Chapter 7 Analysis of Stresses and Strains

7.1 Introduction

axial load $\sigma = P / A$

torsional load in circular shaft $\tau = T \rho / I_p$

bending moment and shear force in beam

$$\sigma = M y / I \qquad \tau = V Q / I b$$

in this chapter, we want to find the normal and shear stresses acting on any inclined section

for uniaxial load and pure shear, this relation are shown in chapters 2 and 3, now we want to derive the transformation relationships that give the stress components for any orientation

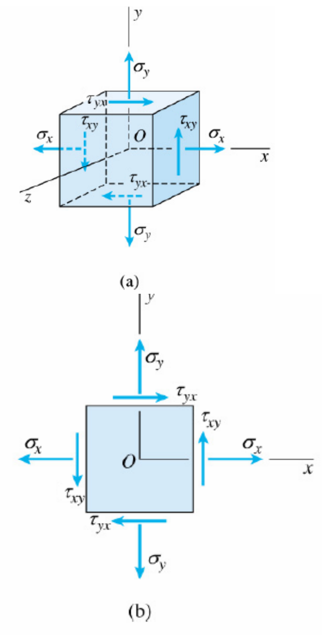
this is referred as **stress transformation**

when an element is rotated from one orientation to another, the stresses acting on the faces of the element are different but they still represent the same state of stress, namely, the stress at the point under consideration

7.2 Plane Stress

consider the infinitesimal element with its edges parallel to x, y, and z axes

if only the x and y faces of the element are subjected to stresses, it is called plane stress, it can be shown as a two dimension stress element

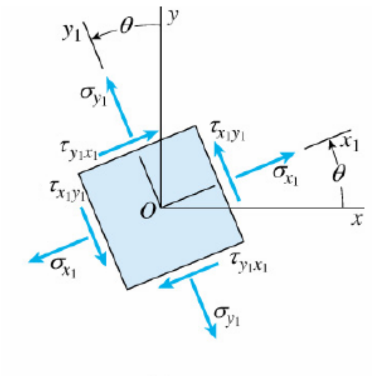


equal normal stresses act on opposite faces, shear stress τ has two subscripts, the first denotes the face on which the stress acts, and the second gives the direction of that face

τ_{xy} : acts on x face directed to y axis
τ_{yx} : acts on y face directed to x axis

sign convention : acts on ± face and directed to ± axis as +

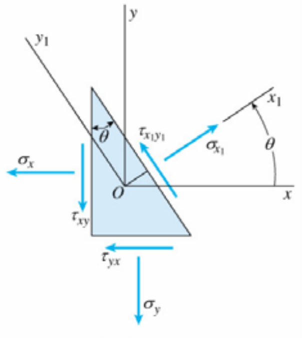
as discussed in chapter 1, $\tau_{xy} = \tau_{yx}$



(c)

consider an element located at the same point and whose faces are perpendicular to x_1, y_1 and z_1 axes, in which z_1 axis coincides with the z axis, and x_1 and y_1 axes are rotated counterclockwise through an angle θ w.r.t. x and y axes, the normal and shear stresses acting on this new element are denoted σ_{x1}, σ_{y1}, τ_{x1y1} and τ_{y1x1}

also $\tau_{x1y1} = \tau_{y1x1}$



(a) Stresses

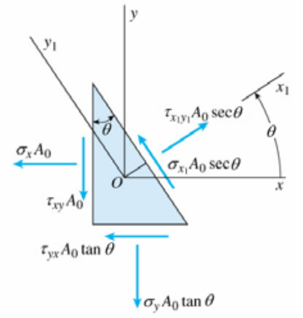
the stresses acting on the rotated x_1y_1 element can be expressed in terms of stress on the xy element by using equation of static equilibrium

choose a wedge-shaped element

force equilibrium in x_1-direction

$$\sigma_{x_1} A_0 \sec\theta - \sigma_x A_0 \cos\theta - \tau_{xy} A_0 \sin\theta$$

$$- \sigma_y A_0 \tan\theta \sin\theta - \tau_{yx} A_0 \tan\theta \operatorname{con}\theta = 0$$



(b) Forces

force equilibrium in y_1-direction

$$\tau_{x1y1} \; A_0 \sec\theta \; + \; \sigma_x A_0 \sin\theta \; - \; \tau_{xy} A_0 \cos\theta$$

$$- \quad \sigma_y A_0 \tan\theta \cos\theta \quad + \quad \tau_{yx} A_0 \tan\theta \sin\theta \quad = \quad 0$$

with $\tau_{xy} = \tau_{yx}$

it is obtained

$$\sigma_{x1} = \sigma_x \cos^2\theta + \sigma_y \sin^2\theta + 2\,\tau_{xy} \sin\theta \cos\theta$$

$$\tau_{x1y1} = -(\sigma_x - \sigma_y) \sin\theta \cos\theta + \tau_{xy} (\cos^2\theta - \sin^2\theta)$$

for $\theta = 0^o \qquad \sigma_{x1} = \sigma_x \qquad \tau_{x1y1} = \tau_{xy}$

for $\theta = 90^o \qquad \sigma_{x1} = \sigma_y \qquad \tau_{x1y1} = -\tau_{xy}$

from trigonometric identities

$$\cos^2\theta = \frac{1}{2}(1 + \cos 2\theta) \qquad \sin^2\theta = \frac{1}{2}(1 - \cos 2\theta)$$

$$\sin\theta \cos\theta = \frac{1}{2}\sin 2\theta$$

the above equations can be expressed in a more convenient form

$$\sigma_{x1} = \frac{\sigma_x + \sigma_y}{2} + \frac{\sigma_x - \sigma_y}{2}\cos 2\theta + \tau_{xy} \sin 2\theta$$

$$\tau_{x1y1} = -\frac{\sigma_x - \sigma_y}{2}\sin 2\theta + \tau_{xy} \cos 2\theta$$

this is the **transformation equations for plane stress**

substituting $\theta + 90^o$ for θ in σ_{x1} equation

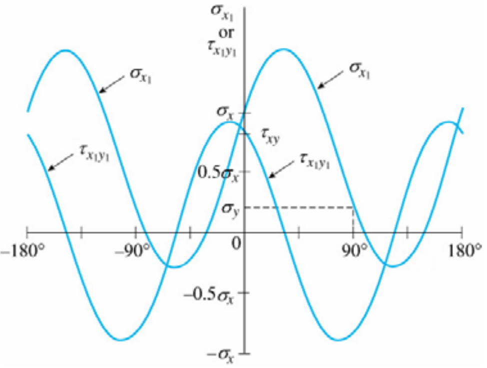
$$\sigma_{y1} = \frac{\sigma_x + \sigma_y}{2} - \frac{\sigma_x - \sigma_y}{2}\cos 2\theta - \tau_{xy} \sin 2\theta$$

also we obtain the following equation for plane stress

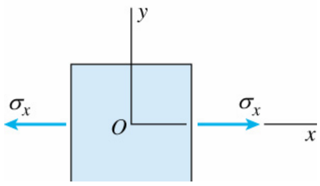
$$\sigma_{x1} \quad + \quad \sigma_{y1} \quad = \quad \sigma_x \quad + \quad \sigma_y$$

i.e. the sum of the normal stresses actin on perpendicular faces for aplane stress element is constant, independent of θ

σ_{x1} and τ_{x1y1} versus the angle of rotation θ can be plotted as



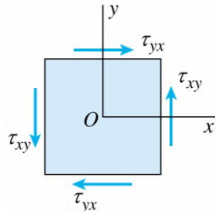
for uniaxial stress case, $\sigma_y \quad = \quad 0, \quad \tau_{xy} \quad = \quad 0$



$$\sigma_{x1} \quad = \quad \sigma_x \cos^2 \theta \quad = \quad \sigma_x (1 + \cos 2\theta) / 2$$

$$\tau_{x1y1} \quad = \quad - \sigma_x \sin \theta \cos \theta \quad = \quad - \sigma_x \sin 2\theta / 2$$

for pure shear stress case, $\sigma_x \quad = \quad \sigma_y \quad = \quad 0$

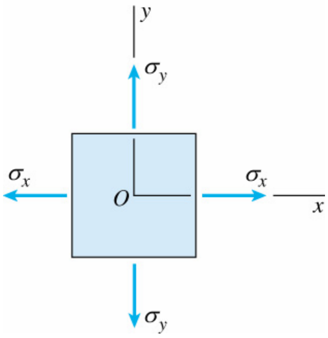


$$\sigma_{x1} \quad = \quad 2 \tau_{xy} \sin \theta \cos \theta \quad = \quad \tau_{xy} \sin 2\theta$$

$$\tau_{x1y1} \quad = \quad \tau_{xy} (\cos^2 \theta - \sin^2 \theta) \quad = \quad \tau_{xy} \cos 2\theta$$

same as derived in previous chapters

for biaxial stress case, $\tau_{xy} \quad = \quad 0$



$$\sigma_{x1} \quad = \quad \frac{\sigma_x + \sigma_y}{2} \quad + \quad \frac{\sigma_x - \sigma_y}{2} \cos 2\theta$$

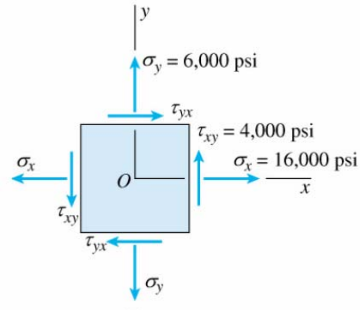
$$\tau_{x1y1} \quad = \quad - \frac{\sigma_x - \sigma_y}{2} \sin 2\theta$$

Example 7-1

$\sigma_x \quad = \quad 110 \text{ MPa} \qquad \sigma_y \quad = \quad 40 \text{ MPa}$

$\tau_{xy} \quad = \quad \tau_{yx} \quad = \quad 28 \text{ MPa}$

determine the stresses for $\theta \quad = \quad 45^{\circ}$



(a)

$$\frac{\sigma_x + \sigma_y}{2} = 75 \text{ MPa} \qquad \frac{\sigma_x - \sigma_y}{2} = 35 \text{ MPa}$$

$$\sin 2\theta = \sin 90^{o} = 1 \qquad \cos 2\theta = \cos 90^{o} = 0$$

$$\sigma_{x1} = \frac{\sigma_x + \sigma_y}{2} + \frac{\sigma_x - \sigma_y}{2} \cos 2\theta + \tau_{xy} \sin 2\theta$$

$$= 75 + 35 \text{ x } 0 + 28 \text{ x } 1 = 103 \text{ MPa}$$

$$\tau_{x1y1} = - \frac{\sigma_x - \sigma_y}{2} \sin 2\theta + \tau_{xy} \cos 2\theta$$

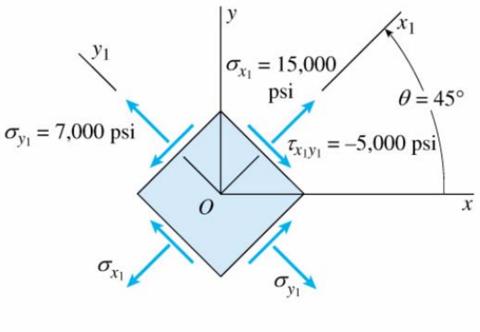
$$= - 35 \text{ x } 1 + 28 \text{ x } 0$$

$$= - 35 \text{ MPa}$$

$$\sigma_{x1} + \sigma_{y1} = \sigma_x + \sigma_y$$

$$\sigma_{y1} = \sigma_x + \sigma_y - \sigma_{x1}$$

$$= 110 + 40 - 103 = 47 \text{ MPa}$$



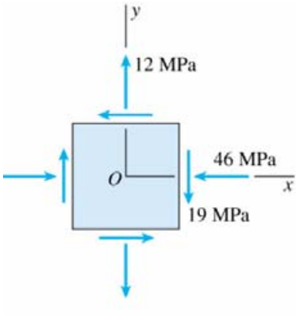
(b)

Example 7-2

$$\sigma_x = - 46 \text{ MPa} \qquad \sigma_y = 12 \text{ MPa}$$

$$\tau_{xy} = \tau_{yx} = - 19 \text{ MPa}$$

determine the stresses for $\theta = - 15^{o}$



(a)

$$\frac{\sigma_x + \sigma_y}{2} = - 17 \text{ MPa} \qquad \frac{\sigma_x - \sigma_y}{2} = - 29 \text{ MPa}$$

$$\sin 2\theta = \sin (- 30^{o}) = - 0.5 \qquad \cos 2\theta = \cos (- 30^{o}) = 0.866$$

$$\sigma_{x1} = \frac{\sigma_x + \sigma_y}{2} + \frac{\sigma_x - \sigma_y}{2} \cos 2\theta + \tau_{xy} \sin 2\theta$$

$$= - 17 + (- 29)\, 0.866 + (- 19)\, (-0.5) = - 32.6 \text{ MPa}$$

$$\tau_{x1y1} = -\frac{\sigma_x - \sigma_y}{2}\sin 2\theta + \tau_{xy}\cos 2\theta$$

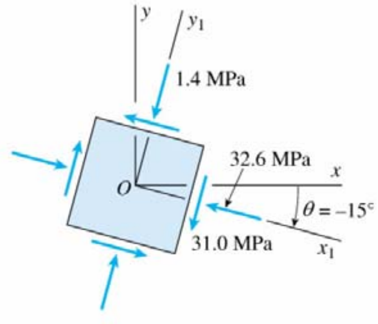
$$= -(-29)(-0.5) + (-19)\,0.866$$

$$= -31 \text{ MPa}$$

$$\sigma_{x1} + \sigma_{y1} = \sigma_x + \sigma_y$$

$$\sigma_{y1} = \sigma_{x1} + \sigma_y - \sigma_x$$

$$= -46 + 12 - (-32.6) = -1.4 \text{ MPa}$$



(b)

7.3 Principal Stresses and Maximum Shear Stresses

σ_{x1} and τ_{x1y1} vary continuously as the element is rotated through the angle θ

for design purpose, the largest positive and negative stresses are usually needed, the maximum and minimum normal stresses are called the principal stresses

consider the stress transformation equation

$$\sigma_{x1} = \frac{\sigma_x + \sigma_y}{2} + \frac{\sigma_x - \sigma_y}{2}\cos 2\theta + \tau_{xy}\sin 2\theta$$

to find the maximum normal stress, we may set $d\,\sigma_{x1} / d\,\theta = 0$

$$\frac{d\,\sigma_{x1}}{d\,\theta} = -(\sigma_x - \sigma_y)\sin 2\theta + 2\,\tau_{xy}\cos 2\theta = 0$$

we get $\tan 2\theta_p = \dfrac{2\,\tau_{xy}}{\sigma_x - \sigma_y}$

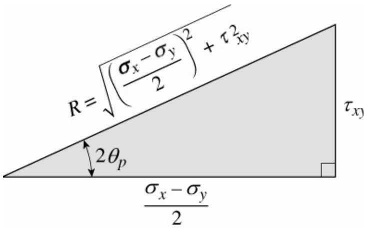
θ_p defines the orientation of the principal plane, two values of $2\theta_p$ from 0 ~ 360° and differ by 180°

$\therefore$ θ_p has two values differ by 90^{o}, we conclude that the principal stresses occur on mutually perpendicular plane

also

$$\cos 2\theta_p = \frac{(\sigma_x - \sigma_y) / 2}{R}$$

$$\sin 2\theta_p = \frac{\tau_{xy}}{R}$$



where $$R = \left[\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2\right]^{1/2}$$

substitute $\cos 2\theta_p$ and $\sin 2\theta_p$ into the expression of σ_{x1}

$$\sigma_1 = (\sigma_{x1})_{max} = \frac{\sigma_x + \sigma_y}{2} + \left[\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2\right]^{1/2}$$

and the smaller principal stress denoted by σ_2 is obtained

$$\sigma_1 + \sigma_2 = \sigma_x + \sigma_y$$

$$\sigma_2 = (\sigma_{x1})_{min} = \frac{\sigma_x + \sigma_y}{2} - \left[\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2\right]^{1/2}$$

the principal stresses can be written as

$$\sigma_{1,2} = \frac{\sigma_x + \sigma_y}{2} \pm \left[\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2\right]^{1/2}$$

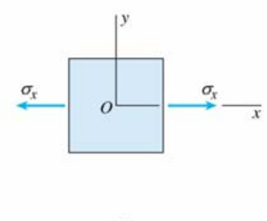
\+ sign gives the larger principal stress

\- sign gives the smaller principal stress

θ_{p1} and θ_{p2} can be determined, but we cannot tell from the equation which angle is θ_{p1} and which is θ_{p2}

an important characteristic concerning the principal plane : the shear is zero on the principal plane

$$\because \qquad \tau_{x1y1} = -\frac{\sigma_x - \sigma_y}{2} \sin 2\theta + \tau_{xy} \cos 2\theta$$



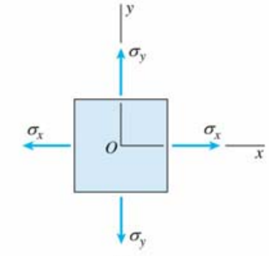
(a)

substitute $2\theta_p$ into this equation, we can get

$$\tau_{x1y1} = 0$$

for uniaxial and biaxial stress states, $\tau_{xy} = 0$

$$\tan 2\theta_p = 0 \qquad \theta_p = 0^{\text{o}} \text{ and } 90^{\text{o}}$$

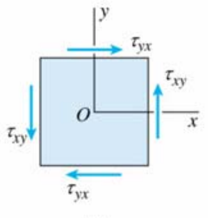


(b)

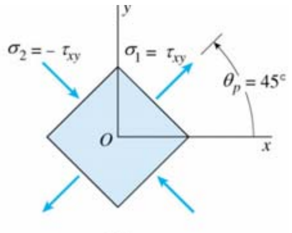
for pure shear stress, $\sigma_x = \sigma_y = 0$

$$\tan 2\theta_p = \infty$$

$$\theta_p = 45^{\text{o}} \text{ and } 135^{\text{o}}$$

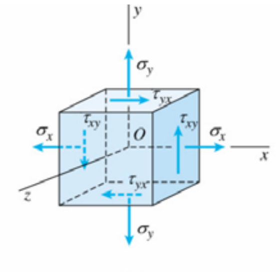


(a)



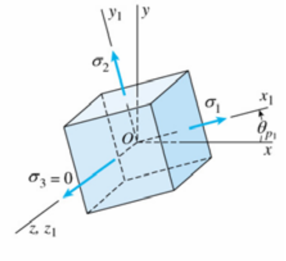
(b)

for the three-dimensional stress element, $\sigma_z = 0$ is also a principal stress, note that there are no shear stresses on the principal plane



Maximum Shear Stress

$$\because \quad \tau_{x1y1} = -\frac{\sigma_x - \sigma_y}{2} \sin 2\theta + \tau_{xy} \cos 2\theta$$



(b)

$$\frac{d\,\tau_{x1y1}}{d\,\theta} = -(\sigma_x - \sigma_y) \cos 2\theta - 2\,\tau_{xy} \sin 2\theta = 0$$

$$\tan 2\theta_s = -\frac{\sigma_x - \sigma_y}{2\,\tau_{xy}}$$

$2\theta_s$ has two values between 0 and 360^o

θ_s has two values between 0 and 180^o, and differ by 90^o

comparing the angle θ_s and θ_p, it is shown that

$$\tan 2\theta_s \;=\; - \frac{1}{\tan 2\theta_p} \;=\; - \cot 2\theta_p$$

i.e. $2\theta_s \perp 2\theta_p \qquad 2\theta_s = 2\theta_p \pm 90^o$

or $\theta_s = \theta_p \pm 45^o$

the plane of maximum shear stress occur at 45^o to the principal plane

similarly we have

$$\cos 2\theta_s = \frac{\tau_{xy}}{R} \qquad \sin 2\theta_s = - \frac{\sigma_x - \sigma_y}{2\,R}$$

and the corresponding maximum shear stress is

$$\tau_{max} = [\,(\frac{\sigma_x - \sigma_y}{2})^2 + \tau_{xy}^{\;2}]^{½}$$

the algebraically minimum shear stress τ_{min} has the same magnitude

a usually expression for the maximum shear stress can be obtained by σ_1 and σ_2

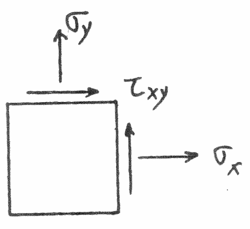
$$\tau_{max} = \frac{\sigma_1 - \sigma_2}{2}$$

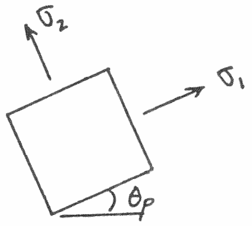
the τ_{max} is equal to one half the difference of the principal stress

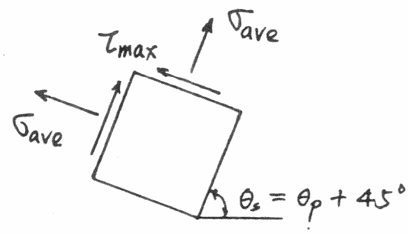
normal stresses also act on the planes of maximum τ, substituting θ_{s1} in the formula of σ_{x1}, it is obtained

$$\sigma_{x1} \;=\; \frac{\sigma_x + \sigma_y}{2} \;=\; \sigma_{ave} \;=\; \sigma_{y1}$$

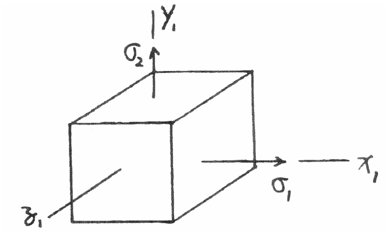
σ_{ave} acts on both the plane of maximum and minimum τ planes







if we make a three-dimensional analysis, we can establish that there are possible positions of element for maximum shear stress



$$(\tau_{max})_{x1} \;=\; \pm\frac{\sigma_1}{2} \qquad \text{rotate the element } 45^o \text{ about } x_1 \text{ axis}$$

$$(\tau_{max})_{y1} \;=\; \pm\frac{\sigma_2}{2} \qquad \text{rotate the element } 45^o \text{ about } y_1 \text{ axis}$$

$$(\tau_{max})_{z1} \;=\; \pm\frac{\sigma_1 - \sigma_2}{2} \qquad \text{rotate the element } 45^o \text{ about } z_1 \text{ axis}$$

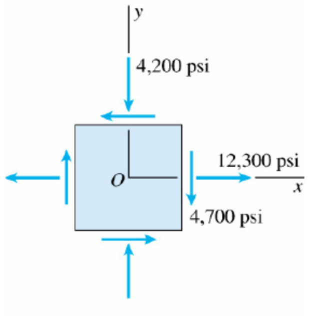
if $\sigma_1 > \sigma_2 > 0$, then $\tau_{max} = \sigma_1 / 2 = (\tau_{max})_{y1}$ for the element

Example 7-3

$\sigma_x = 84$ MPa $\qquad \sigma_y = -30$ MPa

$\tau_{xy} = -32$ MPa

determine the principal stresses and maximum shear stress and their directions



(a)

the principal angles θ_p can be obtained

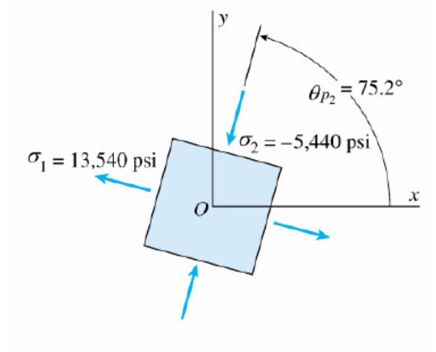
$$\tan 2\theta_p = \frac{2\,\tau_{xy}}{\sigma_x - \sigma_y} = \frac{2\,(-32)}{84-(-30)} = -0.5614$$

$$2\theta_p = 150.6^o \quad \text{or} \quad 330.6^o$$

$$\theta_p = 75.3^o \quad \text{or} \quad 165.3^o$$

$$\frac{\sigma_x + \sigma_y}{2} = (84 - 30) / 2 = 27 \text{ MPa}$$

$$\frac{\sigma_x - \sigma_y}{2} = (84 + 30) / 2 = 57 \text{ MPa}$$



(b)

$$\sigma_{x1} = \frac{\sigma_x + \sigma_y}{2} + \frac{\sigma_x - \sigma_y}{2}\cos 2\theta + \tau_{xy}\sin 2\theta$$

for $2\theta_p = 330.6^o$ $(\theta_p = 165.3^o)$ $\sigma_1 = 92.4$ MPa

for $2\theta_p = 150.3^o$ $(\theta_p = 75.3^o)$ $\sigma_2 = -38.4$ MPa

check $\sigma_1 + \sigma_2 = \sigma_x + \sigma_y$ (O. K.)

alternative method for principal stresses

$$\sigma_{1,2} = \frac{\sigma_x + \sigma_y}{2} \pm \left[\left(\frac{\sigma_x - \sigma_y}{2} \right)^2 + \tau_{xy}^{\;2} \right]^{½}$$

$$= 27 \pm [\,(57)^2 + (-32)^2]^{½} = 27 \pm 65.4$$

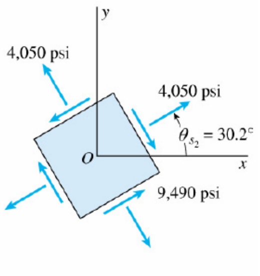
thus $\sigma_1 = 92.4$ MPa $\sigma_2 = -38.4$ MPa

$\theta_{p1} = 165.3^o$ $\theta_{p2} = 75.3^o$

the maximum shear stresses are given by

$$\tau_{max} = \frac{\sigma_1 - \sigma_2}{2} = 65.4 \text{ MPa}$$

$$\theta_{s1} = \theta_{p1} - 45^o = 120.3^o$$



(c)

and $\theta_{s2} = 120.2^\circ - 90^\circ = 30.3^\circ$

and the normal stress acting on the planes of maximum shear stress are

$$\sigma_{ave} = \frac{\sigma_x + \sigma_y}{2} = 27 \text{ MPa}$$

7.4 Mohr's Circle for Plane Stress

the transformation of plane stress can be represented in graphical form, known as Mohr's circle

the equation of Mohr's circle can be derived from the transformation equations for plane stress

$$\sigma_{x1} - \frac{\sigma_x + \sigma_y}{2} = \frac{\sigma_x - \sigma_y}{2} \cos 2\theta + \tau_{xy} \sin 2\theta$$

$$\tau_{x1y1} = -\frac{\sigma_x - \sigma_y}{2} \sin 2\theta + \tau_{xy} \cos 2\theta$$

to eliminate the parameter 2θ, we square both sides of each equation and then add together, it can be obtained

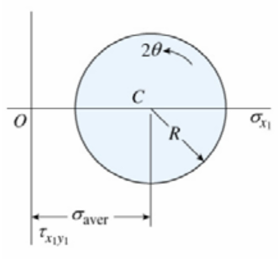
$$(\sigma_{x1} - \frac{\sigma_x + \sigma_y}{2})^2 + \tau_{x1y1}^2 = (\frac{\sigma_x - \sigma_y}{2})^2 + \tau_{xy}^2$$

let $\sigma_{ave} = \dfrac{\sigma_x + \sigma_y}{2}$ $\quad R^2 = (\dfrac{\sigma_x - \sigma_y}{2})^2 + \tau_{xy}^2$

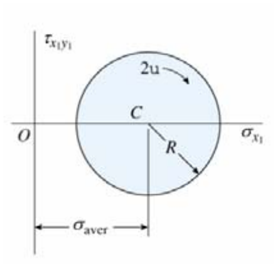
then the above equation can be written

$$\boldsymbol{(\sigma_{x1} - \sigma_{ave})^2 + \tau_{x1y1}^2 = R^2}$$

this is a equation of circle with σ_{x1} and τ_{x1y1} as coordinates, the radius is R and center at $\sigma_{x1} = \sigma_{ave}$, $\tau_{x1y1} = 0$



(a)



(b)

positive shear stress is plotted downward and a positive angle 2θ is plotted counterclockwise

positive shear stress is plotted upward and a positive angle 2θ is plotted clockwise

Construction of Mohr's Circle

(1) locate the center C at $\sigma_{x1} = \sigma_{ave}$, $\tau_{x1y1} = 0$

(2) locate point A which is at $\theta = 0$, $\sigma_{x1} = \sigma_x$, $\tau_{x1y1} = \tau_{xy}$

(3) locate point B which is at $\theta = 90^{\circ}$, $\sigma_{x1} = \sigma_{y1}$, $\tau_{x1y1} = - \tau_{xy}$

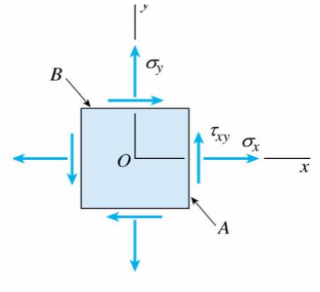
[Note that the line AB must passes through point C]

(4) draw the circle through points A and B with center at C

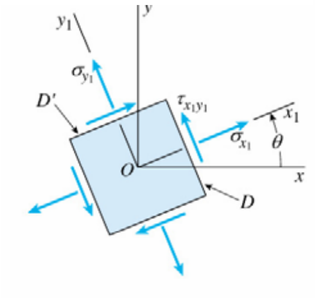
this circle is the **Mohr's circle** with radius R

$$R = \left[\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2 \right]^{1/2}$$

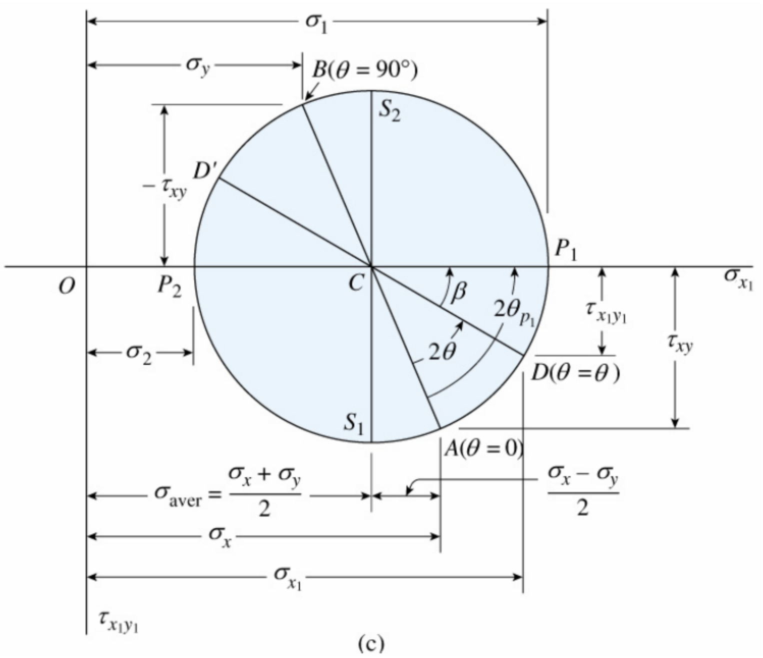
the stress state on an inclined element with an angle θ is represented at point D on the Mohr's circle, which is measured an angle 2θ counter- clockwise from point A



(a)



(b)



(c)

to show the coordinate at D

$$\sigma_{x1} = \sigma_{ave} + R\cos(2\theta_p - 2\theta)$$

$$= \sigma_{ave} + R(\cos 2\theta_p \cos 2\theta + \sin 2\theta_p \sin 2\theta)$$

$$\because \quad R\cos 2\theta_p = R\frac{\sigma_x - \sigma_y}{2R} = \frac{\sigma_x - \sigma_y}{2}$$

$$R\sin 2\theta_p = R\frac{\tau_{xy}}{R} = \tau_{xy}$$

$$\therefore \quad \sigma_{x1} = \sigma_{ave} + \frac{\sigma_x - \sigma_y}{2}\cos 2\theta + \tau_{xy}\sin 2\theta$$

$$\tau_{xy} = R\sin(2\theta_p - 2\theta) = R(\sin 2\theta_p \cos 2\theta - \cos 2\theta_p \sin 2\theta)$$

$$= -\frac{\sigma_x - \sigma_y}{2}\sin 2\theta + \tau_{xy}\cos 2\theta$$

same results as the transformation equations

point D' represents the stress state on the face of the face 90° from the face represented by point D, i.e. y_1 face

$A\ (\sigma_x,\ \tau_{xy})$ $\qquad B\ (\sigma_y,\ -\tau_{xy})$ $\qquad C\ (\sigma_{ave},\ 0)$

$D\ (\sigma_{x1},\ \tau_{x1y1})$ $\qquad D'\ (\sigma_{y1},\ -\tau_{x1y1})$

at point P_1 on the circle, $\sigma_{x1} = \sigma_{max} = \sigma_1$

hence, P_1 represents the stress state at principal plane

the other principal plane $(\sigma_{min} = \sigma_2)$ is represented by P_2

$$\sigma_1 = OC + CP_1 = \frac{\sigma_x + \sigma_y}{2} + R$$

$$\sigma_2 = OC - CP_2 = \frac{\sigma_x + \sigma_y}{2} - R$$

the principal angle θ_{p1} can be obtained by

$$\cos 2\theta_{p1} = \frac{\sigma_x - \sigma_y}{2\,R} \qquad \text{or} \qquad \sin 2\theta_{p1} = \frac{\tau_{xy}}{R}$$

and $\qquad \theta_{p2} = \theta_{p1} + 90^\mathrm{o}$

comparing the Mohr's circle and the stress element, it is observed

Mohr's Circle	stress element
$A \rightarrow P_1\ (2\theta_{p1}\ \circlearrowleft)$	$x \rightarrow x_1\ (\theta_{p1}\ \circlearrowleft)$
$A \rightarrow P_2\ (2\theta_{p1} + 180^\mathrm{o}\ \circlearrowleft)$	$x \rightarrow x_1\ (\theta_{p1} + 90^\mathrm{o}\ \circlearrowleft)$
or $(180^\mathrm{o} - 2\theta_{p1}\ \circlearrowright)$	or $(90^\mathrm{o} - \theta_{p1}\ \circlearrowright)$
$P_1 \rightarrow S\ (90^\mathrm{o}\ \circlearrowright)$	$x_1 \rightarrow \tau_{max}\ (45^\mathrm{o}\ \circlearrowright)$

points S and S' representing the points of maximum and minimum shear stresses, are located on the circle at 90^o from points P_1 and P_2

i.e. the planes of maximum and minimum shear stress are at 45^o to the principal planes, and

$$\tau_{max} = R = \left[\left(\frac{\sigma_x - \sigma_y}{2} \right)^2 + \tau_{xy}^2 \right]^{1/2}$$

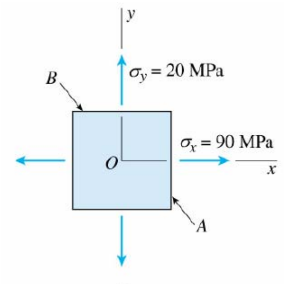
if either σ_x of σ_y is negative, part of the circle will be located to the left of the origin

Mohr's circle makes it possible to visualize the relationships between stresses acting planes at various angles, and it also serves as a simple memory device for obtaining the stress transformation equation

Example 7-4

$\sigma_x = 90$ MPa $\qquad \sigma_y = 20$ MPa

$\tau_{xy} = 0 \qquad \theta = 30^o$

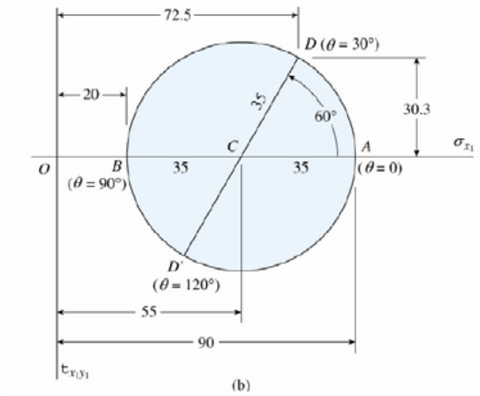


(a)

$$\sigma_{ave} = \frac{\sigma_x + \sigma_y}{2} = \frac{90 + 20}{2} = 55 \text{ MPa}$$

A ($\theta = 0$) $\qquad \sigma_{x1} = 90 \qquad \tau_{x1y1} = 0$

B ($\theta = 90^o$) $\qquad \sigma_{x1} = 20 \qquad \tau_{x1y1} = 0$



(b)

$$R = \left[\left(\frac{\sigma_x - \sigma_y}{2} \right)^2 + \tau_{xy}^2 \right]^{1/2} = 35 \text{ MPa}$$

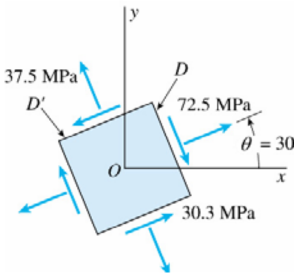
$\theta = 30^o \qquad 2\theta = 60^o \qquad$ (point D)

$$\sigma_{x1} = \sigma_{ave} + R \cos 60^o$$

$$= 55 + 35 \cos 60^o = 72.5 \text{ MPa}$$

$$\tau_{x1y1} = - R \sin 60^o$$

$$= - 35 \sin 60^o = - 30.3 \text{ MPa}$$



$\theta = 120^{\text{o}}$ $\qquad 2\theta = 240^{\text{o}}$ $\quad$ (point D')

$$\sigma_{x1} = \sigma_{ave} - R\cos 60^{\text{o}}$$

$$= 55 - 35 \cos 60^{\text{o}} = 37.5 \text{ MPa}$$

$$\tau_{x1y1} = R\sin 60^{\text{o}}$$

$$= 35 \sin 60^{\text{o}} = 30.3 \text{ MPa}$$

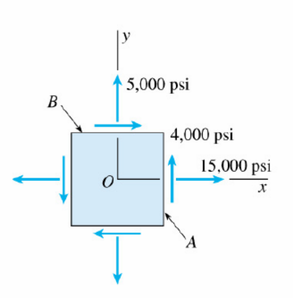
Example 7-5

$\sigma_x = 100$ MPa $\qquad \sigma_y = 34$ MPa

$\tau_{xy} = 28$ MPa

determine the stresses on the face of $\theta = 40^{\text{o}}$

determine σ_1, σ_2 and τ_{max}

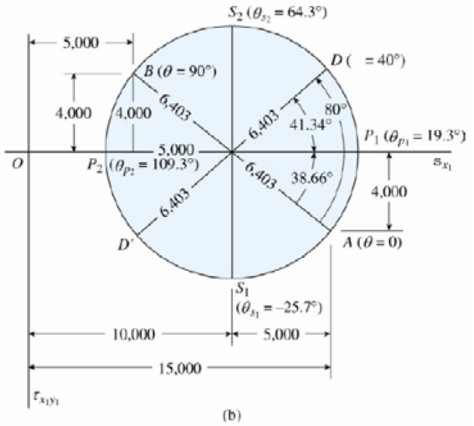


(a)

$$\sigma_{ave} = \frac{\sigma_x + \sigma_y}{2} = \frac{10 + 34}{2} = 67 \text{ MPa}$$

$A\ (\theta = 0)$ $\qquad \sigma_{x1} = 100$ $\qquad \tau_{x1y1} = 28$

$B\ (\theta = 90^{\text{o}})$ $\qquad \sigma_{x1} = 34$ $\qquad \tau_{x1y1} = -28$



(b)

$$R = \left[\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2 \right]^{½} = 43 \text{ MPa}$$

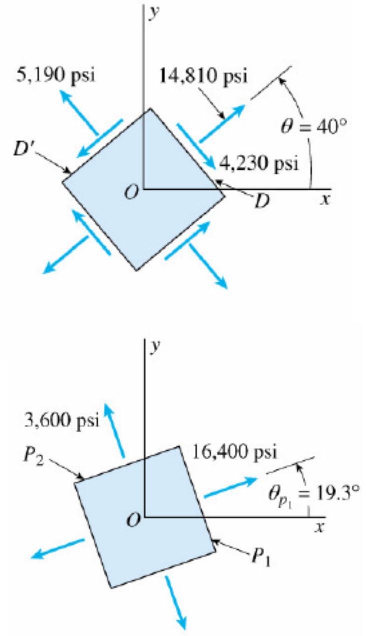
$\theta = 40^{\text{o}}$ $\qquad 2\theta = 80^{\text{o}}$ $\quad$ (point D)

$$\tan \angle ACP_1 = 28 / 33 = 0.848$$

$$\angle ACP_1 = 2\,\theta_{p1} = 40.3^{\text{o}}$$

$$\angle DCP_1 = 80^{\text{o}} - \angle ACP_1 = 39.7^{\text{o}}$$

$$\sigma_{x1} = \sigma_{ave} + R\cos 39.7^{\text{o}} = 100 \text{ MPa}$$



(b)

$\tau_{x1y1} = -R \sin 39.7^{\circ} = -27.5$ MPa

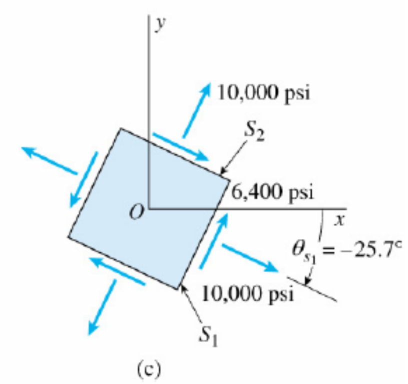
principal stresses are represented by P_1 and P_2

$$\sigma_1 = \sigma_{ave} + R = 110 \text{ MPa}$$

$$2\theta_{p1} = 40.3^{\circ} \qquad \theta_{p1} = 20.15^{\circ}$$

$$\sigma_2 = \sigma_{ave} - R = 24 \text{ MPa}$$

$$\theta_{p2} = \theta_{p1} + 90^{\circ} = 110.15^{\circ}$$



(c)

maximum shear stress

$$\tau_{max} = R = 43 \text{ MPa}$$

$$\theta_s = \theta_{p1} - 45^{\circ} = -24.85^{\circ}$$

$$\sigma_{x1} = \sigma_{ave} = 67 \text{ MPa}$$

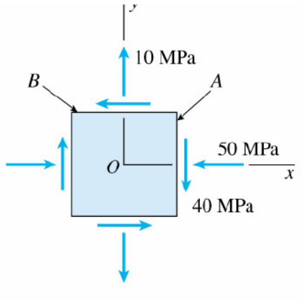
Example 7-6

$\sigma_x = -50$ MPa $\qquad \sigma_y = 10$ MPa

$\tau_{xy} = -40$ MPa $\qquad \theta = 30^{\circ}$

determine σ_{x1}, τ_{x1y1} on $\theta = 45^{\circ}$

determine σ_1, σ_2 and τ_{max}



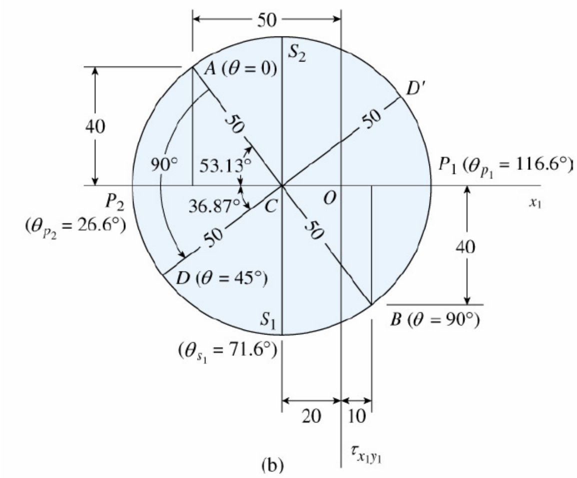
(a)

$$\sigma_{ave} = \frac{\sigma_x + \sigma_y}{2} = \frac{-50 + 10}{2} = -20 \text{ MPa}$$

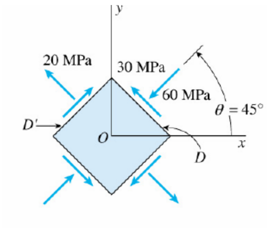
$A\ (\theta = 0) \qquad \sigma_{x1} = -50 \qquad \tau_{x1y1} = -40$

$B\ (\theta = 90^{\circ}) \qquad \sigma_{x1} = 10 \qquad \tau_{x1y1} = 40$

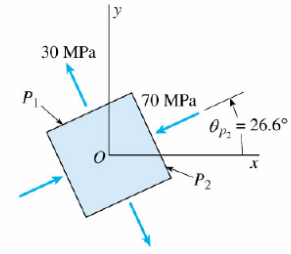
$$R = \left[\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2 \right]^{1/2} = 50 \text{ MPa}$$



(b)



(a)



(b)

$\theta \quad = \quad 45^o \qquad\qquad 2\theta \quad = \quad 90^o$ (point D)

$\tan\ \angle ACP_2 \quad = \quad 40\ /\ 30 \quad = \quad 1.333$

$\angle ACP_2 \quad = \quad 2\ \theta_{p2} \quad = \quad 53.13^o$

$\angle DCP_2 \quad = \quad 90^o \quad - \quad \angle ACP_2 \quad = \quad 36.87^o$

$\sigma_{x1} \quad = \quad -\ 20 \quad - \quad 50 \cos 36.87^o \quad = \quad -\ 60$ MPa

$\tau_{x1y1} \quad = \quad 50 \sin 36.87^o \quad = \quad 30$ MPa

at D' $\qquad \sigma_{x1} \quad = \quad -\ 50 \quad + \quad 10 \quad -\ (-\ 60) \quad = \quad 20$ MPa

$\tau_{x1y1} \quad = \quad -\ 30$ MPa

principal stresses are represented by P_1 and P_2

$\sigma_1 \quad = \quad \sigma_{ave} \quad + \quad R \quad = \quad 30$ MPa

$2\theta_{p1} = 2\theta_{p2} + 180^o = 233.13^o \quad \theta_{p1} \quad = \quad 116.6^o$

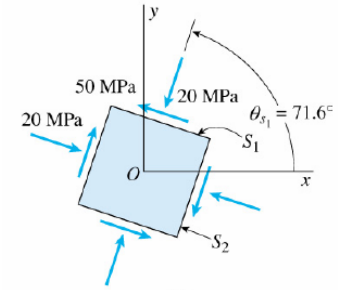
$\sigma_2 \quad = \quad \sigma_{ave} \quad - \quad R \quad = \quad -\ 70$ MPa $\qquad \theta_{p2} \quad = \quad 26.6^o$

maximum shear stress

$\tau_{max} \quad = \quad R \quad = \quad 50$ MPa

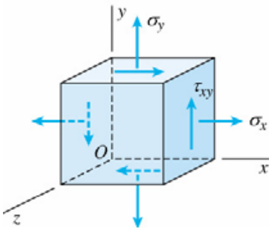
$\theta_s \quad = \quad \theta_{p1} \quad - \quad 45^o \quad = \quad 71.6^o$

$\sigma_{x1} \quad = \quad \sigma_{ave} \quad = \quad -\ 20$ MPa



(c)

7.5 Hook's Law for Plane Stress

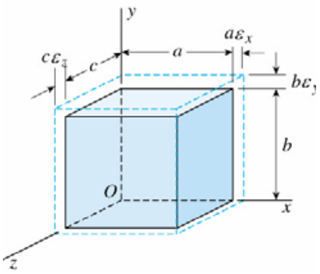


for the plane stress with normal stresses σ_x and σ_y, the normal strains are

$$\varepsilon_x = (\sigma_x - \nu\, \sigma_y) / E$$

$$\varepsilon_y = (\sigma_y - \nu\, \sigma_x) / E$$

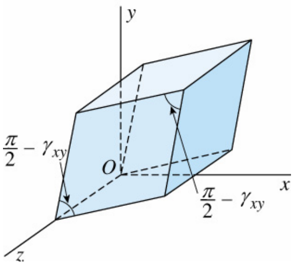
$$\varepsilon_z = -\, \nu\, (\sigma_x + \sigma_y) / E$$



the first two equations can be solved for the stresses in terms of strains

$$\sigma_x = E\, (\varepsilon_x + \nu\, \varepsilon_y) / (1 - \nu^2)$$

$$\sigma_y = E\, (\varepsilon_y + \nu\, \varepsilon_x) / (1 - \nu^2)$$



for the pure shear stress τ_{xy}, the shear strain γ_{xy} is

$$\gamma_{xy} = \tau_{xy} / G$$

or $$\tau_{xy} = G\, \gamma_{xy}$$

the three material parameters with the relation

$$G = E / [2\, (1 + \nu)]$$

volume change

$$V_0 = a\, b\, c$$

$$\begin{aligned} V_1 &= a\, (1 + \varepsilon_x)\, b\, (1 + \varepsilon_y)\, c\, (1 + \varepsilon_z) \\ &= V_0\, (1 + \varepsilon_x)\, (1 + \varepsilon_y)\, (1 + \varepsilon_z) \\ &= V_0\, (1 + \varepsilon_x + \varepsilon_y + \varepsilon_z + \varepsilon_x\varepsilon_y + \varepsilon_y\varepsilon_z + \varepsilon_x\varepsilon_z + \varepsilon_x\varepsilon_y\varepsilon_z) \\ &\simeq V_0\, (1 + \varepsilon_x + \varepsilon_y + \varepsilon_z) \end{aligned}$$

and the volume change is

$$\Delta V = V_1 - V_0 = V_0\,(\varepsilon_x + \varepsilon_y + \varepsilon_z)$$

the unit volume change or dilatation e is defined

$$e = \Delta V / V_0 = \varepsilon_x + \varepsilon_y + \varepsilon_z$$

for uniaxial stress σ_x only

$$e = \Delta V / V_0 = \sigma_x\,(1 - 2\,\nu) / E$$

for plane stress σ_x and σ_y

$$e = \Delta V / V_0 = (\sigma_x + \sigma_y)(1 - 2\,\nu) / E$$

Strain-energy density in plane stress

$$\begin{aligned} u &= \tfrac{1}{2}\,(\sigma_x\,\varepsilon_x + \sigma_y\,\varepsilon_y + \tau_{xy}\,\gamma_{xy}) \\ &= (\sigma_x^2 + \sigma_y^2 - 2\,\nu\,\sigma_x\,\sigma_y) / 2E \quad + \quad \tau_{xy}^2 / 2G \\ &= E\,(\varepsilon_x^2 + \varepsilon_y^2 + 2\,\nu\,\varepsilon_x\,\varepsilon_y) / [2(1 - \nu^2)] \quad + G\,\gamma_{xy}^2 / 2 \end{aligned}$$

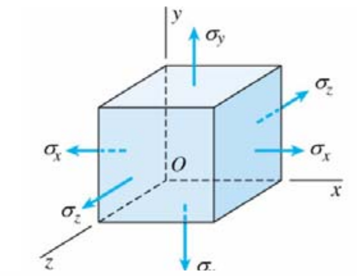
7.6 Triaxial Stress

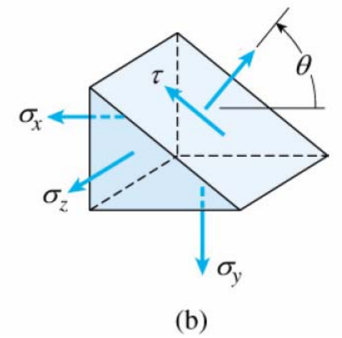
a stress element subjected to normal stress σ_x, σ_y and σ_z is said to be in a triaxial stress state

on the inclined plane parallel to the *z* axis, only σ and τ on this plane, the maximum shear stress occurs on the plane by a 45° rotation about *z* axis is

$$(\tau_{max})_z = \pm\,\frac{\sigma_x - \sigma_y}{2}$$

similarly

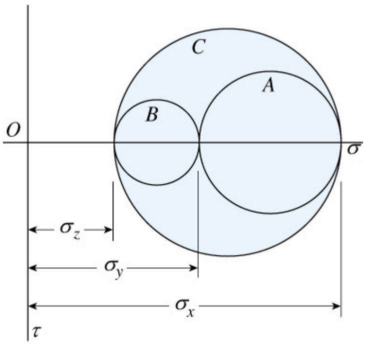




(b)

$$(\tau_{\max})_x = \pm \frac{\sigma_y - \sigma_z}{2}$$

$$(\tau_{\max})_y = \pm \frac{\sigma_x - \sigma_z}{2}$$



the absolute maximum shear stress is the difference between algebraically largest and smallest of the three principle stresses

the Mohr's circles for a 3-D element is shown

Hooke's law for triaxial stress, the normal strains are

$$\varepsilon_x = \sigma_x / E - \nu (\sigma_y + \sigma_z) / E$$

$$\varepsilon_y = \sigma_y / E - \nu (\sigma_x + \sigma_z) / E$$

$$\varepsilon_z = \sigma_z / E - \nu (\sigma_x + \sigma_y) / E$$

stresses in terms of strains are

$$\sigma_x = [E (1 - \nu) \varepsilon_x + \nu (\varepsilon_y + \varepsilon_z)] / (1 + \nu) (1 - 2\nu)$$

$$\sigma_y = [E (1 - \nu) \varepsilon_y + \nu (\varepsilon_x + \varepsilon_z)] / (1 + \nu) (1 - 2\nu)$$

$$\sigma_z = [E (1 - \nu) \varepsilon_z + \nu (\varepsilon_x + \varepsilon_y)] / (1 + \nu) (1 - 2\nu)$$

in the special case of biaxial stress, $\sigma_z = 0$, the result are the same as in section 7.5

the unit volume change is also obtained

$$e = \Delta V / V_0 = \varepsilon_x + \varepsilon_y + \varepsilon_z$$

$$= (\sigma_x + \sigma_y + \sigma_z)(1 - 2 \nu) / E$$

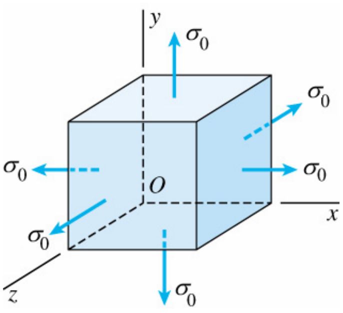
and the strain energy density is

$$u = \tfrac{1}{2} (\sigma_x \varepsilon_x + \sigma_y \varepsilon_y + \sigma_z \varepsilon_z)$$

$$= (\sigma_x^2 + \sigma_y^2 + \sigma_z^2) / 2E - \nu (\sigma_x \sigma_y + \sigma_x \sigma_z + \sigma_y \sigma_z) / E$$

$$= \frac{E\,[(1-\nu)(\varepsilon_x^2 + \varepsilon_y^2 + \varepsilon_z^2) + 2\nu\,(\varepsilon_x\varepsilon_y + \varepsilon_x\varepsilon_z + \varepsilon_y\varepsilon_z)]}{2(1+\nu)(1-2\nu)}$$

for spherical stress



$$\sigma_x = \sigma_y = \sigma_z = \sigma_0$$

then the normal strains are

$$\varepsilon_0 = \sigma_0\,(1-2\nu)\,/\,E$$

and the unit volume change is

$$e = 3\,\varepsilon_0 = 3\sigma_0\,(1-2\nu)\,/\,E$$

define the bulk modulus of elasticity as

$$K = E\,/\,3\,(1-2\nu)$$

then e may expressed as $e = \sigma_0\,/\,K$

and the bulk modulus is $K = \sigma_0\,/\,e$

for an object submerged in water, the stress is spherical state, it is often called hydrostatic stress

7.7 Plane Strain

the normal and shear strains at a point in body vary with direction, for plane strain, the strain components are

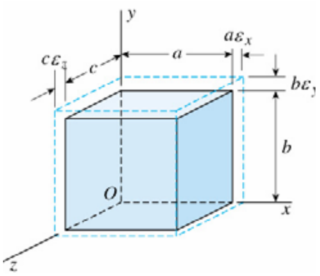
$$\varepsilon_x,\quad \varepsilon_y,\quad \gamma_{xy} \qquad \text{and} \qquad \varepsilon_z = \gamma_{xz} = \gamma_{yz} = 0$$

plane stress : $\sigma_z = \tau_{xz} = \tau_{yz} = 0$

but $\varepsilon_z \neq 0$

plane strain : $\varepsilon_z = \gamma_{xz} = \gamma_{yz} = 0$

but $\sigma_z \neq 0$

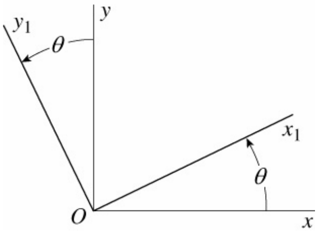


plane stress and plane strain do not occur simultaneously in general

special cases for plane stress ==> plane strain

1. $\sigma_x = -\sigma_y$ $\qquad \varepsilon_z = -\nu(\sigma_x + \sigma_y)/E = 0$
2. $\nu = 0$ $\qquad \varepsilon_z = 0$ for $\sigma_z = 0$

we will derive the strain transformation equations for the case of plane strain, the equations actually are valid even when ε_z exists

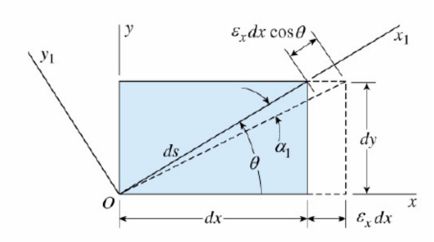


assume ε_x, ε_y, γ_{xy} associated with x and y axes are known, to determine ε_{x1}, ε_{y1}, γ_{x1y1} associated with x_1 and y_1 axes where are rotated counterclockwise through an angle θ form x and y axes

consider first the strain ε_x in x direction

$$\delta x = \varepsilon_x \, dx$$

$$\delta x_1 = \varepsilon_x \, dx \cos\theta$$

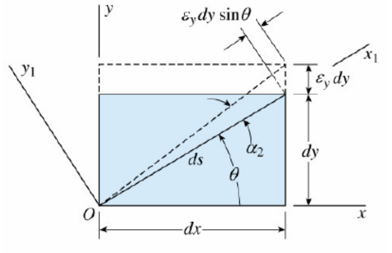


(a)

similarly for ε_y in y direction

$$\delta y = \varepsilon_y \, dy$$

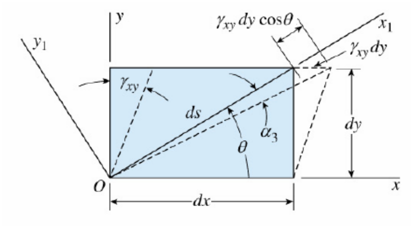
$$\delta x_1 = \varepsilon_y \, dy \sin\theta$$



(b)

consider shear strain γ_{xy} in xy plane

$$\delta x_1 = \gamma_{xy} \, dy \cos\theta$$



(c)

then the total increase $\triangle d$ in x_1 direction is

$$\triangle d = \varepsilon_x \, dx \cos\theta + \varepsilon_y \, dy \sin\theta + \gamma_{xy} \, dy \cos\theta$$

and the strain in x_1 direction is

$$\varepsilon_{x1} = \frac{\triangle d}{ds} = \varepsilon_x \frac{dx}{ds} \cos\theta + \varepsilon_y \frac{dy}{ds} \sin\theta + \gamma_{xy} \frac{dy}{ds} \cos\theta$$

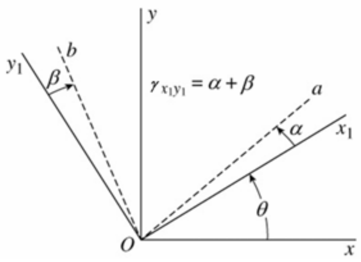
but $dx / ds = \cos\theta$ and $dy / ds = \sin\theta$

thus $\varepsilon_{x1} = \varepsilon_x \cos^2\theta + \varepsilon_y \sin^2\theta + \gamma_{xy} \sin\theta \cos\theta$

substituting $\theta + 90^{o}$ for θ, the ε_{y1} is obtained

$$\varepsilon_{y1} = \varepsilon_x \sin^2\theta + \varepsilon_y \cos^2\theta - \gamma_{xy} \sin\theta \cos\theta$$

hence $\varepsilon_{x1} + \varepsilon_{y1} = \varepsilon_x + \varepsilon_y$



to obtain the shear strain γ_{x1y1}, this strain is equal to the decrease in angle between lines that where initially along x_1 and y_1 axes

$$\gamma_{x1y1} = \alpha + \beta$$

$$\alpha = -\alpha_1 + \alpha_2 - \alpha_3$$

$$= -\varepsilon_x \frac{dx}{ds} \sin\theta + \varepsilon_y \frac{dy}{ds} \cos\theta - \gamma_{xy} \frac{dy}{ds} \sin\theta$$

$$= -(\varepsilon_x - \varepsilon_y) \sin\theta \cos\theta - \gamma_{xy} \sin^2\theta$$

similarly

$$\beta = -(\varepsilon_x - \varepsilon_y) \sin\theta \cos\theta + \gamma_{xy} \cos^2\theta$$

then the shear strain γ_{x1y1} is

$$\gamma_{x1y1} = -2(\varepsilon_x - \varepsilon_y) \sin\theta \cos\theta + \gamma_{xy} (\cos^2\theta - \sin^2\theta)$$

use some trigonometric identities, the transformation equations for plane strain are

$$\varepsilon_{x1} = \frac{\varepsilon_x + \varepsilon_y}{2} + \frac{\varepsilon_x - \varepsilon_y}{2}\cos 2\theta + \frac{\gamma_{xy}}{2}\sin 2\theta$$

$$\frac{\gamma_{x1y1}}{2} = -\frac{\varepsilon_x - \varepsilon_y}{2}\sin 2\theta + \frac{\gamma_{xy}}{2}\cos 2\theta$$

the equations are the counterparts of plane stress

stresses	strains
$\sigma_x \quad \sigma_y$	$\varepsilon_x \quad \varepsilon_y$
τ_{xy}	$\gamma_{xy}/2$
$\sigma_{x1} \quad \sigma_{y1}$	$\varepsilon_{x1} \quad \varepsilon_{y1}$
τ_{x1y1}	$\gamma_{x1y1}/2$

principal strains exist on perpendicular planes with angles θ_p

$$\tan 2\theta_p = \frac{\gamma_{xy}}{\varepsilon_x - \varepsilon_y}$$

the principal strains can be calculated

$$\varepsilon_{1,2} = \frac{\varepsilon_x + \varepsilon_y}{2} \pm \left[\left(\frac{\varepsilon_x - \varepsilon_y}{2}\right)^2 + \left(\frac{\gamma_{xy}}{2}\right)^2\right]^{½}$$

the maximum shear strains exists at 45° to the direction of the principal strains

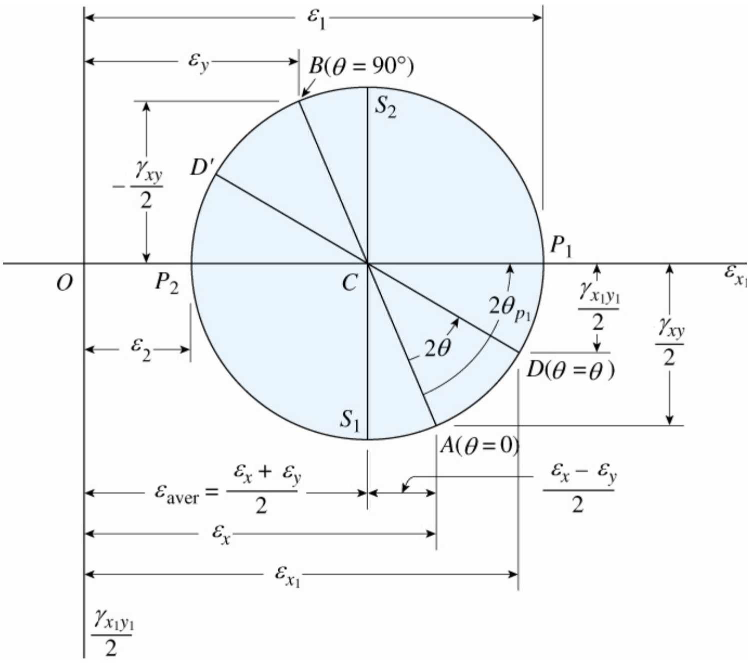
$$\frac{\gamma_{max}}{2} = \left[\left(\frac{\varepsilon_x - \varepsilon_y}{2}\right)^2 + \left(\frac{\gamma_{xy}}{2}\right)^2\right]^{½}$$

and the normal strains in the directions of maximum shear strains are

$$\varepsilon_{max} = \frac{\varepsilon_x + \varepsilon_y}{2}$$

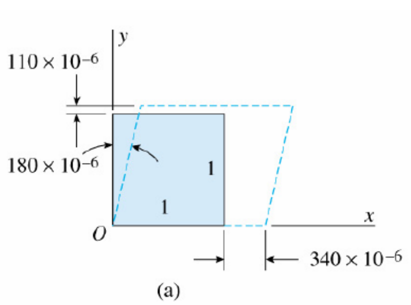
the principal strains and principal stresses occur in the same directions

Mohr's Circle for Plane Strain



plane strains at a point can be measured by strain rosette, then the stresses at this point can be calculated, also the principal strains and principal stresses can be obtained

Example 7-7



ε_x = 340 x 10^{-6} ε_y = 110 x 10^{-6}

γ_{xy} = 180 x 10^{-6}

determine the strains for $\theta = 30^{o}$, principal strains, maximum shear strain

$$\frac{\varepsilon_x + \varepsilon_y}{2} = \frac{(340 + 110)10^{-6}}{2} = 225 \text{ x } 10^{-6}$$

$$\frac{\varepsilon_x - \varepsilon_y}{2} = \frac{(340 - 110)10^{-6}}{2} = 115 \text{ x } 10^{-6}$$

$$\gamma_{xy} / 2 = 90 \text{ x } 10^{-6}$$

$$\theta = 30^{\circ} \qquad 2\theta = 60^{\circ}$$

$$\varepsilon_{x1} = \frac{\varepsilon_x + \varepsilon_y}{2} + \frac{\varepsilon_x - \varepsilon_y}{2} \cos 2\theta + \frac{\gamma_{xy}}{2} \sin 2\theta$$

$$= 225 \text{ x } 10^{-6} + (115 \text{ x } 10^{-6}) \cos 60^{\circ} + (90 \text{ x } 10^{-6}) \sin 60^{\circ}$$

$$= 360 \text{ x } 10^{-6}$$

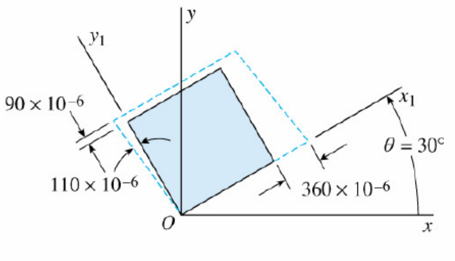
$$\frac{\gamma_{x1y1}}{2} = -\frac{\varepsilon_x - \varepsilon_y}{2} \sin 2\theta + \frac{\gamma_{xy}}{2} \cos 2\theta$$

$$= -55 \text{ x } 10^{-6}$$

$$\gamma_{x1y1} = -110 \text{ x } 10^{-6}$$

$$\varepsilon_{y1} = \varepsilon_x + \varepsilon_y - \varepsilon_{x1}$$

$$= 90 \text{ x } 10^{-6}$$



(b)

$$\varepsilon_{1,2} = \frac{\varepsilon_x + \varepsilon_y}{2} \pm \left[\left(\frac{\varepsilon_x - \varepsilon_y}{2}\right)^2 + \left(\frac{\gamma_{xy}}{2}\right)^2 \right]^{1/2}$$

$$= 225 \text{ x } 10^{-6} \pm [(115 \text{ x } 10^{-6})^2 + (90 \text{ x } 10^{-6})^2]^{1/2}$$

$$= (225 \pm 146) \text{ x } 10^{-6}$$

$$\varepsilon_1 = 370 \text{ x } 10^{-6} \qquad \varepsilon_2 = 80 \text{ x } 10^{-6}$$

the angles of principal directions are

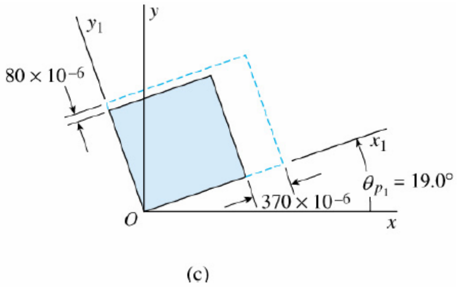
$$\tan 2\theta_p = \frac{\gamma_{xy}}{\varepsilon_x - \varepsilon_y} = \frac{180}{340 - 110} = 0.7826$$

$$2\theta_p = 38^{\circ} \quad \text{and} \quad 218^{\circ}$$

$\theta_p = 19^o$ and 109^o

$\varepsilon_1 = 370 \text{ x } 10^{-6}$ $\quad \theta_p = 19^o$

$\varepsilon_2 = 80 \text{ x } 10^{-6}$ $\quad \theta_p = 109^o$



(c)

note that $\varepsilon_1 + \varepsilon_2 = \varepsilon_x + \varepsilon_y$

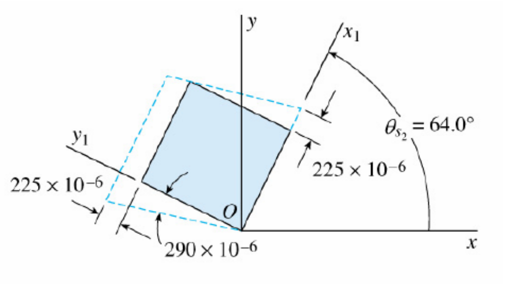
the maximum shear strain is

$$\frac{\gamma_{max}}{2} = \left[\left(\frac{\varepsilon_x - \varepsilon_y}{2}\right)^2 + \left(\frac{\gamma_{xy}}{2}\right)^2 \right]^{½} = 146 \text{ x } 10^{-6}$$

$\gamma_{max} = 292 \text{ x } 10^{-6}$

$\theta_{s1} = \theta_{p1} - 45^o = -26^o$

$\theta_{s2} = \theta_{s1} + 90^o = 64^o$

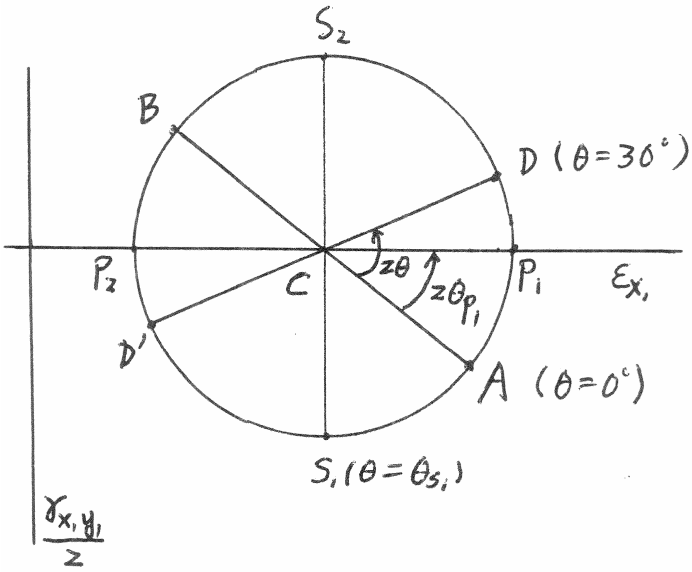


(d)

the normal strains at this direction is

$$\varepsilon_{ave} = \frac{\varepsilon_x + \varepsilon_y}{2} = 225 \text{ x } 10^{-6}$$

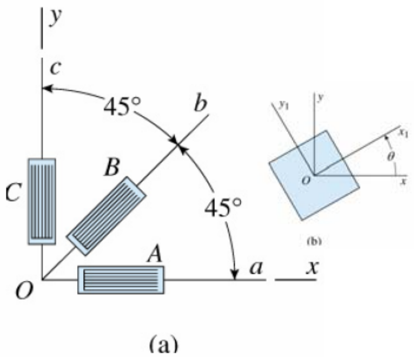
all of this results can be obtained from Mohr's circle



Example 7-8

the plane strains measured by a 45° strain rosette are ε_a, ε_b and ε_c

determine ε_x, ε_y and γ_{xy}



$$\varepsilon_x = \varepsilon_a \qquad \varepsilon_y = \varepsilon_b$$

$$\varepsilon_{x1} = \frac{\varepsilon_x + \varepsilon_y}{2} + \frac{\varepsilon_x - \varepsilon_y}{2} \cos 2\theta + \frac{\gamma_{xy}}{2} \sin 2\theta$$

for $\theta = 45^{\circ}$, $\varepsilon_{x1} = \varepsilon_b$

$$\varepsilon_b = \frac{\varepsilon_a + \varepsilon_c}{2} + \frac{\varepsilon_a - \varepsilon_c}{2} \cos 90^{\circ} + \frac{\gamma_{xy}}{2} \sin 90^{\circ}$$

solve for γ_{xy}, we get

$$\gamma_{xy} = 2\,\varepsilon_b - \varepsilon_a - \varepsilon_c$$

the strains ε_x, ε_y and γ_{xy} can be determined from the strain-gage reading

also the strains ε_{x1}, ε_{y1} and γ_{x1y1} can be calculate at any angle θ