Solution: Again $\frac{\partial e^x}{\partial x} = e^x$; thus, $\text{Var}(X) \approx e^{2\mu_X}\sigma_X^2$.

These approximations can be extended to nonlinear functions of more than one random variable.

Given a set of independent random variables $X_1, X_2, \ldots, X_k$ with means $\mu_1, \mu_2, \ldots, \mu_k$ and variances $\sigma_1^2, \sigma_2^2, \ldots, \sigma_k^2$, respectively, let

$$Y = h(X_1, X_2, \ldots, X_k)$$

be a nonlinear function; then the following are approximations for $E(Y)$ and $\text{Var}(Y)$:

$$E(Y) \approx h(\mu_1, \mu_2, \ldots, \mu_k) + \sum_{i=1}^{k} \frac{\sigma_i^2}{2}\left[\frac{\partial^2 h(x_1, x_2, \ldots, x_k)}{\partial x_i^2}\right]\Bigg|_{x_i=\mu_i,\ 1\le i\le k},$$

$$\text{Var}(Y) \approx \sum_{i=1}^{k}\left[\frac{\partial h(x_1, x_2, \ldots, x_k)}{\partial x_i}\right]^2\Bigg|_{x_i=\mu_i,\ 1\le i\le k} \sigma_i^2.$$

Example 4.26: Consider two independent random variables X and Z with means μ_X and μ_Z and variances σ_X^2 and σ_Z^2, respectively. Consider a random variable

$$Y = X/Z.$$

Give approximations for $E(Y)$ and $\text{Var}(Y)$.

Solution: For $E(Y)$, we must use $\frac{\partial y}{\partial x} = \frac{1}{z}$ and $\frac{\partial y}{\partial z} = -\frac{x}{z^2}$. Thus,

$$\frac{\partial^2 y}{\partial x^2} = 0 \text{ and } \frac{\partial^2 y}{\partial z^2} = \frac{2x}{z^3}.$$

As a result,

$$E(Y) \approx \frac{\mu_X}{\mu_Z} + \frac{\mu_X}{\mu_Z^3}\sigma_Z^2 = \frac{\mu_X}{\mu_Z}\left(1 + \frac{\sigma_Z^2}{\mu_Z^2}\right),$$

and the approximation for the variance of Y is given by

$$\text{Var}(Y) \approx \frac{1}{\mu_Z^2}\sigma_X^2 + \frac{\mu_X^2}{\mu_Z^4}\sigma_Z^2 = \frac{1}{\mu_Z^2}\left(\sigma_X^2 + \frac{\mu_X^2}{\mu_Z^2}\sigma_Z^2\right).$$

4.4 Chebyshev's Theorem

In Section 4.2 we stated that the variance of a random variable tells us something about the variability of the observations about the mean. If a random variable has a small variance or standard deviation, we would expect most of the values to be grouped around the mean. Therefore, the probability that the random variable assumes a value within a certain interval about the mean is greater than for a similar random variable with a larger standard deviation. If we think of probability in terms of area, we would expect a continuous distribution with a large value of σ to indicate a greater variability, and therefore we should expect the area to be more spread out, as in Figure 4.2(a). A distribution with a small standard deviation should have most of its area close to μ, as in Figure 4.2(b).

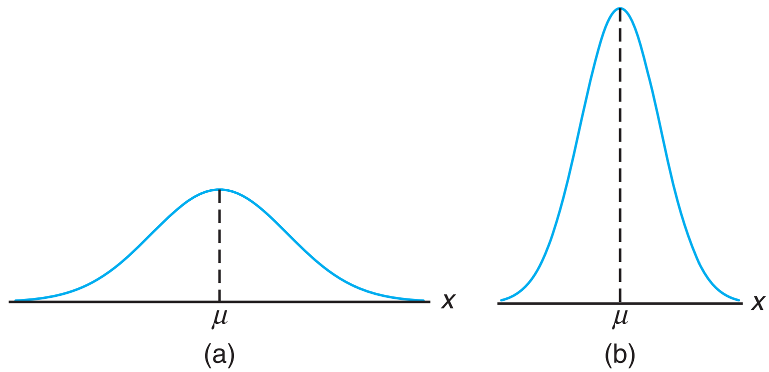

Figure 4.2: Variability of continuous observations about the mean.

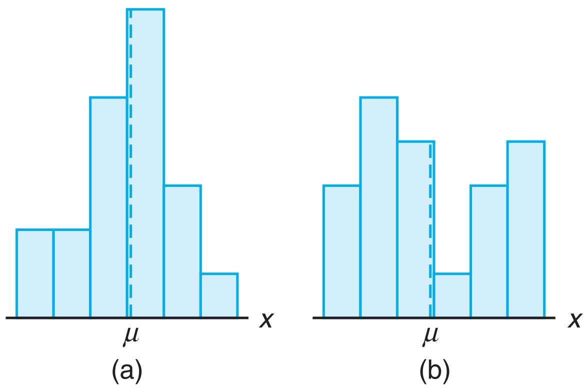

Figure 4.3: Variability of discrete observations about the mean.

We can argue the same way for a discrete distribution. The area in the probability histogram in Figure 4.3(b) is spread out much more than that in Figure 4.3(a) indicating a more variable distribution of measurements or outcomes.

The Russian mathematician P. L. Chebyshev (1821–1894) discovered that the fraction of the area between any two values symmetric about the mean is related to the standard deviation. Since the area under a probability distribution curve or in a probability histogram adds to 1, the area between any two numbers is the probability of the random variable assuming a value between these numbers.

The following theorem, due to Chebyshev, gives a conservative estimate of the probability that a random variable assumes a value within k standard deviations of its mean for any real number k.

Theorem 4.10: **(Chebyshev's Theorem)** The probability that any random variable X will assume a value within k standard deviations of the mean is at least $1 - 1/k^2$. That is,

$$P(\mu - k\sigma < X < \mu + k\sigma) \geq 1 - \frac{1}{k^2}.$$

For $k = 2$, the theorem states that the random variable X has a probability of at least $1 - 1/2^2 = 3/4$ of falling within two standard deviations of the mean. That is, three-fourths or more of the observations of any distribution lie in the interval $\mu \pm 2\sigma$. Similarly, the theorem says that at least eight-ninths of the observations of any distribution fall in the interval $\mu \pm 3\sigma$.

Example 4.27: A random variable X has a mean $\mu = 8$, a variance $\sigma^2 = 9$, and an unknown probability distribution. Find

(a) $P(-4 < X < 20)$,

(b) $P(|X - 8| \geq 6)$.

Solution: (a) $P(-4 < X < 20) = P[8 - (4)(3) < X < 8 + (4)(3)] \geq \frac{15}{16}$.

(b) $P(|X - 8| \geq 6) = 1 - P(|X - 8| < 6) = 1 - P(-6 < X - 8 < 6)$

$$= 1 - P[8 - (2)(3) < X < 8 + (2)(3)] \leq \frac{1}{4}.$$

Chebyshev's theorem holds for any distribution of observations, and for this reason the results are usually weak. The value given by the theorem is a lower bound only. That is, we know that the probability of a random variable falling within two standard deviations of the mean can be *no less* than 3/4, but we never know how much more it might actually be. Only when the probability distribution is known can we determine exact probabilities. For this reason we call the theorem a *distribution-free* result. When specific distributions are assumed, as in future chapters, the results will be less conservative. The use of Chebyshev's theorem is relegated to situations where the form of the distribution is unknown.

Exercises

4.53 Referring to Exercise 4.35 on page 127, find the mean and variance of the discrete random variable $Z = 3X - 2$, when X represents the number of errors per 100 lines of code.

4.54 Using Theorem 4.5 and Corollary 4.6, find the mean and variance of the random variable $Z = 5X + 3$, where X has the probability distribution of Exercise 4.36 on page 127.

4.55 Suppose that a grocery store purchases 5 cartons of skim milk at the wholesale price of \$1.20 per carton and retails the milk at \$1.65 per carton. After the expiration date, the unsold milk is removed from the shelf and the grocer receives a credit from the distributor equal to three-fourths of the wholesale price. If the probability distribution of the random variable X, the number of cartons that are sold from this lot, is

x	0	1	2	3	4	5
$f(x)$	$\frac{1}{15}$	$\frac{2}{15}$	$\frac{2}{15}$	$\frac{3}{15}$	$\frac{4}{15}$	$\frac{3}{15}$

find the expected profit.

4.56 Repeat Exercise 4.43 on page 127 by applying Theorem 4.5 and Corollary 4.6.

4.57 Let X be a random variable with the following probability distribution:

x	-3	6	9
$f(x)$	$\frac{1}{6}$	$\frac{1}{2}$	$\frac{1}{3}$

Find $E(X)$ and $E(X^2)$ and then, using these values, evaluate $E[(2X+1)^2]$.

4.58 The total time, measured in units of 100 hours, that a teenager runs her hair dryer over a period of one year is a continuous random variable X that has the density function

$$f(x) = \begin{cases} x, & 0 < x < 1, \\ 2-x, & 1 \le x < 2, \\ 0, & \text{elsewhere.} \end{cases}$$

Use Theorem 4.6 to evaluate the mean of the random variable $Y = 60X^2 + 39X$, where Y is equal to the number of kilowatt hours expended annually.

4.59 If a random variable X is defined such that

$$E[(X-1)^2] = 10 \text{ and } E[(X-2)^2] = 6,$$

find μ and σ^2.

4.60 Suppose that X and Y are independent random variables having the joint probability distribution

	$f(x,y)$	x = 2	x = 4
y	1	0.10	0.15
	3	0.20	0.30
	5	0.10	0.15

Find
(a) $E(2X - 3Y)$;
(b) $E(XY)$.

4.61 Use Theorem 4.7 to evaluate $E(2XY^2 - X^2Y)$ for the joint probability distribution shown in Table 3.1 on page 96.

4.62 If X and Y are independent random variables with variances $\sigma_X^2 = 5$ and $\sigma_Y^2 = 3$, find the variance of the random variable $Z = -2X + 4Y - 3$.

4.63 Repeat Exercise 4.62 if X and Y are not independent and $\sigma_{XY} = 1$.

4.64 Suppose that X and Y are independent random variables with probability densities and

$$g(x) = \begin{cases} \frac{8}{x^3}, & x > 2, \\ 0, & \text{elsewhere,} \end{cases}$$

and

$$h(y) = \begin{cases} 2y, & 0 < y < 1, \\ 0, & \text{elsewhere.} \end{cases}$$

Find the expected value of $Z = XY$.

4.65 Let X represent the number that occurs when a red die is tossed and Y the number that occurs when a green die is tossed. Find
(a) $E(X+Y)$;
(b) $E(X-Y)$;
(c) $E(XY)$.

4.66 Let X represent the number that occurs when a green die is tossed and Y the number that occurs when a red die is tossed. Find the variance of the random variable
(a) $2X - Y$;
(b) $X + 3Y - 5$.

4.67 If the joint density function of X and Y is given by

$$f(x,y) = \begin{cases} \frac{2}{7}(x+2y), & 0 < x < 1,\ 1 < y < 2, \\ 0, & \text{elsewhere,} \end{cases}$$

find the expected value of $g(X,Y) = \frac{X}{Y^3} + X^2Y$.

4.68 The power P in watts which is dissipated in an electric circuit with resistance R is known to be given by $P = I^2R$, where I is current in amperes and R is a constant fixed at 50 ohms. However, I is a random variable with $\mu_I = 15$ amperes and $\sigma_I^2 = 0.03$ amperes^2. Give numerical approximations to the mean and variance of the power P.

4.69 Consider Review Exercise 3.77 on page 108. The random variables X and Y represent the number of vehicles that arrive at two separate street corners during a certain 2-minute period in the day. The joint distribution is

$$f(x,y) = \left(\frac{1}{4^{(x+y)}}\right)\left(\frac{9}{16}\right),$$

for $x = 0, 1, 2, \ldots$ and $y = 0, 1, 2, \ldots$.
(a) Give $E(X)$, $E(Y)$, $\text{Var}(X)$, and $\text{Var}(Y)$.
(b) Consider $Z = X + Y$, the sum of the two. Find $E(Z)$ and $\text{Var}(Z)$.

4.70 Consider Review Exercise 3.64 on page 107. There are two service lines. The random variables X and Y are the proportions of time that line 1 and line 2 are in use, respectively. The joint probability density function for (X,Y) is given by

$$f(x,y) = \begin{cases} \frac{3}{2}(x^2+y^2), & 0 \le x,\ y \le 1, \\ 0, & \text{elsewhere.} \end{cases}$$

(a) Determine whether or not X and Y are independent.

(b) It is of interest to know something about the proportion of $Z = X + Y$, the sum of the two proportions. Find $E(X + Y)$. Also find $E(XY)$.
(c) Find $\text{Var}(X)$, $\text{Var}(Y)$, and $\text{Cov}(X, Y)$.
(d) Find $\text{Var}(X + Y)$.

4.71 The length of time Y, in minutes, required to generate a human reflex to tear gas has the density function

$$f(y) = \begin{cases} \frac{1}{4}e^{-y/4}, & 0 \leq y < \infty, \\ 0, & \text{elsewhere.} \end{cases}$$

(a) What is the mean time to reflex?
(b) Find $E(Y^2)$ and $\text{Var}(Y)$.

4.72 A manufacturing company has developed a machine for cleaning carpet that is fuel-efficient because it delivers carpet cleaner so rapidly. Of interest is a random variable Y, the amount in gallons per minute delivered. It is known that the density function is given by

$$f(y) = \begin{cases} 1, & 7 \leq y \leq 8, \\ 0, & \text{elsewhere.} \end{cases}$$

(a) Sketch the density function.
(b) Give $E(Y)$, $E(Y^2)$, and $\text{Var}(Y)$.

4.73 For the situation in Exercise 4.72, compute $E(e^Y)$ using Theorem 4.1, that is, by using

$$E(e^Y) = \int_7^8 e^y f(y)\, dy.$$

Then compute $E(e^Y)$ not by using $f(y)$, but rather by using the second-order adjustment to the first-order approximation of $E(e^Y)$. Comment.

4.74 Consider again the situation of Exercise 4.72. It is required to find $\text{Var}(e^Y)$. Use Theorems 4.2 and 4.3 and define $Z = e^Y$. Thus, use the conditions of Exercise 4.73 to find

$$\text{Var}(Z) = E(Z^2) - [E(Z)]^2.$$

Then do it not by using $f(y)$, but rather by using the first-order Taylor series approximation to $\text{Var}(e^Y)$. Comment!

4.75 An electrical firm manufactures a 100-watt light bulb, which, according to specifications written on the package, has a mean life of 900 hours with a standard deviation of 50 hours. At most, what percentage of the bulbs fail to last even 700 hours? Assume that the distribution is symmetric about the mean.

4.76 Seventy new jobs are opening up at an automobile manufacturing plant, and 1000 applicants show up for the 70 positions. To select the best 70 from among the applicants, the company gives a test that covers mechanical skill, manual dexterity, and mathematical ability. The mean grade on this test turns out to be 60, and the scores have a standard deviation of 6. Can a person who scores 84 count on getting one of the jobs? [*Hint*: Use Chebyshev's theorem.] Assume that the distribution is symmetric about the mean.

4.77 A random variable X has a mean $\mu = 10$ and a variance $\sigma^2 = 4$. Using Chebyshev's theorem, find
(a) $P(|X - 10| \geq 3)$;
(b) $P(|X - 10| < 3)$;
(c) $P(5 < X < 15)$;
(d) the value of the constant c such that

$$P(|X - 10| \geq c) \leq 0.04.$$

4.78 Compute $P(\mu - 2\sigma < X < \mu + 2\sigma)$, where X has the density function

$$f(x) = \begin{cases} 6x(1 - x), & 0 < x < 1, \\ 0, & \text{elsewhere,} \end{cases}$$

and compare with the result given in Chebyshev's theorem.

Review Exercises

4.79 Prove Chebyshev's theorem.

4.80 Find the covariance of random variables X and Y having the joint probability density function

$$f(x, y) = \begin{cases} x + y, & 0 < x < 1,\ 0 < y < 1, \\ 0, & \text{elsewhere.} \end{cases}$$

4.81 Referring to the random variables whose joint probability density function is given in Exercise 3.47 on page 105, find the average amount of kerosene left in the tank at the end of the day.

4.82 Assume the length X, in minutes, of a particular type of telephone conversation is a random variable