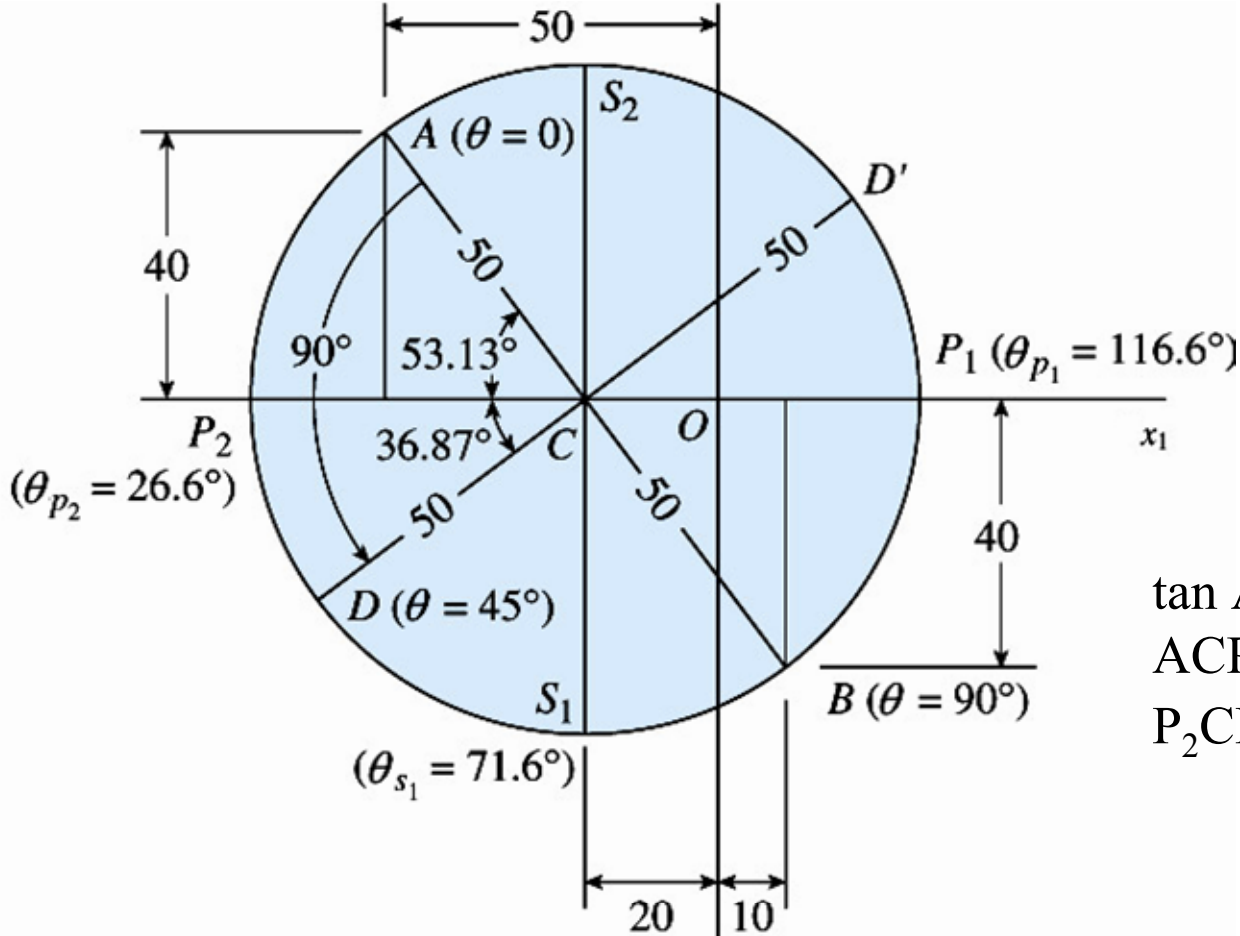
Stresses on an element inclined at $\theta=45°$

These stresses are given by the coordinates of point D ($2\theta = 90°$ of point A). To calculate its magnitude we need to determine the angles ACP2 and P2CD.

$\tan ACP_2 = 40/30 = 4/3$
$ACP_2 = 53.13°$
$P_2CD = 90° - 53.13° = 36.87°$

Then, the coordinates of point D are

$$\sigma_{x1} = \sigma_{Average} + R \cdot Cos36.87° = (-20) + 50 \cdot Cos36.87° = -60MPa$$

$$\tau_{x1y1} = R \cdot Sin36.87° = 50(Sin36.87°) = 30MPa$$

$$\sigma_{y1} = \sigma_{Average} - R \cdot Cos36.87° = (-20) + 50 \cdot Cos36.87° = 20MPa$$

And of course, the sum of the normal stresses is -50MPa+10MPa = -60MPa +20MPa

Principal Stresses

They are represented by points P_1 and P_2 on Mohr's circle.

$$\sigma_1 = \sigma_{Average} + R = -20 + 50 = 30 MPa$$

$$\sigma_2 = \sigma_{Average} - R = -20 - 50 = -70 MPa$$

The angle ACP_1 is $2\theta_{P1} = 180^\circ + 53.13^\circ$ $= 233.13^\circ$ or $\theta_{P1} = 116.6^\circ$

The angle ACP_2 is $2\theta_{P2} = 53.13^\circ$ or θ_{P2} $= 26.6^\circ$
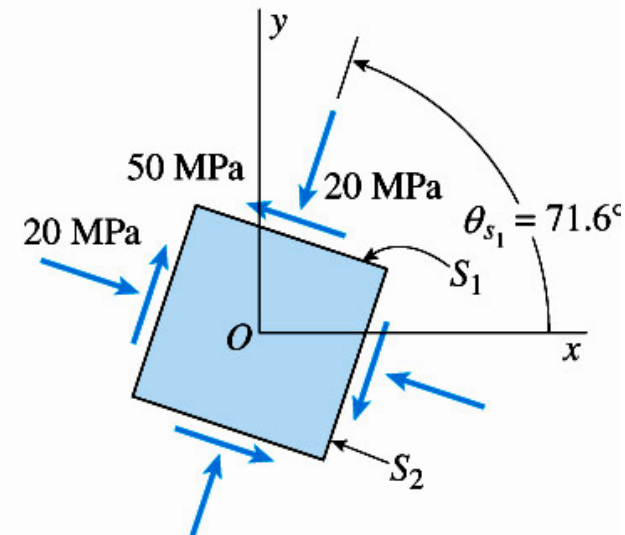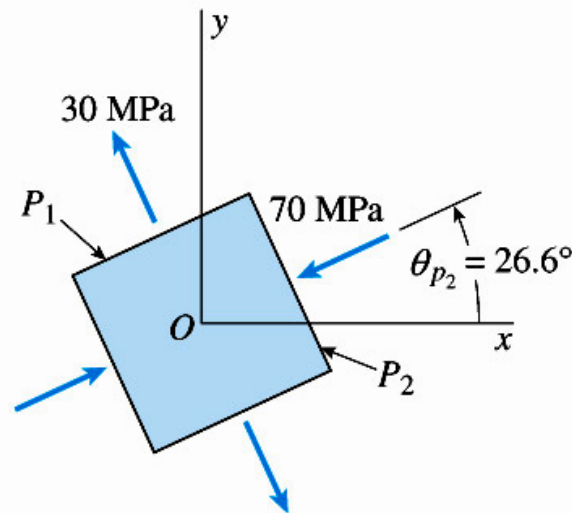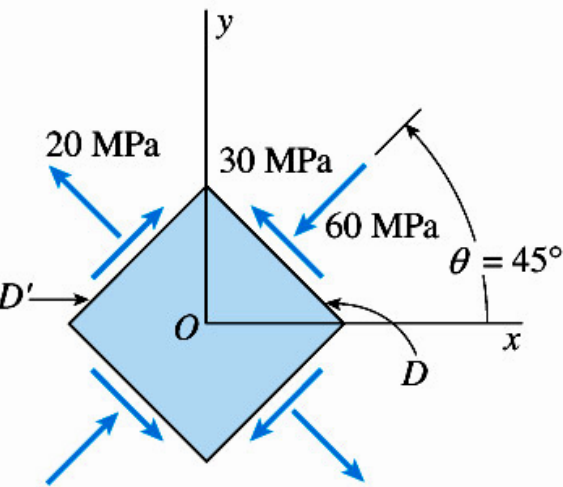
Maximum Shear Stresses

These are represented by point S_1 and S_2 in Mohr's circle.

The angle ACS_1 is $2\theta_{S1} = 90^\circ + 53.13^\circ = 143.13^\circ$ or $\theta = 71.6^\circ$.

The magnitude of the maximum shear stress is 50MPa and the normal stresses corresponding to point S1 is the average stress -20MPa.

3-D stress state
$$\begin{bmatrix} -50 & -40 & 0 \\ -40 & 10 & 0 \\ 0 & 0 & 0 \end{bmatrix} MPa$$
Transform to
$$\begin{bmatrix} 30 & 0 & 0 \\ 0 & -70 & 0 \\ 0 & 0 & 0 \end{bmatrix} MPa$$

In matrix notation the transformation is known as the *Eigen-values*.

The principal stresses are the "new-axes" coordinate system. The angles between the "old-axes" and the "new-axes" are known as the *Eigen-vectors*.

principal stress	Cosine of angle between X and the principal stress	Cosine of angle between Y and the principal stress	Cosine of angle between Z and the principal stress
30	-0.44721359	0.894427193	0
-70	0.894427	0.447214	0
0	0	0	1

Hooke's Law for Plane Stress

The stress transformations equations were derived solely from equilibrium conditions and they are material independent.

Here the material properties will be considered (strain) taking into account the following:

a) The material is uniform throughout the body (homogeneous)

b) The material has the same properties in all directions (isotropic)

c) The material follows Hooke's law (linearly elastic material)

Hooke's law: Linear relationship between stress and strain

For uniaxial stress:

(**E** = modulus of elasticity or Young's modulus)
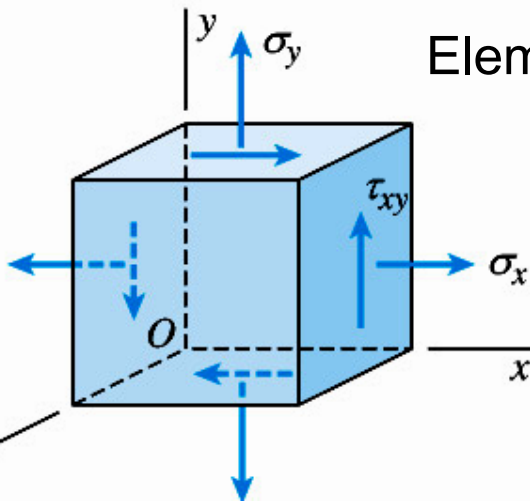
$$\sigma = E\varepsilon$$

Poisson's ratio: $$\upsilon = -\frac{lateral\ strain}{axial\ strain} = -\frac{\varepsilon_{transverse}}{\varepsilon_{longitudinal}}$$

For pure shear : (**G** = Shear modulus of elasticity)

$$\tau = G\gamma$$

Element of material in plane stress ($\sigma_z = 0$).

Consider the normal strains ε_x, ε_y, ε_z in plane stress.

All are shown with positive elongation.

The strains can be expressed in terms of the stresses by superimposing the effect of the individual stresses.

For instance the strain ε_x in the *x* direction:

a) Due to the stress σ_x is equal to σ_x/E.

b) Due to the stress σ_y is equal to $-v\sigma_y/E$.

The resulting strain is:

$$\varepsilon_x = \frac{\sigma_x}{E} - v\frac{\sigma_y}{E}$$

$$\varepsilon_y = -v\frac{\sigma_x}{E} + \frac{\sigma_y}{E}$$
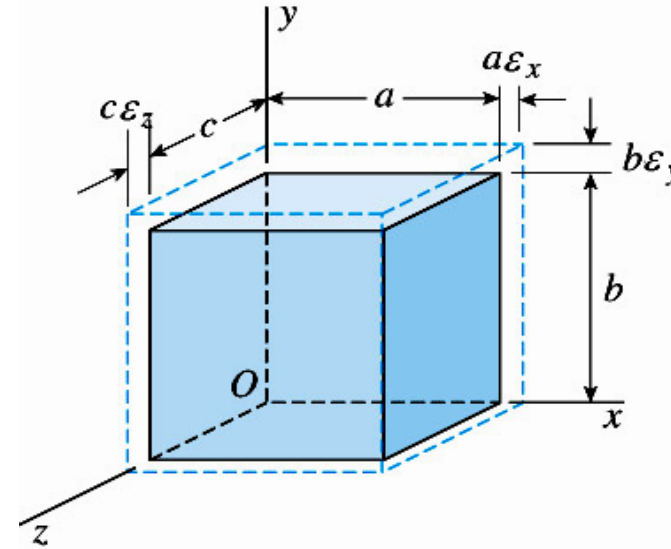
$$\varepsilon_z = -v\frac{\sigma_x}{E} - v\frac{\sigma_y}{E}$$

The shear stress causes a distortion of the element such that each *z* face becomes a rhombus.

$$\gamma_{XY} = \frac{\tau_{XY}}{G}$$

The normal stresses σ_x and σ_y have no effect on the shear strain γ_{xy}

$$\varepsilon_x = \frac{\sigma_x}{E} - \nu\frac{\sigma_y}{E} \qquad \varepsilon_y = -\nu\frac{\sigma_x}{E} + \frac{\sigma_y}{E} \qquad \varepsilon_z = -\nu\frac{\sigma_x}{E} - \nu\frac{\sigma_y}{E} \qquad \gamma_{XY} = \frac{\tau_{XY}}{G}$$
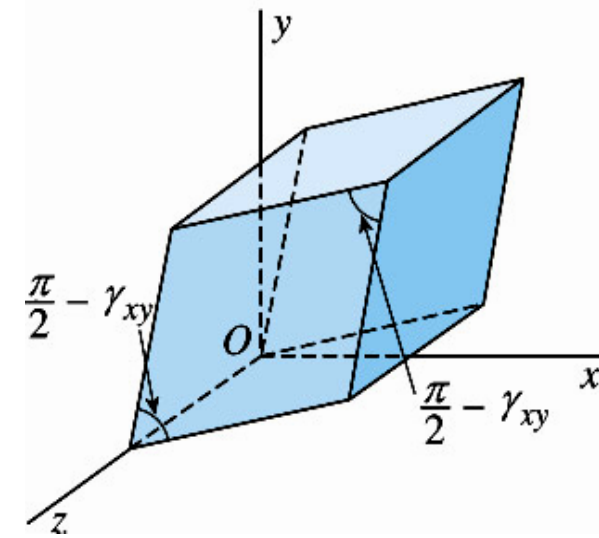
or rearranging the equations:

$$\sigma_x = \frac{E\varepsilon_x}{\left(1-\nu^2\right)} + \frac{\nu E\varepsilon_y}{\left(1-\nu^2\right)} \qquad \sigma_Y = \frac{\nu E\varepsilon_x}{\left(1-\nu^2\right)} + \frac{E\varepsilon_y}{\left(1-\nu^2\right)} + \qquad \tau_{XY} = G\gamma_{XY}$$

These equations are known collectively as ***Hooke's Law for plane stress***

These equations contain three material constants (***E, G and*** ν) but only two are independent because of the relationship:

$$G = \frac{E}{2(1+\nu)}$$

Special cases of Hooke's law ($\sigma_z = 0$)

Uniaxial stress :

$$\sigma_y = 0 \qquad \tau_{xy} = 0$$

$$\varepsilon_x = \frac{\sigma_x}{E} \qquad \varepsilon_y = \varepsilon_z = -\upsilon \frac{\sigma_x}{E}$$

Pure Shear :

$$\sigma_x = \sigma_y = 0$$

$$\varepsilon_x = \varepsilon_y = \varepsilon_z = 0 \qquad \gamma_{xy} = \frac{\tau_{xy}}{G}$$

Biaxial stress :

$$\tau_{xy} = 0$$

$$\varepsilon_x = \frac{\sigma_x}{E} - \upsilon \frac{\sigma_y}{E} \qquad \varepsilon_y = -\upsilon \frac{\sigma_x}{E} + \frac{\sigma_y}{E} \qquad \varepsilon_z = -\upsilon \frac{\sigma_x}{E} - \upsilon \frac{\sigma_y}{E}$$
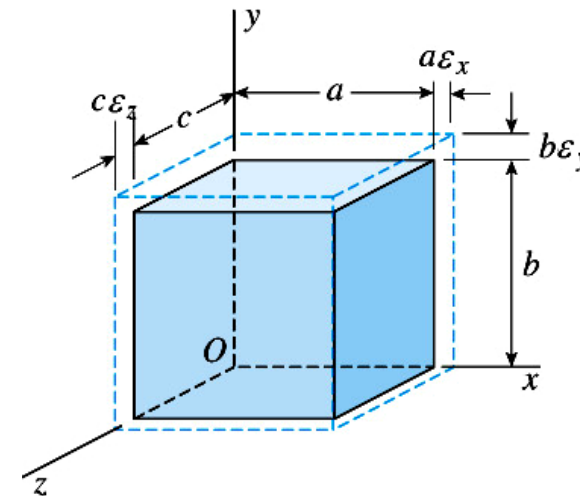
Volume Change

When a solid object undergoes strains, both its dimensions and its volume will change.

Consider an object of dimensions *a, b, c*. The original volume is $V_o = abc$ and its final volume is

$V_1 = (a + a\varepsilon_x)\ (b + b\varepsilon_y)\ (c + c\varepsilon_z)$

$V_1 = abc\ (1+\varepsilon_x)\ (1+\varepsilon_y)\ (1+\varepsilon_z)$

Upon expanding the terms:

$$V_1 = V_o (1 + \varepsilon_x + \varepsilon_y + \varepsilon_z + \varepsilon_x\varepsilon_y + \varepsilon_x\varepsilon_z + \varepsilon_y\varepsilon_z + \varepsilon_x\varepsilon_y\varepsilon_z)$$

For small strains:

$$V_1 = V_o (1 + \varepsilon_x + \varepsilon_y + \varepsilon_z)$$

The volume change is

$$\Delta V = V_1 - V_o = V_o (\varepsilon_x + \varepsilon_y + \varepsilon_z)$$

The unit volume change e, also known as dilatation is defines as:

$$e = \Delta V / V_o = \varepsilon_x + \varepsilon_y + \varepsilon_z$$

Positive strains are considered for elongations and negative strains for shortening, i.e. positive values of e for an increase in volume.

$$\varepsilon_x = \frac{\sigma_x}{E} - v\frac{\sigma_y}{E} \qquad \varepsilon_y = -v\frac{\sigma_x}{E} + \frac{\sigma_y}{E} \qquad \varepsilon_z = -v\frac{\sigma_x}{E} - v\frac{\sigma_y}{E}$$

$$e = \frac{\Delta V}{V} = (\sigma_x + \sigma_y)\frac{(1 - 2v)}{E}$$

For uniaxial stress $\sigma_y = 0$

$$e = \sigma_x \frac{(1 - 2v)}{E}$$

We can notice that the maximum possible value of Poisson's ratio is **0.5**, because a larger value means that the volume decreases when the material is in tension (contrary to physical behavior).
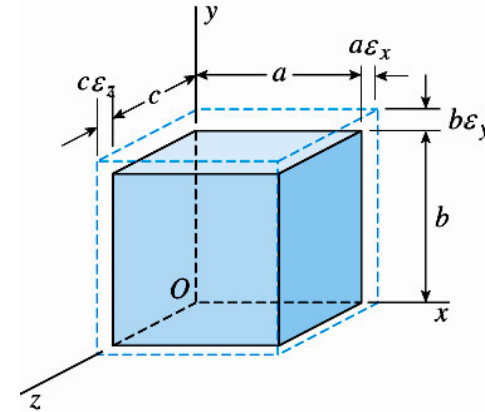
STRAIN ENERGY DENSITY IN PLANE STRESS

The strain energy density u is the strain energy stored in a unit volume of the material. Because the normal and shear strains occur independently, we can add the strain energy of these two elements to obtain the total energy.

Work done = Force x distance

Work done in the x - direction $= \dfrac{(\sigma_x)(bc)}{2}(a\varepsilon_x)$

Work done in the y - direction $= \dfrac{(\sigma_y)(ac)}{2}(b\varepsilon_y)$

The sum of the energies due to normal stresses:

$$U = \frac{abc}{2}\left(\sigma_x\varepsilon_x + \sigma_y\varepsilon_y\right)$$

Then the strain energy density (strain per unit volume)

$$u_1 = \frac{1}{2}\left(\sigma_x\varepsilon_x + \sigma_y\varepsilon_y\right)$$

Similarly, the strain energy density associated with the shear strain:

$$u_2 = \frac{1}{2}\tau_{xy}\gamma_{xy}$$

By combining the strain energy densities for the normal and shear strains:

$$u = \frac{1}{2}\left(\sigma_x\varepsilon_x + \sigma_y\varepsilon_y + \tau_{xy}\gamma_{xy}\right)$$

The strain energy density in terms of stress alone:

$$u = \frac{\sigma_X^2}{2E} + \frac{\sigma_Y^2}{2E} - \upsilon\frac{\sigma_X\sigma_Y}{E} + \frac{\tau_{XY}^2}{2G}$$

The strain energy density in terms of strain alone:

$$u = \frac{E}{2(1-\upsilon^2)}\left(\varepsilon_X^2 + \varepsilon_Y^2 - 2\upsilon\varepsilon_X\varepsilon_Y\right) + \frac{G}{2}\gamma_{XY}^2$$

For the special case of uniaxial stress:

$$\sigma_y = 0 \quad \tau_{xy} = 0 \quad \varepsilon_y = -\upsilon\varepsilon_x \quad \gamma_{xy} = 0$$

$$u = \frac{\sigma_x^2}{2E} \quad \text{or} \quad u = \frac{E\varepsilon_x^2}{2}$$

For the special case of pure shear:

$$\sigma_x = 0 \quad \sigma_y = 0 \quad \varepsilon_y = \varepsilon_x = 0$$

$$u = \frac{\tau_{xy}^2}{2G} \quad \text{or} \quad u = \frac{G\gamma_{xy}^2}{2}$$

TRIAXIAL STRESS

An element of the material subjected to normal stresses σ_x, σ_y and σ_z acting in three mutually perpendicular directions is said to be in a state of triaxial stress. Since there is *no shear* in *x*, *y* or *z* faces then the stresses σ_x, σ_y and σ_z are the principal stresses in the material.
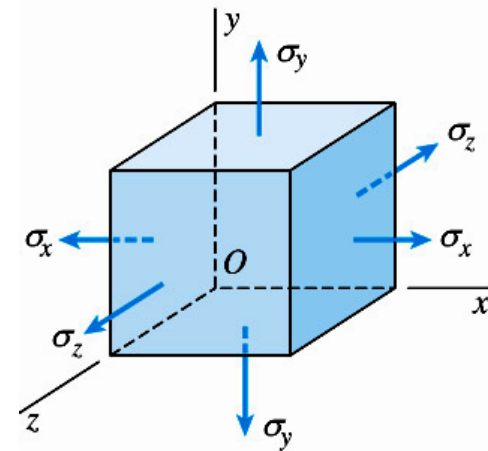
If an inclined plane parallel to the **z-axis** is cut through the element, the only stress of the inclined face are the normal stress σ and the shear stress τ, both of which act parallel to the **xy** plane.
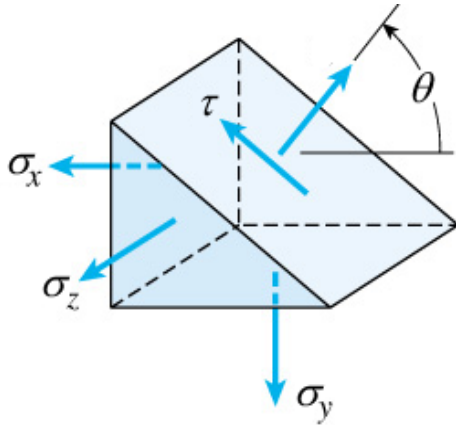
Because these stresses are independent of the σ_z, we can use the transformation equations of plane stress, as well as the Mohr's circle for plane stress, when determining the stresses σ and τ in triaxial stress.

The same general conclusion hold for normal and shear stresses acting on inclined planes cut through the element parallel to the **x** and **y** axes.

Maximum Shear Stress   For a material in triaxial stress, the maximum shear stresses occur on elements oriented at angles of **45⁰** to the **x**, **y** and **z** axes.

for the inclined plane // z-axis

$$(\tau_{MAX})_Z = \pm \frac{\sigma_X - \sigma_Y}{2}$$

for the inclined plane // x-axis

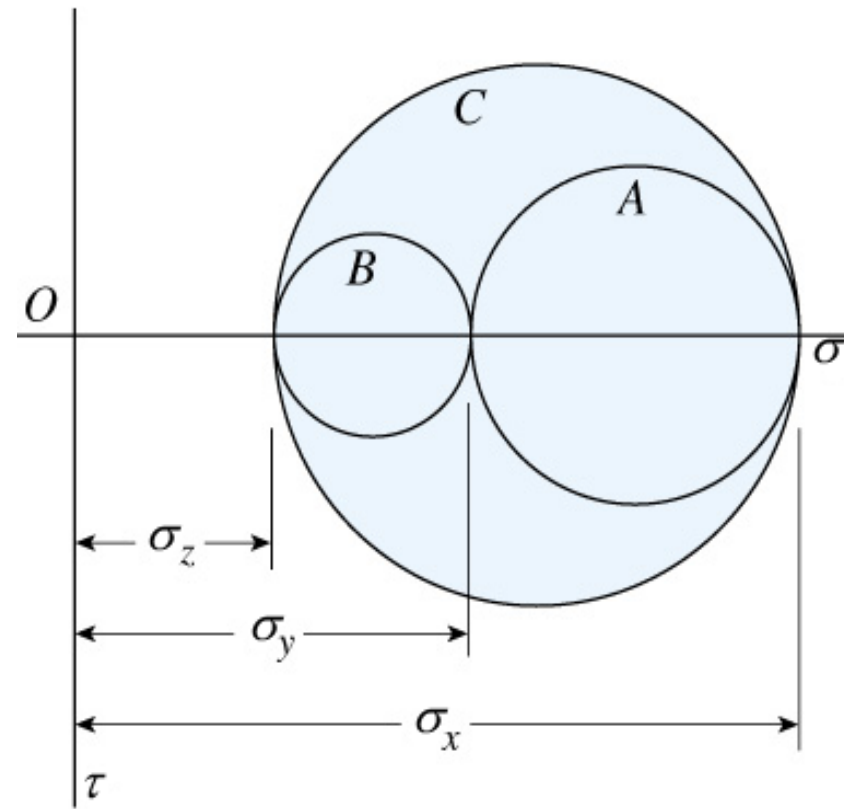$$(\tau_{MAX})_X = \pm \frac{\sigma_Y - \sigma_Z}{2}$$

for the inclined plane // y-axis

$$(\tau_{MAX})_Y = \pm \frac{\sigma_X - \sigma_Z}{2}$$

The absolute maximum of the shear stress is the numerically largest of the above.

The stresses acting on elements oriented at various angles to the x, y and z axes can be visualized with the aid of the Mohr's circle.

In this case $\sigma_x > \sigma_y > \sigma_z$

Hooke's Law for Triaxial Stress

If Hooke's law is obeyed, it is possible to obtain the relationship between normal stresses and normal strains using the same procedure as for plane stress.

$$\varepsilon_x = \frac{\sigma_x}{E} - v\frac{\sigma_y}{E} - v\frac{\sigma_z}{E}$$

$$\varepsilon_y = -v\frac{\sigma_x}{E} + \frac{\sigma_y}{E} - v\frac{\sigma_z}{E}$$

$$\varepsilon_z = -v\frac{\sigma_x}{E} - v\frac{\sigma_y}{E} + \frac{\sigma_z}{E}$$

Or

$$\sigma_x = \frac{E}{(1+v)(1-2v)}\left[(1-v)\varepsilon_x + v\varepsilon_y + v\varepsilon_z\right]$$

$$\sigma_y = \frac{E}{(1+v)(1-2v)}\left[v\varepsilon_x + (1-v)\varepsilon_y + v\varepsilon_z\right]$$

$$\sigma_z = \frac{E}{(1+v)(1-2v)}\left[v\varepsilon_x + v\varepsilon_y + (1-v)\varepsilon_z\right]$$

They are known as the **Hooke's law for triaxial stress**.

Unit Volume Change

The unit volume change (or dilatation) for an element in triaxial stress is obtained in the same manner as for plane stress.

$$e = \frac{\Delta V}{V_O} = \varepsilon_X + \varepsilon_Y + \varepsilon_Z$$

If Hooke's laws apply, then

$$e = \frac{\Delta V}{V_O} = \frac{(1-2v)}{E}(\sigma_X + \sigma_Y + \sigma_Z)$$

Strain Energy Density

The strain energy density for an element in triaxial stress is obtained by the same method used for plane stress.

$$u = \frac{\sigma_x \varepsilon_x}{2} + \frac{\sigma_y \varepsilon_y}{2} + \frac{\sigma_z \varepsilon_z}{2}$$

In terms of stresses :

$$u = \frac{1}{2E}\left(\sigma_x^2 + \sigma_y^2 + \sigma_z^2\right) - \frac{\upsilon}{E}\left(\sigma_x \sigma_y + \sigma_x \sigma_z + \sigma_y \sigma_z\right)$$

In terms of the strains:

$$u = \frac{E}{2(1+\upsilon)(1-2\upsilon)}\left[(1-\upsilon)\left(\varepsilon_x^2 + \varepsilon_y^2 + \varepsilon_z^2\right) + 2\upsilon\left(\varepsilon_x \varepsilon_y + \varepsilon_x \varepsilon_z + \varepsilon_y \varepsilon_z\right)\right]$$
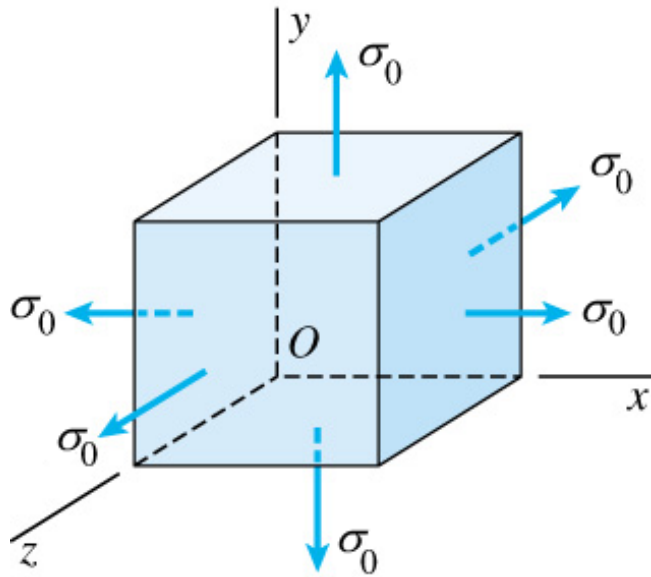
Spherical Stress

A special case of triaxial stress, called spherical stress, occurs whenever all three normal stresses are equal:

$$\sigma_x = \sigma_y = \sigma_z = \sigma_0$$

The Mohr's circle is reduced to a single point. Any plane cut through the element will be free of shear stress and will be subjected to the same normal stress *so* and it is a principal plane.

The normal strains in spherical stress are also the same in all directions, provided the material is isotropic and if Hooke's law applies:

$$\varepsilon_O = \frac{\sigma_O}{E}(1-2\upsilon)$$ The volume change

$$e = 3\varepsilon_O = 3\sigma_O \frac{(1-2\upsilon)}{E}$$

Element in spherical stress.

K = bulk or volume modulus of elasticity

$$K = \frac{E}{3(1-2\upsilon)} \qquad K = \frac{\sigma_0}{e}$$

If $\upsilon = 1/3$ then $K = E$

If $\upsilon = 0$ then $K = E/3$

If $\upsilon = 1/2$ then K = infinite (rigid material having no change in volume)

These formulas also apply to and element in uniform compression, for example rock deep within the earth or material submerged in water (hydrostatic stress).
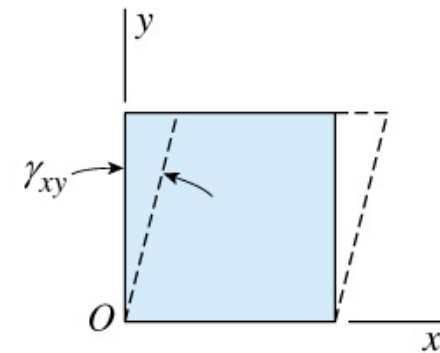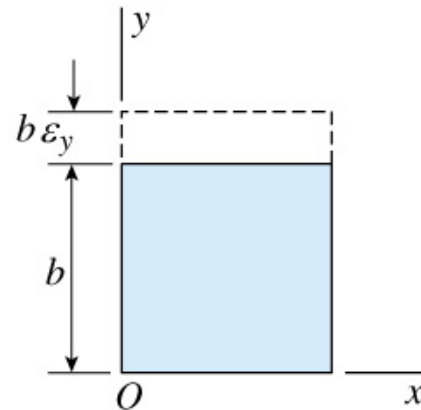
PLAIN STRAIN

Strains are measured by strain gages.
A material is said to be in a state of plain strain if the only deformations are those in the **xy** plane, i.e. it has only three strain components ε_x, ε_y and γ_{xy}.

Plain stress is analogous to plane stress, but under ordinary conditions they do not occur simultaneously
Exception when $\sigma_x = -\sigma_y$ and when $v = 0$

Strain components ε_x, ε_y, and γ_{xy} in the **xy** plane (plane strain).
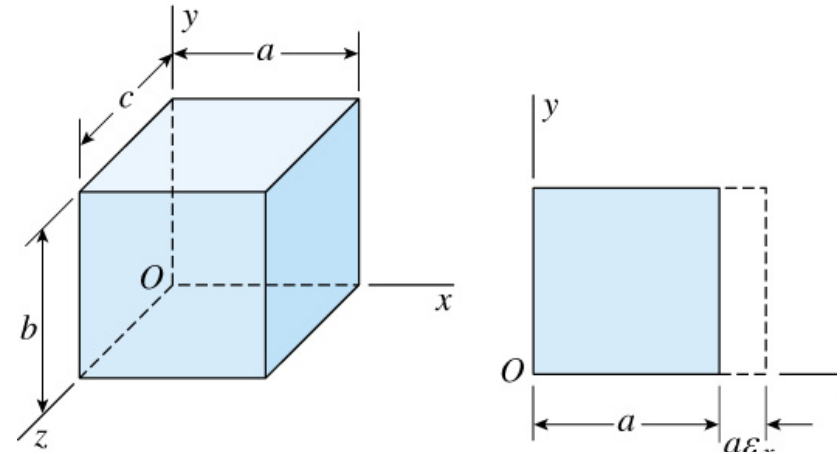
	Plane stress	Plane strain
Stresses	$\sigma_z = 0$ $\quad$ $\tau_{xz} = 0$ $\quad$ $\tau_{yz} = 0$ σ_x, σ_y, and τ_{xy} may have nonzero values	$\tau_{xz} = 0$ $\quad$ $\tau_{yz} = 0$ σ_x, σ_y, σ_z, and τ_{xy} may have nonzero values
Strains	$\gamma_{xz} = 0$ $\quad$ $\gamma_{yz} = 0$ ε_x, ε_y, ε_z, and γ_{xy} may have nonzero values	$\varepsilon_z = 0$ $\quad$ $\gamma_{xz} = 0$ $\quad$ $\gamma_{yz} = 0$ ε_x, ε_y, and γ_{xy} may have nonzero values

Comparison of plane stress and plane strain.

APPLICATION OF THE TRANSFORMATION EQUATIONS

The transformation equations for plane stress are valid even when σ_z is not zero, because σ_z does not enter the equations of equilibrium.Therefore, *the transformations equations for plane stress can also be used for stresses in plane strain*.

Similarly, the strain transformation equations that will be derived for the case of plain strain in the *xy* plane are valid even when ε_z is not zero, because the strain ε_z does not affect the geometric relationship used for the derivation.Therefore, *the transformations equations for plane strain can also be used for strains in plane stress*.

Transformation Equations for Plain Strain

We will assume that the strain ε_x, ε_y and γ_{xy} associated with the *xy* plane are known.
We need to determine the normal and shear strains (ε_{x1} and γ_{x1y1}) associated with the x_1y_1 axis. ε_{y1} can be obtained from the equation of ε_{x1} by substituting $\theta+90$ for θ.
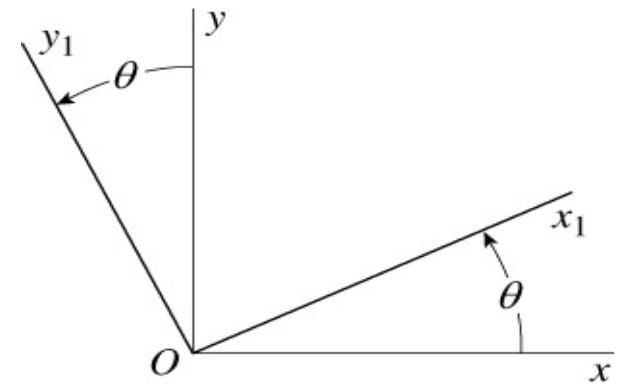
For an element of size dx, dy

In the x direction:

the strain ε_x produces an elongation $\varepsilon_x\, dx$.
The diagonal increases in length by $\varepsilon_x\, dx\cos\theta$.
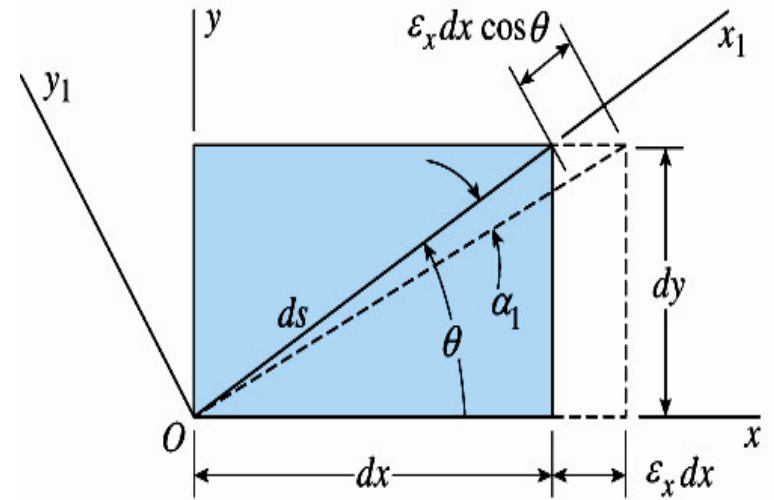

In the y direction:
the strain ε_y produces an elongation $\varepsilon_y\, dy$.
The diagonal increases in length by $\varepsilon_y\, dy\sin\theta$.
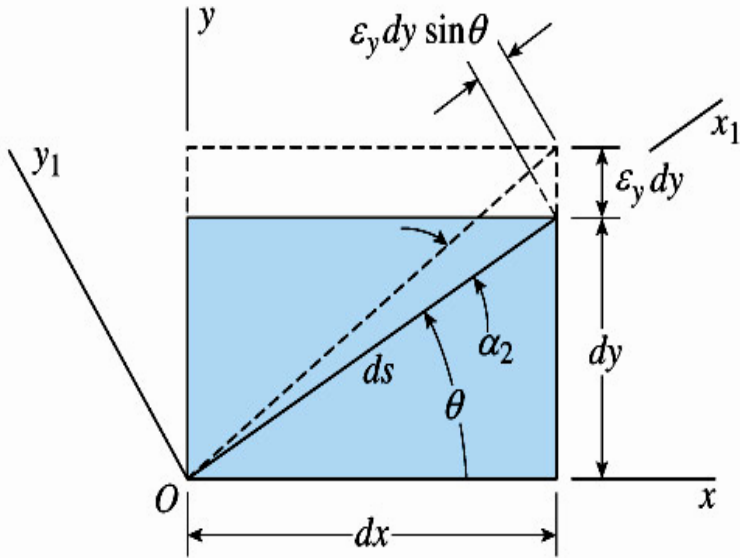
The shear strain γ_{xy} in the plane xy produces a distortion of the element such that the angle at the lower left corner decreases by an amount equal to the shear strain. Consequently, the upper face moves to the right by an amount $\gamma_{xy}\, dy$. This deformation results in an increase in the length of the diagonal equal to: $\gamma_{xy}\, dy\cos\theta$
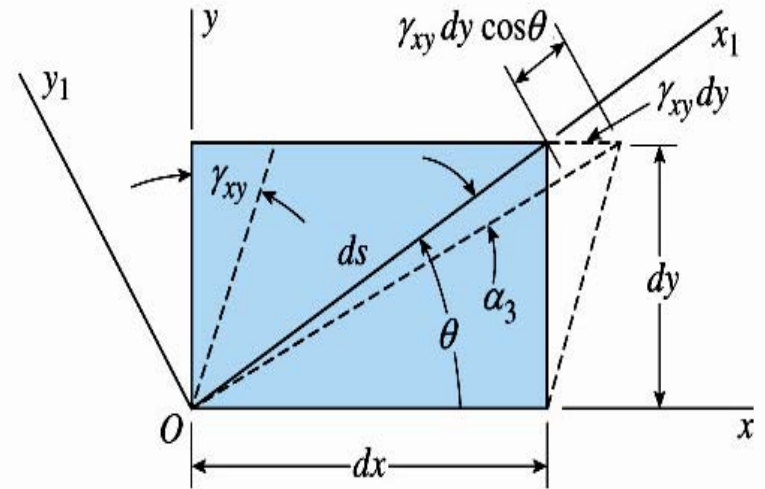
The total increase $\Delta \delta$ of the diagonal is the sum of the preceding three expressions, thus:

$$\Delta d = \varepsilon_x dx (Cos\,\theta) + \varepsilon_y dy (Sin\,\theta) + \gamma_{xy} dy (Cos\,\theta)$$

But

$$\varepsilon_{x1} = \frac{\Delta d}{ds} = \varepsilon_x \left(\frac{dx}{ds} \right)(Cos\,\theta) + \varepsilon_y \left(\frac{dy}{ds} \right)(Sin\,\theta) + \gamma_{xy} \left(\frac{dy}{ds} \right)(Cos\,\theta)$$

$$\frac{dx}{ds} = Cos\,\theta \qquad \frac{dy}{ds} = Sin\,\theta$$

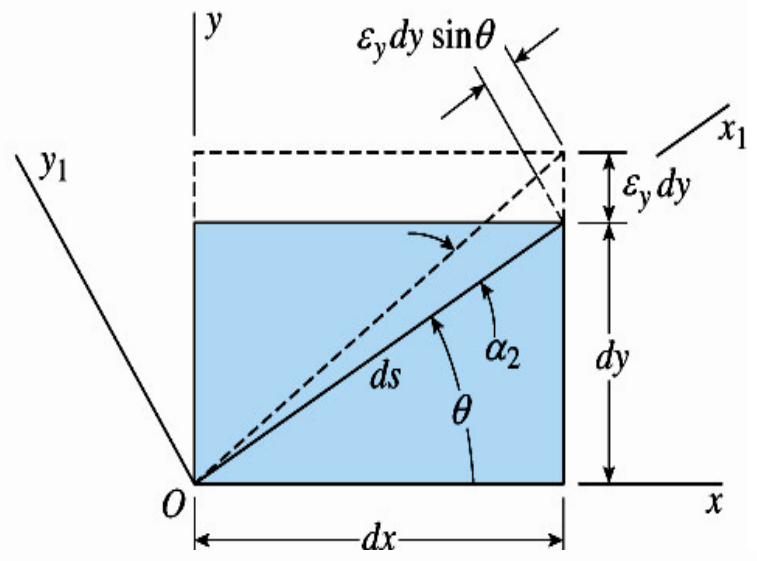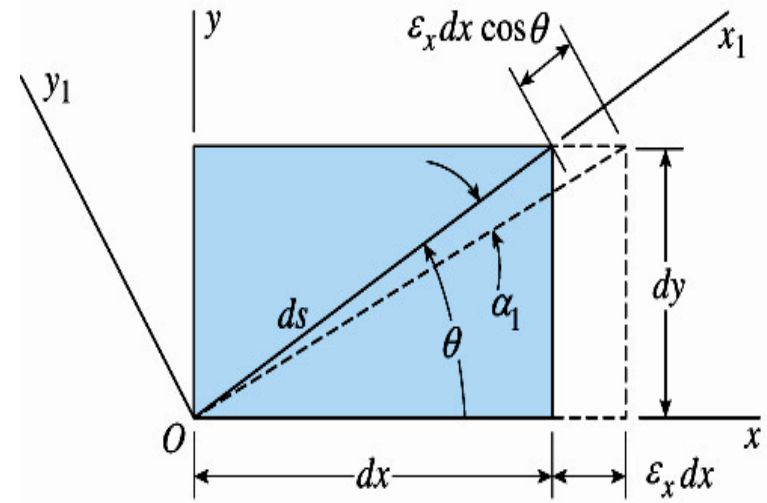$$\varepsilon_{x1} = \varepsilon_x Cos^2\theta + \varepsilon_y Sin^2\theta + \gamma_{xy} Sin\,\theta Cos\,\theta$$

Shear Strain γ_{x1y1} associated with x_1y_1 axes.

This strain is equal to the decrease in angle between lines in the material that were initially along the **x1** and **y1** axes.
Oa and **Ob** were the lines initially along the **x1** and **y1** axis respectively. The deformation caused by the strains ε_x, ε_y and γ_{xy} caused the **Oa** and **Ob** lines to rotate and angle α and β from the x_1 and y_1 axis respectively. The shear strain γ_{x1y1} is the decrease in angle between the two lines that originally were at right angles, therefore, $\gamma_{x1y1} = \alpha + \beta$.
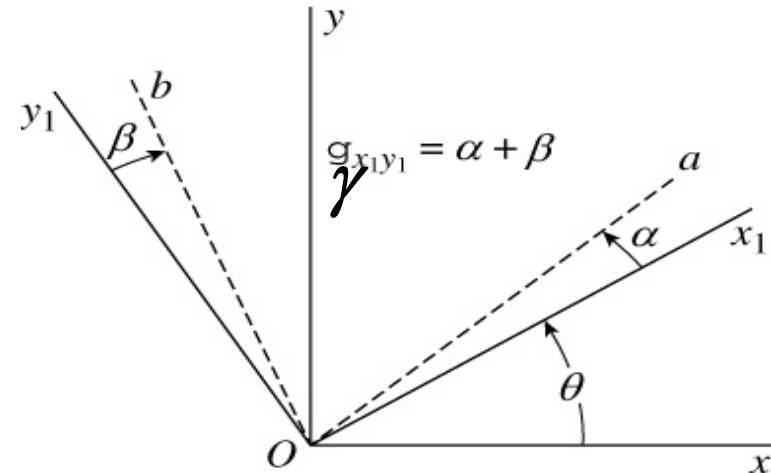
The angle α can be found from the deformations produced by the strains ε_x, ε_y and γ_{xy} . The strains ε_x and γ_{xy} produce a clockwise rotation, while the strain ε_y produces a counterclockwise rotation.

Let us denote the angle of rotation produced by ε_x , ε_y and γ_{xy} as α_1 , α_2 and α_3 respectively.

$$\alpha_1 = \varepsilon_x Sin\theta \frac{dx}{ds}$$

$$\frac{dx}{ds} = Cos\theta \qquad \frac{dy}{ds} = Sin\theta$$

$$\alpha_2 = \varepsilon_y Cos\theta \frac{dy}{ds}$$

$$\alpha_1 = \gamma_{xy} Sin\theta \frac{dy}{ds}$$
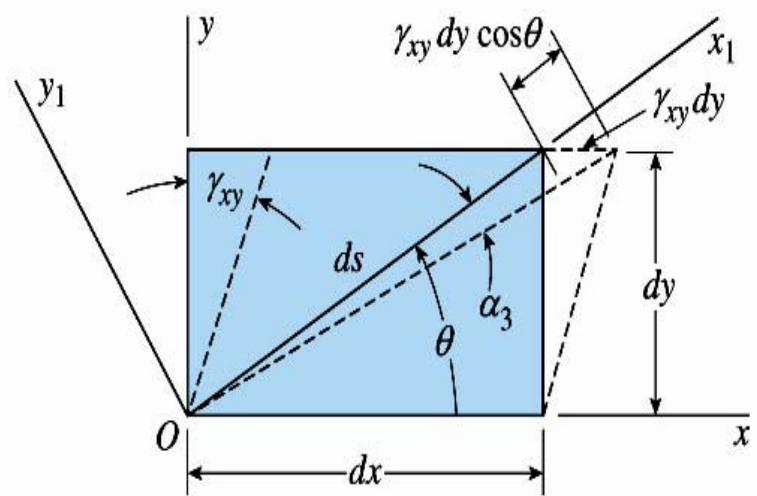
$$\alpha = -\alpha_1 + \alpha_2 - \alpha_3 = -\left(\varepsilon_x - \varepsilon_y\right) Sin\theta Cos\theta - \gamma_{xy} Sin^2\theta$$

The rotation of line **Ob** which initially was at **90⁰** to the line **Oa** can be found by substituting $\theta + 90$ for θ in the expression for α. Because β is positive when clockwise. Thus:

$$\beta = \left(\varepsilon_x - \varepsilon_y\right)Sin(\theta + 90)Cos(\theta + 90) + \gamma_{xy}Sin^2(\theta + 90)$$

$$\beta = -\left(\varepsilon_x - \varepsilon_y\right)Sin\,\theta Cos\,\theta + \gamma_{xy}Cos^2\theta$$

$$\gamma_{x1y1} = \alpha + \beta = -\left(\varepsilon_x - \varepsilon_y\right)Sin\,\theta Cos\,\theta + \frac{\gamma_{xy}}{2}\left[Cos^2\theta - Sin^2\theta\right]$$

Transformation Equations for Plain Strain

Using the following trigonometric identities:

Cos²θ = ½ (1+ Cos 2θ)

Sin²θ = ½ (1- Cos 2θ)

Sin θ cos θ = ½ Sin 2θ

$$\varepsilon_{x1} = \frac{\left(\varepsilon_x + \varepsilon_y\right)}{2} + \frac{\left(\varepsilon_x - \varepsilon_y\right)}{2}\cos 2\theta + \frac{\gamma_{xy}}{2}\sin 2\theta$$

$$\frac{\gamma_{x1y1}}{2} = -\frac{\left(\varepsilon_x - \varepsilon_y\right)}{2}\sin 2\theta + \frac{\gamma_{xy}}{2}\cos 2\theta$$

$$In\,var\,iant = \varepsilon_x + \varepsilon_y = \varepsilon_{x1} + \varepsilon_{y1}$$

$$\varepsilon_{Average} = \frac{\left(\varepsilon_x + \varepsilon_y\right)}{2}$$

PRINCIPAL STRAINS

The angle for the principal strains is :

$$\tan 2\theta_P == \frac{\gamma_{xy}/2}{(\varepsilon_x - \varepsilon_y)/2} = \frac{\gamma_{xy}}{\varepsilon_x - \varepsilon_y}$$

The value for the principal strains are

$$\varepsilon_1 = \frac{(\varepsilon_x + \varepsilon_y)}{2} + \sqrt{\left(\frac{\varepsilon_x - \varepsilon_y}{2}\right)^2 + \left(\frac{\gamma_{xy}}{2}\right)^2}$$

$$\varepsilon_2 = \frac{(\varepsilon_x + \varepsilon_y)}{2} - \sqrt{\left(\frac{\varepsilon_x - \varepsilon_y}{2}\right)^2 + \left(\frac{\gamma_{xy}}{2}\right)^2}$$
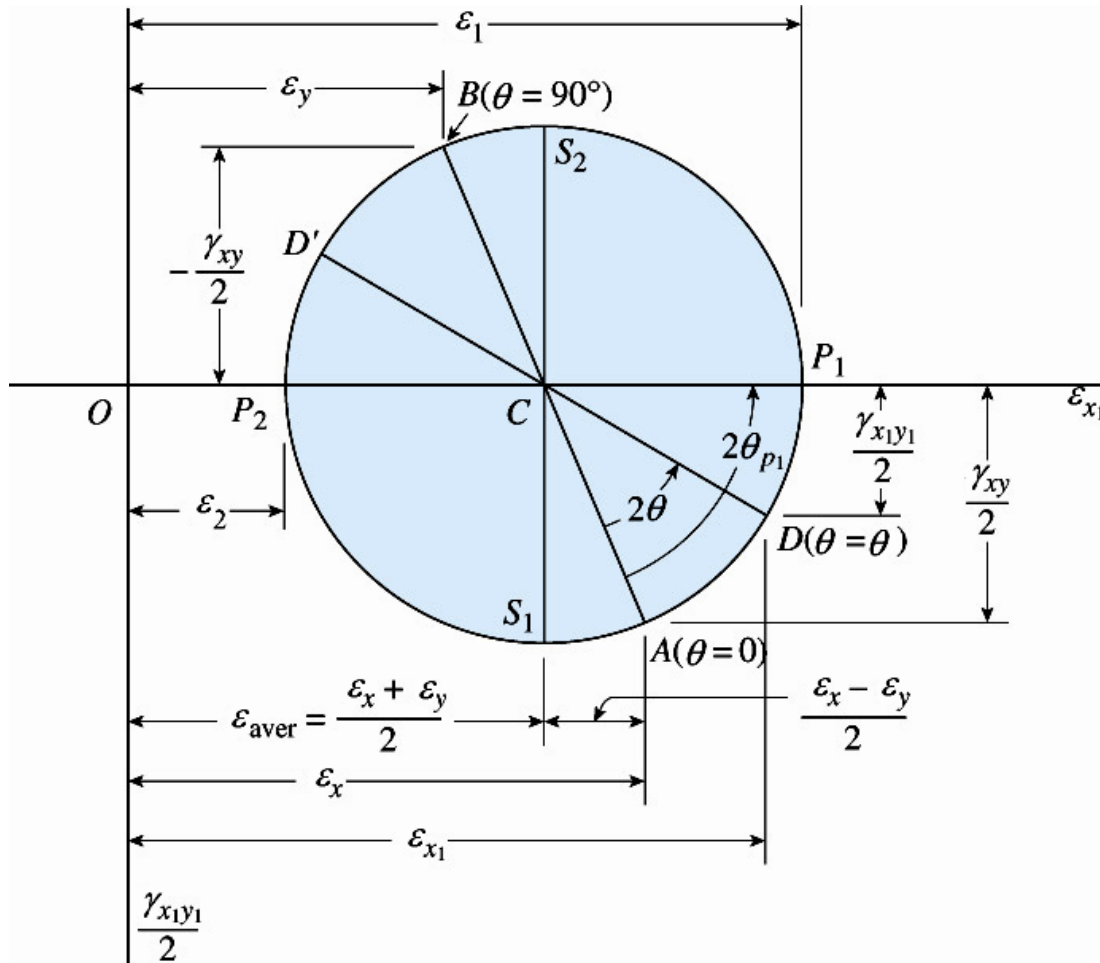
Maximum Shear

The maximum shear strains in the **xy** plane are associated with axes at **45°** to the directions of the principal strains

$$\frac{\gamma_{Max}}{2} = +\sqrt{\left(\frac{\varepsilon_x - \varepsilon_y}{2}\right)^2 + \left(\frac{\gamma_{xy}}{2}\right)^2} \quad \text{or} \quad \gamma_{Max} = (\varepsilon_1 - \varepsilon_2)$$

For isotropic materials, at a given point in an stressed body, the principal strains and principal stresses occur in the same directions.

MOHR'S CIRCLE FOR PLANE STRAIN

It is constructed in the same manner as the Mohr's circle for plane stress with the following similarities:

$$\sigma_x \parallel \varepsilon_x$$
$$\sigma_y \parallel \varepsilon_y$$
$$\tau_{xy} \parallel \frac{\gamma_{xy}}{2}$$
$$\sigma_{x1} \parallel \varepsilon_{x1}$$
$$\tau_{x1y1} \parallel \frac{\gamma_{x1y1}}{2}$$

Strain Measurements

An electrical-resistance *strain gage* is a device for measuring normal strains (ε) on the surface of a stressed object.

The gages are small (less than ½ inch) made of wires that are bonded to the surface of the object. Each gage that is stretched or shortened when the object is strained at the point, changes its electrical resistance. This change in resistance is converted into a measurement of strain.

From three measurements it is possible to calculate the strains in any direction. A group of three gages arranged in a particular pattern is called a *strain rosette*. Because the rosette is mounted in the surface of the body, where the material is in plane stress, we can use the transformation equations for plane strain to calculate the strains in various directions.
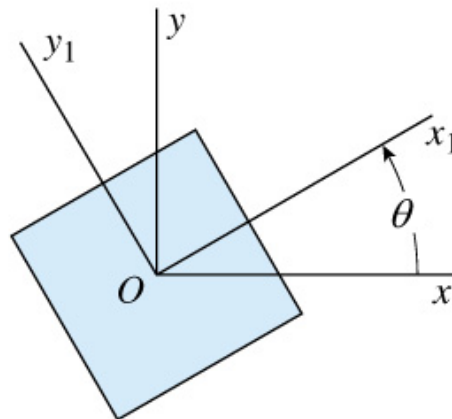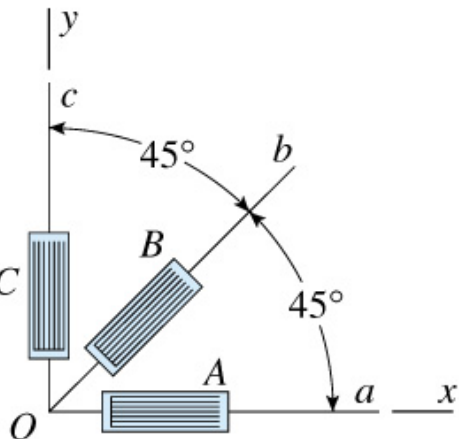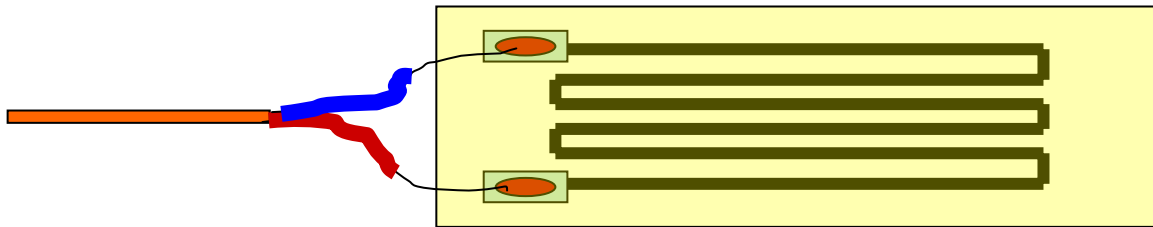
45° strain rosette, and element oriented at an angle θ to the *xy* axes.

General Equations
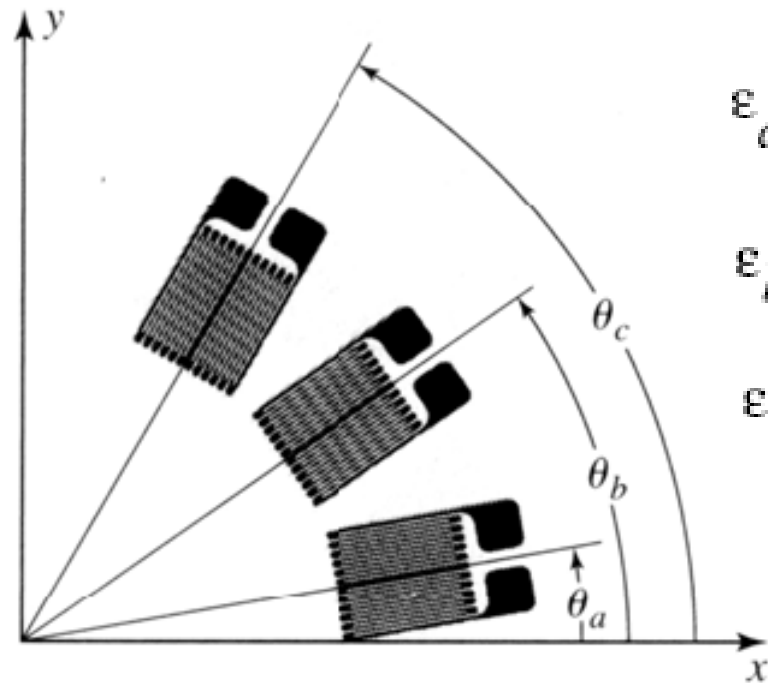
$$\varepsilon_a = \varepsilon_{xx}\cos^2\theta_a + \varepsilon_{yy}\sin^2\theta_a + \gamma_{xy}\sin\theta_a\cos\theta_a$$
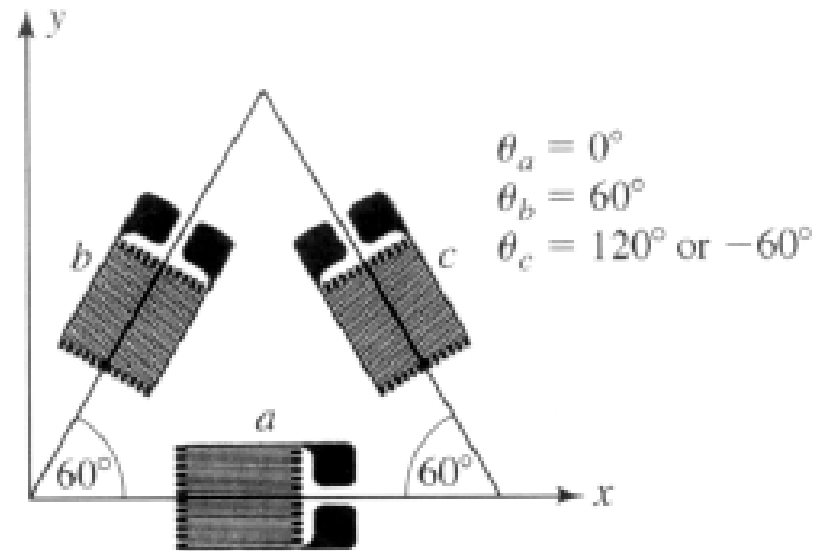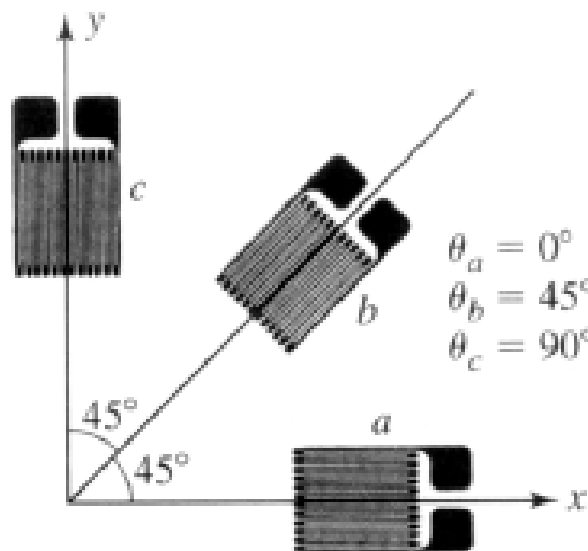
$$\varepsilon_b = \varepsilon_{xx}\cos^2\theta_b + \varepsilon_{yy}\sin^2\theta_b + \gamma_{xy}\sin\theta_b\cos\theta_b$$

$$\varepsilon_c = \varepsilon_{xx}\cos^2\theta_c + \varepsilon_{yy}\sin^2\theta_c + \gamma_{xy}\sin\theta_c\cos\theta_c$$

Other Strain Rosette

$\theta_a = 0°$
$\theta_b = 45°$
$\theta_c = 90°$

$\theta_a = 0°$
$\theta_b = 60°$
$\theta_c = 120°$ or $-60°$

Example

An element of material in plane strain undergoes the following strains: $\varepsilon_x = 340 \times 10^{-6}$; $\varepsilon_y = 110 \times 10^{-6}$; $\gamma_{xy} = 180 \times 10^{-6}$. Determine the following quantities:
(a) the strains of an element oriented at an angle $\theta = 30^o$;
(b) the principal strains and
(c) the maximum shear strains.

(a) Element oriented at an angle $\theta = 30^o$ ($2\theta = 60^o$)

$$\varepsilon_{x1} = \frac{(\varepsilon_x + \varepsilon_y)}{2} + \frac{(\varepsilon_x - \varepsilon_y)}{2} \cos 2\theta + \frac{\gamma_{xy}}{2} \sin 2\theta$$
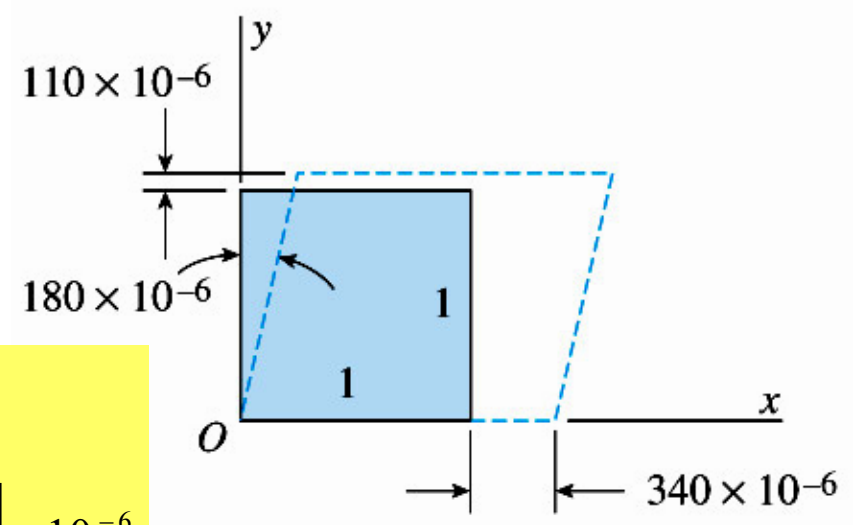
$$\varepsilon_{x1} = \left[\frac{340 + 110}{2} + \frac{340 - 110}{2} Cos\ 60 + \frac{180}{2} Sin\ 60 \right] \times 10^{-6}$$

$$\varepsilon_{x1} = 360 \times 10^{-6}$$

$$\frac{\gamma_{x1y1}}{2} = -\frac{(\varepsilon_x - \varepsilon_y)}{2} \sin 2\theta + \frac{\gamma_{xy}}{2} \cos 2\theta$$

$$\frac{\gamma_{x1y1}}{2} = \left[-\frac{340 - 110}{2} Sin\ 60 + \frac{180}{2} Cos\ 60 \right] \times 10^{-6} = -55 \times 10^{-6}$$

$$\varepsilon_{Average} = 225\ \mu$$

110×10^{-6}

180×10^{-6}

340×10^{-6}

$$\varepsilon_x + \varepsilon_y = \varepsilon_{x1} + \varepsilon_{y1}$$

$$340\mu + 110\mu = 360\mu + \varepsilon_{y1}$$

$$\varepsilon_{y1} = 90\mu$$

(b) *Principal Strains and Angle of Rotation*

$$\varepsilon_{1,2} = \frac{(\varepsilon_x + \varepsilon_y)}{2} \pm \sqrt{\left(\frac{\varepsilon_x - \varepsilon_y}{2}\right)^2 + \left(\frac{\gamma_{xy}}{2}\right)^2}$$

$$\varepsilon_1 = 225\mu + \sqrt{\left(\frac{340\mu - 110\mu}{2}\right)^2 + \left(\frac{180}{2}\right)^2} = 370\mu$$

$$\varepsilon_2 = 225\mu - \sqrt{\left(\frac{340\mu - 110\mu}{2}\right)^2 + \left(\frac{180}{2}\right)^2} = 80\mu$$

$$\tan 2\theta_P = \frac{\gamma_{xy}}{\varepsilon_x - \varepsilon_y} = \frac{-55\mu}{340\mu - 110\mu} = 0.7826$$

$$\theta_P = 19^0$$
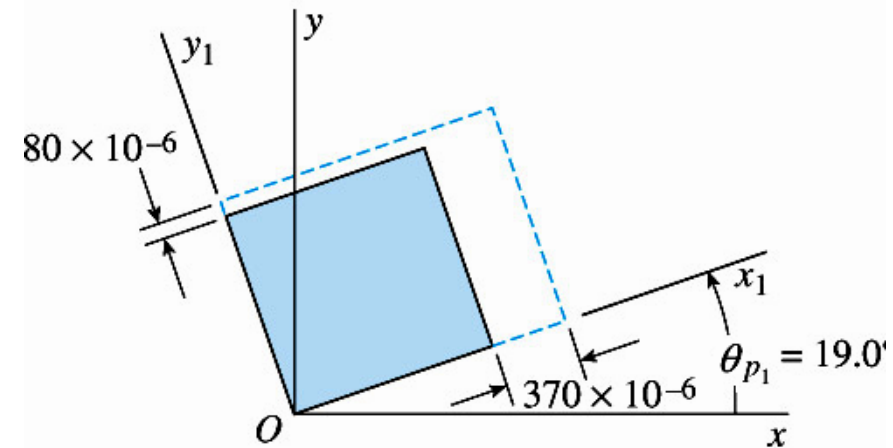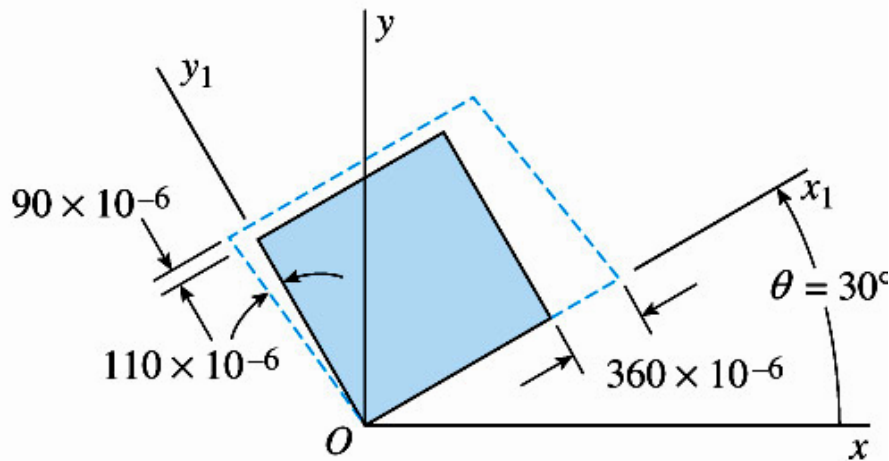
(c) In-Plane Maximum Shear Strain

$$\frac{\gamma_{Max}}{2} = +\sqrt{\left(\frac{\varepsilon_x - \varepsilon_y}{2}\right)^2 + \left(\frac{\gamma_{xy}}{2}\right)^2} = \sqrt{\left(\frac{340\mu - 110\mu}{2}\right)^2 + \left(\frac{180\mu}{2}\right)^2} = 145\mu$$

$$\gamma_{Max} = \left(\varepsilon_1 - \varepsilon_2\right) = 370\mu - 80\mu = 290\mu$$

$$\varepsilon_{Average} = 225\ \mu$$
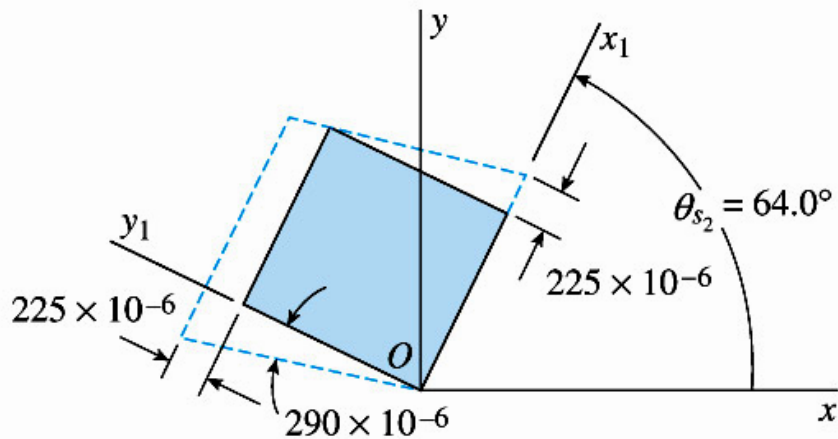
(d) Out-of-Plane Maximum Shear Strain

$$\gamma_{Max} = \left(\varepsilon_1 - \varepsilon_3\right) = 370\mu - 0\mu = 370\mu$$

Transformation Equations

$$\varepsilon_{X1} = \varepsilon_X \cos^2\theta + \varepsilon_Y \sin^2\theta + \left(\frac{\gamma_{XY}}{2}\right) 2\sin\theta\cos\theta$$

$$\varepsilon_{Y1} = \varepsilon_X \sin^2\theta + \varepsilon_Y \cos^2\theta - \left(\frac{\gamma_{XY}}{2}\right) 2\sin\theta\cos\theta$$

$$\frac{\gamma_{X1Y1}}{2} = -\varepsilon_X \sin\theta\cos\theta + \varepsilon_Y \sin\theta\cos\theta + \frac{\gamma_{XY}}{2}\left(\cos^2\theta - \sin^2\theta\right)$$

$$\begin{bmatrix} \varepsilon_{X1} \\ \varepsilon_{Y1} \\ \dfrac{\gamma_{X1Y1}}{2} \end{bmatrix} = [T] \times \begin{bmatrix} \varepsilon_X \\ \varepsilon_Y \\ \dfrac{\gamma_{XY}}{2} \end{bmatrix}$$

$$[T] = \begin{bmatrix} \cos^2\theta & \sin^2\theta & 2\sin\theta\cos\theta \\ \sin^2\theta & \cos^2\theta & -2\sin\theta\cos\theta \\ -\sin\theta\cos\theta & \sin\theta\cos\theta & \left(\cos^2\theta - \sin^2\theta\right) \end{bmatrix}$$

$$\begin{bmatrix} \varepsilon_X \\ \varepsilon_Y \\ \dfrac{\gamma_{XY}}{2} \end{bmatrix} = [T]^{-1} \times \begin{bmatrix} \varepsilon_{X1} \\ \varepsilon_{Y1} \\ \dfrac{\gamma_{X1Y1}}{2} \end{bmatrix}$$

$$\begin{bmatrix} \varepsilon_{X1} \\ \varepsilon_{Y1} \\ \dfrac{\gamma_{X1Y1}}{2} \end{bmatrix} = \begin{bmatrix} \cos^2\theta & \sin^2\theta & 2\sin\theta\cos\theta \\ \sin^2\theta & \cos^2\theta & -2\sin\theta\cos\theta \\ -\sin\theta\cos\theta & \sin\theta\cos\theta & \left(\cos^2\theta - \sin^2\theta\right) \end{bmatrix} \begin{bmatrix} \varepsilon_X \\ \varepsilon_Y \\ \dfrac{\gamma_{XY}}{2} \end{bmatrix}$$

Example   For Θ=30 degrees

$$\begin{bmatrix} \varepsilon_{X1} \\ \varepsilon_{Y1} \\ \dfrac{\gamma_{X1Y1}}{2} \end{bmatrix} = \begin{bmatrix} \cos^2 30 & \sin^2 30 & 2\sin 30 \cos 30 \\ \sin^2 30 & \cos^2 30 & -2\sin 30 \cos 30 \\ -\sin 30 \cos 30 & \sin 30 \cos 30 & \left(\cos^2 30 - \sin^2 30\right) \end{bmatrix} \begin{bmatrix} \varepsilon_X \\ \varepsilon_Y \\ \dfrac{\gamma_{XY}}{2} \end{bmatrix}$$

$$\begin{bmatrix} \varepsilon_{X1} \\ \varepsilon_{Y1} \\ \dfrac{\gamma_{X1Y1}}{2} \end{bmatrix} = \begin{bmatrix} 0.75 & 0.25 & 0.876 \\ 0.25 & 0.75 & -0.876 \\ -0.438 & 0.438 & 0.5 \end{bmatrix} \begin{bmatrix} 340 \\ 110 \\ 90 \end{bmatrix} \mu = \begin{bmatrix} 361.3 \\ 88.6 \\ -55.8 \end{bmatrix} \mu$$

$$[\varepsilon] = \begin{bmatrix} \varepsilon_{xx} & \frac{1}{2}\gamma_{yx} & \frac{1}{2}\gamma_{zx} \\ \frac{1}{2}\gamma_{xy} & \varepsilon_{yy} & \frac{1}{2}\gamma_{zy} \\ \frac{1}{2}\gamma_{xz} & \frac{1}{2}\gamma_{yz} & \varepsilon_{zz} \end{bmatrix} = Strain\_Tensor$$

Example

$$[\varepsilon] = \begin{bmatrix} \varepsilon_{xx} & \frac{1}{2}\gamma_{yx} & \frac{1}{2}\gamma_{zx} \\ \frac{1}{2}\gamma_{xy} & \varepsilon_{yy} & \frac{1}{2}\gamma_{zy} \\ \frac{1}{2}\gamma_{xz} & \frac{1}{2}\gamma_{yz} & \varepsilon_{zz} \end{bmatrix} = \begin{bmatrix} 340 & 180/2 & 0 \\ 180/2 & 110 & 0 \\ 0 & 0 & 0 \end{bmatrix}\mu$$

$$Eigen\_Values \Rightarrow \begin{bmatrix} 371 & 0 & 0 \\ 0 & 79 & 0 \\ 0 & 0 & 0 \end{bmatrix}\mu$$
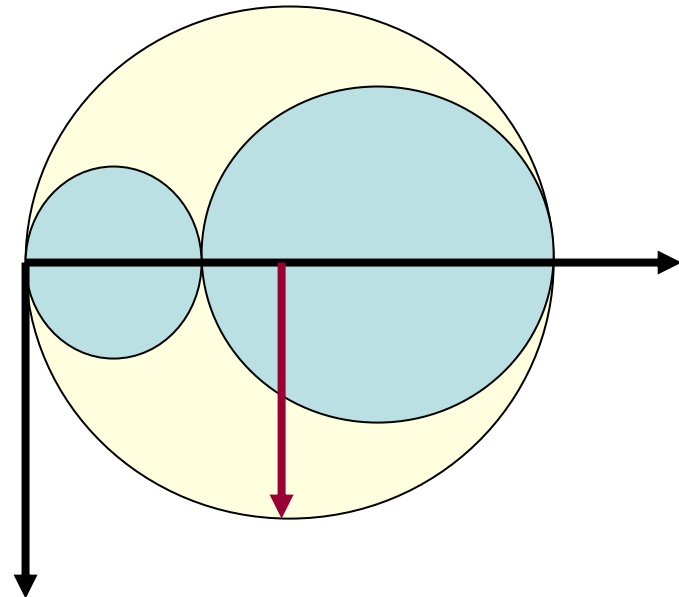
Example

A *45⁰* strain rosette (rectangular rosette) consists of three electrical-resistance strain gages, arranged to measure strains in two perpendicular directions and also at a *45⁰* angle (as shown below). The rosette is bonded to the surface of the structure before it is loaded. Gages *A*, *B* and *C* measure the normal strains ε_a, ε_b and ε_c in the directions of the lines *Oa*, *Ob* and *Oc*, respectively.

Explain how to obtain the strains ε_{x1}, ε_{y1} and γ_{x1y1}, associated with an element oriented at an angle θ to the *xy* axes.
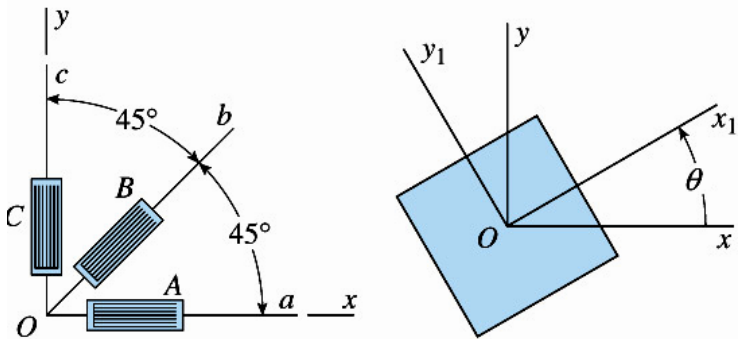
$$\varepsilon_a = \varepsilon_x \cos^2 \theta + \varepsilon_y \sin^2 \theta + \left(\frac{\gamma_{xy}}{2}\right) 2 \sin \theta \cos \theta$$

Angles with respect to x-axis: (a) is zero ; (b) is 45 degrees CCW ; (c) 90 degrees CCW

$$\varepsilon_a = \varepsilon_x \cos^2 0 + \varepsilon_y \sin^2 0 + \left(\frac{\gamma_{xy}}{2}\right) 2 \sin 0 \cos 0$$

$$\varepsilon_b = \varepsilon_x \cos^2 45 + \varepsilon_y \sin^2 45 + \left(\frac{\gamma_{xy}}{2}\right)(2 \sin 45 \cos 45)$$

$$\varepsilon_b = \varepsilon_x \cos^2 90 + \varepsilon_y \sin^2 90 + \left(\frac{\gamma_{xy}}{2}\right)(2 \sin 90 \cos 90)$$

$$\varepsilon_x = \varepsilon_a$$

$$\varepsilon_y = \varepsilon_b$$

$$\gamma_{xy} = 2\varepsilon_b - \varepsilon_a - \varepsilon_c$$

The following results are obtained from a 60^0 strain gauge rosette:

Strain in direction of strain gauge A = 750μ;

Strain in direction of SG B, 60^0 to A = 350μ;

Strain in direction of SG C, 120^0 to A = 100 μ.

Determine the principal strains and their directions.
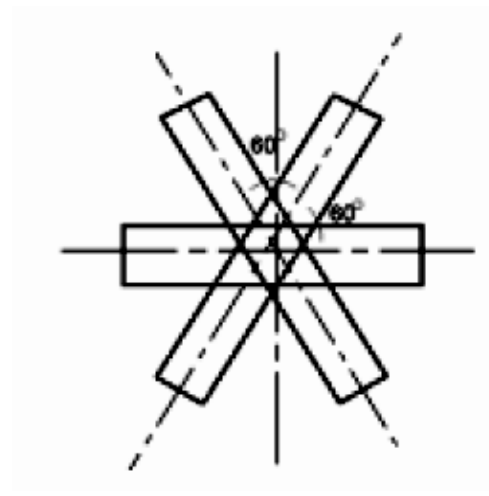
$$\varepsilon_a = \varepsilon_x = 750 \ \mu$$

$$\varepsilon_b = \varepsilon_x \cos^2 60 + \varepsilon_y \sin^2 60 + \left(\frac{\gamma_{xy}}{2}\right)(2 \sin 60 \cos 60)$$

$$\varepsilon_b = \varepsilon_x (0.25) + \varepsilon_y (0.75) + \left(\frac{\gamma_{xy}}{2}\right)(0.433)$$

$$\varepsilon_b = \varepsilon_x \cos^2 120 + \varepsilon_y \sin^2 120 + \left(\frac{\gamma_{xy}}{2}\right)(2 \sin 120 \cos 120)$$

$$\varepsilon_b = \varepsilon_x (0.25) + \varepsilon_y (0.75) + \left(\frac{\gamma_{xy}}{2}\right)(-0.433)$$

$$\varepsilon_x = 750, \ \varepsilon_y = 50 \text{ and } \gamma_{xy} = 289$$

$$\Rightarrow \begin{vmatrix} \varepsilon_y = 50\mu \\ \gamma_{xy} = 289\mu \end{vmatrix}$$

$$\theta = 11.2^o \text{ and } 101.2^o$$

$$\varepsilon_{11.2} = 779\mu\varepsilon$$

$$\varepsilon_{101.2} = 21\mu\varepsilon$$

$$[\varepsilon] = \begin{bmatrix} \varepsilon_{xx} & \frac{1}{2}\gamma_{yx} & \frac{1}{2}\gamma_{zx} \\ \frac{1}{2}\gamma_{xy} & \varepsilon_{yy} & \frac{1}{2}\gamma_{zy} \\ \frac{1}{2}\gamma_{xz} & \frac{1}{2}\gamma_{yz} & \varepsilon_{zz} \end{bmatrix} = \begin{bmatrix} 750 & \frac{289}{2} & 0 \\ \frac{289}{2} & 50 & 0 \\ 0 & 0 & 0 \end{bmatrix} \mu$$

$$Eigen\_Values \Rightarrow \begin{bmatrix} 779 & 0 & 0 \\ 0 & 21 & 0 \\ 0 & 0 & 0 \end{bmatrix} \mu$$

$$Eigen\_Vectors \Rightarrow \begin{bmatrix} 0.981 & 0.1945 & 0 \\ -0.1945 & 0.981 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

ArcCos(angle)=0.981

Angle =11.2degress

Example

(A) Using the transformation equations define the maximum and minimum principal strains, maximum shearing strain and principal angles given $\varepsilon_X = 3500\mu$; $\varepsilon_Y = 700\mu$ and $\gamma_{XY} = -1050\mu$

(B) Repeat using the Mohr's circle.

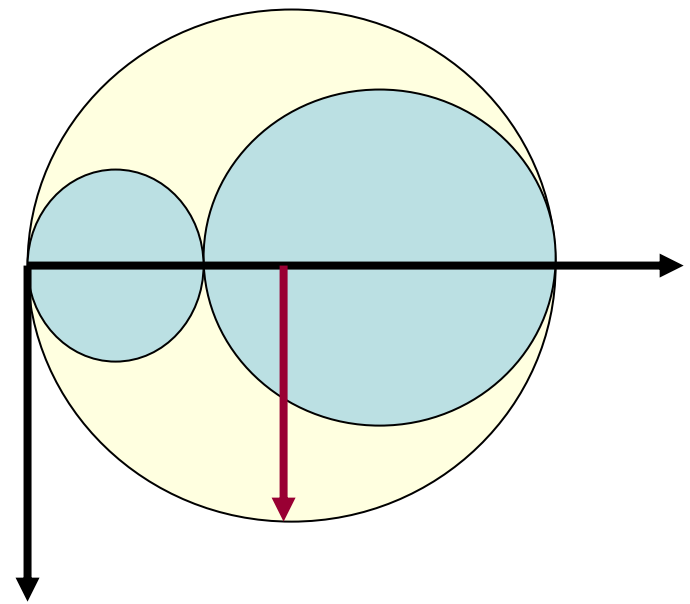$$[\varepsilon] = \begin{bmatrix} 3500 & -\dfrac{1050}{2} & 0 \\ -\dfrac{1050}{2} & 700 & 0 \\ 0 & 0 & 0 \end{bmatrix} \mu$$

$$Eigen\_Values \Rightarrow \begin{bmatrix} 3595.2 & 0 & 0 \\ 0 & 604.8 & 0 \\ 0 & 0 & 0 \end{bmatrix} \mu$$

$$Eigen\_Vectors \Rightarrow \begin{bmatrix} 0.984 & -0.178 & 0 \\ 0.178 & 0.984 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

ArcCos(angle)=0.984

Angle =10.28degress

(c) **In-Plane Maximum Shear Strain**

$$\gamma_{Max} = (\varepsilon_1 - \varepsilon_2) = 3595.2\mu - 604.8\mu = 2990.4\mu$$

(d) **Out-of-Plane Maximum Shear Strain**

$$\gamma_{Max} = (\varepsilon_1 - \varepsilon_3) = 3595.2\mu - 0\mu = 3595.2\mu$$

Example

The state of stress at a point in a structural member is determined to be as shown. Knowing that for this material **E=210GPa** and **v=0.3**, use the Mohr's circle to determine: (1) the principal stresses; (2) the in-plane maximum shear stress; (3) the absolute maximum shear stress; (4) principal angles; (5) the strains and the principal strains; (6) the maximum shear strain; (7) the principal angles.

14MPa

11.2MPa

56MPa
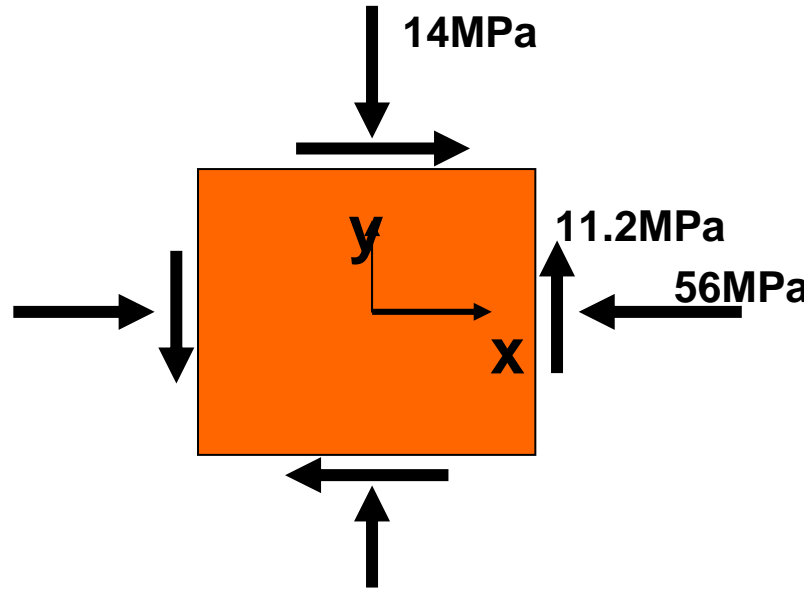
1. Principal Stresses

$$\sigma_{1,2} = \left(\frac{\sigma_x + \sigma_y}{2}\right) \pm \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \left(\tau_{xy}\right)^2} = \sigma_{Average} \pm R$$

$$R = \sqrt{\left(\frac{-56 - (-14)}{2}\right)^2 + (11.2)^2} = 23.8 MPa$$

$$\sigma_{Average} = \left(\frac{-56 + (-14)}{2}\right) = -35 MPa$$

$$\sigma_1 = -35 + 23.8 = -11.2 MPa$$
$$\sigma_2 = -35 - 23.8 = -58.8 MPa$$

$$\sigma_1 = 0 MPa$$
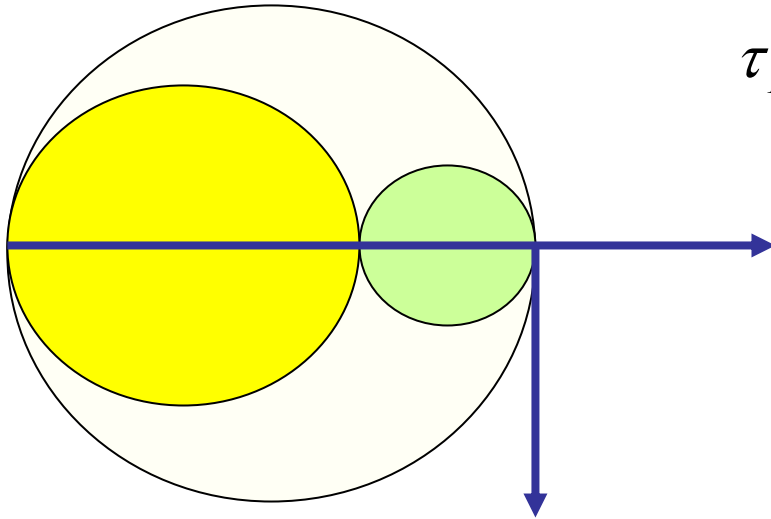$$\sigma_2 = -11.2 MPa$$
$$\sigma_3 = -58.8 MPa$$

$$[\sigma] = Stress\_Tensor = \begin{bmatrix} -56 & 11.2 & 0 \\ 11.2 & -14 & 0 \\ 0 & 0 & 0 \end{bmatrix} MPa$$

$$Eigen\_Values \Rightarrow \begin{bmatrix} 0 & 0 & 0 \\ 0 & -11.2 & 0 \\ 0 & 0 & -58.8 \end{bmatrix} MPa$$

2. In-Plane Maximum Shear Stress

$$\tau_{Max} = R = 23.8MPa$$

3. The Absolute Maximum Shear Stress (Out of plane)

$$\tau_{Max} = \frac{\sigma_1 - \sigma_3}{2} = \frac{0 - (-58.8)}{2} = 29.4MPa$$

4. Angle between the x-axis and the Principal Stresses

$$(In\_Plane) = \tan(2\theta_P) = \frac{\tau_{xy}}{(\sigma_x - \sigma_y)/2} = \frac{11.2}{(-56 - (-14))/2}$$

$$\tan(2\theta_P) = \frac{11.2}{-21} = -0.533$$

$$2\theta_P = -28.07\,\text{deg}$$

$$Eigen\_Vectors \Rightarrow \begin{bmatrix} 0 & 0 & 0 \\ 0.2425 & 0.9701 & 0 \\ 0.9701 & -0.2425 & 1 \end{bmatrix}$$

For σ_1=0
ArcCos(angle)=0.0
Angle =90degress

For σ_2=-11.2MPa
ArcCos(angle)=0.2425
Angle =76degress

For σ_3=-58.8MPa
ArcCos(angle)=0.9701
Angle =14degress

5. Strains and Principal Strains

$$\varepsilon_x = \frac{\sigma_x}{E} - v\frac{\sigma_y}{E} - v\frac{\sigma_z}{E}$$

$$\varepsilon_y = -v\frac{\sigma_x}{E} + \frac{\sigma_y}{E} - v\frac{\sigma_z}{E}$$

$$\varepsilon_z = -v\frac{\sigma_x}{E} - v\frac{\sigma_y}{E} + \frac{\sigma_z}{E}$$

$$\varepsilon_x = \frac{-56}{210000} - (0.3)\frac{-14}{210000} = (-266.6 + 20)\mu = -246.6\mu$$

$$\varepsilon_y = -(0.3)\frac{-56}{210000} + \frac{-14}{210000} = (80 - 66.6)\mu = 13.4\mu$$

$$\varepsilon_z = -(0.3)\frac{-56}{210000} - (0.3)\frac{-14}{210000} = (80 + 20)\mu = 100\mu$$

$$\gamma_{XY} = \frac{\tau_{XY}}{G}$$

$$\gamma_{xy} = \frac{\tau_{xy}}{G} = \frac{11.2}{80770} = 139\mu$$

$$G = \frac{E}{2(1+v)} = \frac{210}{2(1+0.3)} = 80.77 GPa$$

$$[\varepsilon] = \begin{bmatrix} \varepsilon_{xx} & \frac{1}{2}\gamma_{yx} & \frac{1}{2}\gamma_{zx} \\ \frac{1}{2}\gamma_{xy} & \varepsilon_{yy} & \frac{1}{2}\gamma_{zy} \\ \frac{1}{2}\gamma_{xz} & \frac{1}{2}\gamma_{yz} & \varepsilon_{zz} \end{bmatrix} = \begin{bmatrix} -246.6 & 139/2 & 0 \\ 139/2 & 13.4 & 0 \\ 0 & 0 & 100 \end{bmatrix}\mu$$

$$Eigen\_Values \Rightarrow \begin{bmatrix} 100 & 0 & 0 \\ 0 & 30.8 & 0 \\ 0 & 0 & -264.0 \end{bmatrix}\mu$$

$$\varepsilon_{1,2} = \frac{(\varepsilon_x + \varepsilon_y)}{2} \pm \sqrt{\left(\frac{\varepsilon_x - \varepsilon_y}{2}\right)^2 + \left(\frac{\gamma_{xy}}{2}\right)^2}$$

$$\varepsilon_{1,2} = \varepsilon_{Average} \pm R$$

$$\varepsilon_{Average} = \frac{(-246.6 + 13.4)}{2} = -116.6\mu$$

$$R = \sqrt{\left(\frac{-246.6 - 13.4}{2}\right)^2 + \left(\frac{139}{2}\right)^2} = 147.4\mu$$

$$\varepsilon_1 = 30.8\mu$$
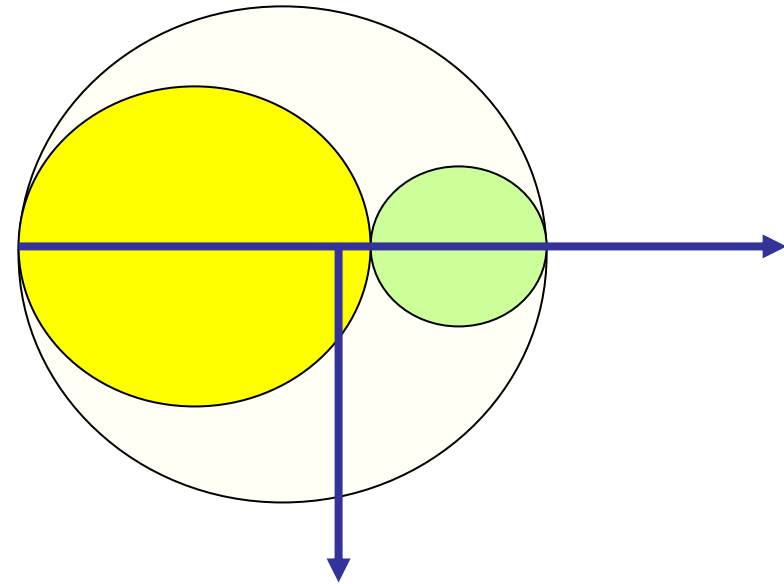
$$\varepsilon_2 = -264\mu$$

6. Maximum Shear Strain and Absolute Maximum Shear Strain

$$\frac{\gamma_{Max}}{2}(in\_plane) = R = \sqrt{\left(\frac{-246.6 - 13.4}{2}\right)^2 + \left(\frac{139}{2}\right)^2} = 147.4\mu$$

$$\gamma_{Max} = 294.8\mu$$

$$\frac{\gamma_{Max}}{2}(out\_of\_plane) = \frac{\varepsilon_1 - \varepsilon_3}{2} = \frac{100 - (-264)}{2} = 182\mu$$

$$\gamma_{Max} = 364\mu$$

7. In-plane and Out-of-plane angles

$$\tan 2\theta_P = = \frac{\gamma_{xy}/2}{\left(\varepsilon_x - \varepsilon_y\right)/2} = \frac{\gamma_{xy}}{\varepsilon_x - \varepsilon_y} = \frac{139}{(-246.6 - 13.4)} = -0.534$$

$$\tan(2\theta_P) = -0.533$$

$$2\theta_P = -28.07\,deg$$

$$Eigen\_Vectors \Rightarrow \begin{bmatrix} 0 & 0 & 0 \\ 0.2425 & 0.9701 & 0 \\ 0.9701 & -0.2425 & 1 \end{bmatrix}$$

For $\varepsilon_1 = 100\mu$
ArcCos(angle)=0.0
Angle =90degress

For $\varepsilon_2 = 30.8\mu$
ArcCos(angle)=0.2425
Angle =76degress

For $\varepsilon_3 = -264\mu$
ArcCos(angle)=0.9701
Angle =14degress