Probability Theory

Continuous Probability Distribution

We have seen that a random variable which can assume all possible values within a given interval is called a continuous random variable.

Or

A variable that can assume any possible value between two points is called a continuous random variable.

With a given interval of values, there is an infinite number of values. Between any two values, say, 70.5 kg and 71.5kg. or even between 70.99 kg and 70.01 kg there are infinite number of weights, one of which is 71 kg. Therefore the probability that an item chosen at random will weight exactly 71 kg is extremely remote and thus we assign a probability of zero to the event. However if we talk about the probability that the item weights at least 70 kg but not more than 72 kg. We dealing with an interval rather than a point value of our random variable.

So in Simple A r.v X is defined to be continuous if it can take assume every possible value in an interval [a,b]

Examples:

The height of a person, the temperature at a place, the amount of a rain fall, time to failure for an electric system etc.

Probability Density Function

The probability function of the continuous random variable is called probability density function(p.d.f) or simply the density function. It is denoted by f(x), where f(x) is the probability that a r.v X takes the values between "a" and "b"

$$P[a \leq x \leq b] = \int_{-\infty}^{+\infty} f(x)\, dx$$

Properties of Probability Density Function

I. It is non-negative $f(x) \geq 0 \quad for\ all\ x$
II. Total area $= \int_{-\infty}^{+\infty} f(x)\, dx = 1$
III. $P(c \leq x \leq d) = \int_{c}^{d} f(x)\, dx$
IV. $P(X = k) = \int_{k}^{k} f(x)\, dx = 0 \quad where\ k\ is\ constant$,We see that the probability that a continuous r.v assumes a particular point value is zero weather or not particular value is within the range of the variable
V. $P(a \leq x \leq b) = P(a < X < b)$. This means that weather we include an end point of the interval or not when the random variable is continuous

What is P (X = k)?

Let's now revisit this question that we can interpret probabilities as integrals. It is now clear that for a continuous random variable X, we will always have Pr(X = x) = 0, since the area under a single point of a curve is always zero. In other words, if X is a continuous random variable, the probability that X is equal to a particular value will always be zero. We again note this important difference between continuous and discrete random variables.

Example#1

Let X be a random variable having the function

$$f(x) = \begin{cases} cx & 0 \le x \le 2 \\ 0 & otherise \end{cases}$$

Find

(i) the value of the constant c so that function f(x) may be density function

(ii) $\boldsymbol{P(1/2) \le X \le 3/2)}$

(iii) $\boldsymbol{P(X > 1)}$

Solution

(i)

The function f(x) will be density function if $f(x) \ge 0$ and $\int_{-\infty}^{+\infty} f(x)\, dx = 1$

So

$$\int_0^2 cx\, dx = c\frac{x^2}{2}\Big|_0^2 = \frac{c}{2}(4-0) = 1$$

$$c = \frac{1}{2}$$

Since c$> 0, the\ first\ condition\ is\ satisfied\ .thus\ the\ denisty\ function\ of\ \ X\ is$

f(x) = $x/2$, $0 \le x \le 2$

(ii)

$$P(1/2 \le x \le 3/2) = \frac{1}{2}\int_{1/2}^{3/2} x\ \ dx = \frac{1}{2}.\frac{x^2}{2}dx\Big|_{1/2}^{3/2} = \frac{1}{4}(\frac{9}{4} - \frac{1}{4}) = \frac{1}{2}$$

(iii)

$$P(X > 1) = \frac{1}{2}\int_1^2 x\, dx = \frac{1}{2}.\frac{x^2}{2}\Big|_1^2 = \frac{1}{4}(4-1) = \frac{3}{4}$$

Example#2

A continuous random variable X that can assume values between x=0 and x=2 has a density function given by f(x)=x/2.Show that the area under the curve is equal to 1.

Solution

Do it Yourself.

Example#3

A continuous random variable X has a density Function f(x)= c(4-x) for x=1 to x=3, zero otherwise find c.

Solution

Do it Yourself.

Example 4#

A continuous random variable X has a density function $f(x) = \frac{x+1}{8}$ for x=2 to x=4.

Find

(i) $P(X < 3.5)$
(ii) $P(2.4 \leq X \leq 3.5)$
(iii) $P(X = 1.5)$

Solution

Do it Yourself.

Example#5

A continuous Random Variable has the function

$$f(x)\begin{cases} kx & 0 \leq x \leq 2 \\ k(4-x) & 2 \leq x \leq 4 \\ 0 & otherwise \end{cases}$$

Find the value of the constant k and also find out P$(0.5 \leq X \leq 2.5)$.

Solution

Do it Yourself.

Distribution Function For Continuous Random Variable

The Cumulative Distribution Function In the continuous Case

F(x) = $P(X \leq x) = P(-\infty \leq X \leq x) = \int_{-\infty}^{x} f(t)\, dt$

$$\frac{d\,F(x)}{dx} = \frac{d}{dx}\int_{-\infty}^{x} f(t)\, dt = f(x)$$

The distribution function can be used to find

$$P[a \leq x \leq b] = P(X \leq b) - P(X \leq a) = F(b) - F(a)$$

Example#1

Obtain the distribution function for the density function

F(x) = $\frac{x}{2}$ 0≤x≤2

Solution

F(x) = $P(X \leq x)$ = c

For any x such that $-\infty < x < 0, F(x) = \int_{-\infty}^{x} 0\, dx$ = 0

If $0 < x \leq 2$ we have , $F(x) = \int_{-\infty}^{0} 0\, dx + \int_{0}^{x} \frac{x}{2} dx = \frac{x^2}{4}$

And finally for x>2 we have F(x)= $\int_{-\infty}^{0} 0\, dx + \int_{0}^{2} \frac{x}{2} dx + \int_{2}^{\infty} 0\, dx = 1$

Hence

F(x) = $\begin{cases} 0 & for\ x < 0 \\ \frac{x^2}{} & for\ 0 \leq x \leq 2 \\ 1 & for\ x > 2 \end{cases}$

Type equation here.

Example#2

Find the value of k so that the function the function f(x) defined as follows,may be a density function

$$f(x) = \begin{matrix} kx & 0\le x\le 2 \\ 0 & elsewhere \end{matrix}$$

find (i) the value of constant (ii) P(X>1) (iii) Compute the distribution function F(x)

Solution

Do it Yourself

Example#3

A Continuous r.v X Has The d.f F(x) as Follows

$$F(x) = \begin{cases} 0 & for\ x < 0 \\ \frac{2x^2}{5} & for\ 0 \le x \le 1 \\ -\frac{3}{5} + \frac{2}{5}\left(3x - \frac{x^2}{2}\right) & for\ 1 \le x \le 2 \end{cases}$$

And 1 $\quad for\ x > 2$

Find the p.d And P($|x| < 1.5$)

By the definition

f(x) =$\frac{d}{dx}F(x)$

f(x)= $\frac{4x}{5} \quad For\ 0 \le x \le 1$

=$\frac{2}{5}(3-x)\ \ for\ 1 \le x \le 2$

= else where

Now P($|x| < 1.5$)=($-1.5 < X < 1.5$)

$$= \int_{-\infty}^{-1.5} 0\,dx + \int_{-1.5}^{0} 0\,dx + \int_{0}^{1} \frac{4x}{5}\,dx + \int_{1}^{1.5} \frac{2}{5}(3-x)\,dx$$

= 0.75

Example#4

A r.v X is of Continuous type with p.d.f

f(x)= 2x , $0<x<1$

(iv) Find (i) $P(X = \frac{1}{2})$

(v) $P\left(X \le \frac{1}{2}\right)$

(vi) $P(X > \frac{1}{4})$

(vii) $P\left(\frac{1}{4} \le X < \frac{1}{2}\right)$

Solution

Example#5

If X is a continuous random variable with density function

$$f(x) = \begin{matrix} \frac{x}{8} & 0 \le x \le 4 \\ 0 & elsewhere \end{matrix}$$

Find (i) the distribution function F(x)

(ii) find P(2≤X≤3) using the distribution function.

Solution

(i)

Do it Yourself

(ii)

The distribution function can be used to find

$$P[a \le x \le b] = P(X \le b) - P(X \le a) = F(b) - F(a)$$

P[2≤X≤3]= $P(X \le 3) - P(X \le 2) = F(3) - F(2)$

= 5/16 Ans

Example#6

A continuous r.v has the density function function

$f(x)=1/a \quad -a/2<x<a/2$

Find the cumulative distribution function of X.

Solution

Do it Yourself

Example#7

Consider the density function

$$f(x)\begin{cases} k\sqrt{x} & for\ 0 \le x \le 1 \\ 0 & elsewhere \end{cases}$$

(i) Evaluate k

(ii) Find F(x) and use it to evaluate $P(0.3< X < 0.6)$

Solution

Do it Yourself

Example#8

Consider the density function

$$f(x) = \begin{matrix} \frac{x^2}{3} & -1 \le x \le 2 \\ 0 & elsewhere \end{matrix}$$

(a) Verify that f(x) is a density function

(b) Find $P(0 < x \le 1)$

Solution

Do it Yourself

Example#9

A continuous r.v has the density function function

$$f(x) = \begin{matrix} K(3 - x^2) & -1 \leq x \leq 1 \\ 0 & elsewhere \end{matrix}$$

(a) Determine k

(b) Use the value of F(x) to find P$(X \leq \frac{1}{2})$

Solution

Do it yourself

Example#10

A r.v has the density function function

$$f(x) = \begin{matrix} 5(1 - y)^4 & 0 \leq y \leq 1 \\ 0 & elsewhere \end{matrix}$$

(a) Verify that above is valid density function.

(b) Use the value of F(x) to find P$(X \leq 0.1)$

Solution

Do it Yourself

Mathematical Expectation

An important Concept in Probability and statistics is that of the mathematical expectation, expected value or expectation of a random variable. The expected value of a random variable tells where the center mass of the probability function is located, it gives a quick picture of the long run "average" result when the experiment is performed over and over again.

We already discussed the mathematical expectation for discreate random variable.

So for a continuous random variable X with density function f(x), the expectation of X is defined as

$$E(X) = \int_{-\infty}^{+\infty} xf(x)\, dx$$

The expectation of X is usually called the mean of X and is denoted by μ.Whenever several random variables such As X,Y……… are being studied together, the symbol μ should be subscripted to indicate the particular variable involved, that is E(X) = μ_x and E(Y) = μ_y.When only one variable is being studied,the subscripts are often omitted.

If X is a continuous random variable with density function f(x), the expectation of the function g(X) is given by

$$E(g(x)) = \int_{-\infty}^{+\infty} g(x)f(x)\, dx$$

Example#1

The random variable X Has the density function given by

$f(x) = \frac{2x}{3} \quad 1 \le x \le 2$

Find E(X) and E(X^2)

Example#2

If a Continuous Random variable has pdf

$f(x) = \frac{3}{4}(3-x)(x-5) \quad 3 \le x \le 5$

Calculate A.M=?, Var =?, S.D =? of x.

Solution

Do it Yourself

Example#3

Let X be a random variable defined by density function

$$f(x) = \begin{cases} 3x^2 & 0 \leq y \leq 1 \\ 0 & elsewhere \end{cases}$$

Find

(i) E(X)
(ii) E(3X-2)
(iii) E(X^2)

Solution

Do it Yourself

Expectation of joint probability function

Question#1

Let X and Y be two discrete random variables with following joint probability function

y\x	1	2	3	h(y)
1	0.03	0.04	0.03	0.1
2	0.15	0.20	0.15	0.5
3	0.12	0.16	0.12	0.4
g(x)	0.3	0.4	0.3	1

Find

(i)

E(X), E(Y),E(X+Y), and E(XY)

(ii)

Show that E(X+Y) = E(X) + E(Y)

(iii)

Is E(XY)=E(X)E(Y)? if so,what does that mean

Solution

Do it Yourself

Question#2

$$f(x,y) = \begin{cases} 4xy & 0 \leq x \leq 1, 0 \leq y \leq 1 \\ 0 & \text{otherwise} \end{cases}$$

Show that

(i) $E(X+Y) = E(X) + E(Y)$
(ii) $E(XY) = E(X)E(Y)$

Solution

Do it Yourself

Joint Probability Distribution of Two Random Variables

Let X and Y be two Discrete random variables where X can assume any one of the m values $x_1, x_2, \ldots\ldots x_m$ and Y can assume any one of the n values $y_{1,}y_2, \ldots\ldots.. y_n$.The probability of the event that X= x_i and Y = y_i, given by P(X=x_i, X=y_i) = f(x_i, y_i) is called the joint probability function of X and y

The joint Probability Function f(x_i, y_i) can be put in the form of a table listing all possible values of x and y together with the probabilities as shown in the following tabl.

x\y	$\boldsymbol{y_1}$	$\boldsymbol{y_2}$	**…**	$\boldsymbol{y_n}$	**Total P(X=$\boldsymbol{x_i}$)=g($\boldsymbol{x_i}$)**
$\boldsymbol{x_1}$	f(x_1, y_1)	f(x_1, y_2)	…	f(x_1, y_n)	g(x_1)
$\boldsymbol{x_2}$	f(x_2, y_1)	f(x_2, y_2)	…	f(x_2, y_n)	g(x_2)
:	:	:	:	:	
$\boldsymbol{x_m}$	f(x_m, y_1)	f(x_m, y_2)	…	f(x_m, y_n)	g(x_m)
Total P(Y=$\boldsymbol{y_j}$)=h($\boldsymbol{y_j}$)	**h($\boldsymbol{y_1}$)**	**h($\boldsymbol{x_2}$)**	**…**	**h($\boldsymbol{x_n}$)**	**1**

This function is characterized by the properties

(i) f(x_i, y_j) $\geq$ 0 for i=1,2,………..m ; j=1,2,……….n

(ii) $\Sigma_j \Sigma_i$ f(x_i, y_j) = 1

If X and Y are continuous random variable the joint density function of X and Y, f(x_i, y_i) has the properties of

(i) f(x_i, y_i) ≥ 0, $-\infty < x < \infty, -\infty < y < \infty$

(ii) $\iint_{-\infty}^{\infty}$ f$(x, y)\ dx\, dy = 1$

The joint Distribution Function of X and Y is defined by

F(x, y) = P(X$\leq x, Y \leq y$) =$\sum_{u\leq x} \sum_{v\leq y} f(u, v)$

If X and Y are continuous random variables, then joint density function F(x, y) is defined as

F(x, y) =P(X$\leq x, Y \leq y$) =$\int_{-\infty}^{x} \int_{-\infty}^{y} f(x, y)\, dx\, dy$

Question#1

Two marbles are selected at random from a box containing 3 black ball,2 red and 3 green marbles. If X is the number of black marbles and Y is the number of red marbles selected, then find the joint probability function f(x, y) and also find P(X+Y≤1)

Solution

Thus 8 marbles out of which 2 marbles are selected in $\binom{8}{2} = 28$ways

The possible values of X are 0 1 2 and those of Y are also 0 1 2

So possible pairs of values are (0,0),(0,1),(0.2),(1,0),(1,1),(2,0),(1,2),(2,1),(2,2)

The first component is the value of X and second Component is the value of Y

We want to find f(x,y) for each value (x,y)

f(0,0) =P(X=0,Y=0)= $\binom{3}{0}\binom{2}{0}\binom{3}{2}/\binom{8}{2} = 3/28$

represents the probability that no black ,no red ,two green marbles are selected

f(0,1)= $\binom{3}{0}\binom{2}{1}\binom{3}{1}/\binom{8}{2} = 6/28$

f(0,2)= $\binom{3}{0}\binom{2}{2}\binom{3}{0}/\binom{8}{2} = 1/28$

f(1,0)= $\binom{3}{1}\binom{2}{0}\binom{3}{1}/\binom{8}{2} = 9/28$

f(1,1) =$\binom{3}{1}\binom{2}{1}\binom{3}{0} = 6/28$

f(2,0)= $\binom{3}{2}\binom{2}{0}\binom{3}{0} = 3/28$

(x,y)	(0,0)	(0,1)	(0,2)	(1,0)	(1,1)	(2,0)
f(x,y)	3/28	6/28	1/28	9/28	6/28	3/28

The remaining Probabilities f(1,2),f(2,1)f(2,2)=0

Thus all the probabilities are presented in the following table

x\y	0	1	2	P(X=x)=g(x)
0	3/28	6/28	1/28	10/28
1	9/28	6/28	0	15/28
2	3/28	0	0	3/28
P(Y=y)=h(y)	15/28	12/28	1/28	1

The above joint probabilities can also represented by the formula

f(x,y) = $\binom{3}{x}\binom{2}{y}\binom{3}{2-x-y}/\binom{8}{2}, x = 0,1,2 \; and \;\; y = 0,1,2$

To Compute P(X+Y≤1) we see that x+y≤1 for the cells (0,0),(0,1),(1,0)

P(X+Y≤1)=f(0,0)+f(0,1)+f(1,0)

$= 3/28 + 6/28 + 9/28 = 18/28$

Question#2

Suppose that random variable X and Y have the Joint Density Function

f(x,y)= $\begin{cases} c(2x+y) & 2 < x < 6\,;\, 0 < y < 5 \\ 0 & otherwise \end{cases}$

Find (i) the constant c

Solution

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)dxdy = 1$$

or

$$\int_{2}^{6}\int_{0}^{5} c(2x+y)dxdy = 1$$

$$c\int_{2}^{6}(2xy+\frac{y^2}{2})\Big|_{0}^{5}dx = 1$$

$$c\int_{2}^{6}(10x+\frac{25}{2})dx = 1$$

$$c(\frac{10x^2}{2}+\frac{25x}{2}\Big|_{2}^{6}$$

210c=1

c=1/210

So the Joint density function of X and Y is

$$\begin{cases} (2x+y)/210 & 2 < x < 6, 0 < y < 5 \\ 0 & otherwise \end{cases}$$

Marginal Distributions

From the joint probability distribution of the random variables X and Y, we can find the probability distribution of X alone or that of Y alone. Such Probability distributions are called marginal probability distributions.

If X and Y are discrete random variables and f(x_i, y_i) is their joint probability function, the marginal probability functions of X and Y are defined respectively by

g(x_i) = $\sum_j$ f(x_i, y_j) and h(y_j)=$\sum_i$ f(x_i, y_j)

Note that

$\sum_i^m g(x_i) = 1$ and $\sum_i^n h(y_j) = 1$ which is written as $\Sigma_j \Sigma_i$ f(x_i, y_j) = 1

This is simply the statement that the total probability of all the entries is 1

If X and Y are Continuous random variable f(x, y) is their joint density function, the marginal density function of X and Y are defined respectively by

g(x) = $\int_{-\infty}^{\infty} f(x, y)\, dy$ and h(y)=$\int_{-\infty}^{\infty} f(x, y)\, dx$

Taking the data of Question#1

Question#1

Find its Marginal probability function of X and Y (g(x) and h(y))

Solution

Taking the data of Question#1

The marginal probability functions of X and Y denoted respectively by g(x) and h(y) are given in the following tables

g(x_i) = $\sum_j$ f(x_i, y_j)

g(0)=f(0,0)+f(0,1)+f(0,2)

=3/28 + 6/28 + 1/28

=10/28

g(1)=f(1,0) +f(1,1) +f(1,2)

= 9/28+ 6/28+0

= 15/28

g(2)=f(2,0)+f(2,1)+f(2,2)

= 3/28+0+0

= 3/28

x	0	1	2
g(x)	$10/28$	$15/28$	$3/28$

Same as it is for h(y)

h(y_j)=$\sum_i$ f(x_i, y_j)

y	0	1	2
h(y)	$15/28$	$12/28$	$1/28$

Question#2

Find the marginal Density function of X and Y

Solution

Taking the data of Question#2

g(x) = $\int_{-\infty}^{\infty} f(x,y)\,dy$

g(x) = $\int_0^5 (2x+y)/210\,dy$

= 4x+5/84

So the marginal Density Function of X is

$$\begin{cases}(4x+5/84 & 2<x<6\\ 0 & otherwise\end{cases}$$

h(y)=$\int_{-\infty}^{\infty} f(x,y)\,dx$

h(y)=$\int_2^6 2x+y/210\,dx$

=16+2y/105

So the marginal Density Function of Y is

$$\begin{cases}(16+2y/105 & 0<y<5\\ 0 & otherwise\end{cases}$$

Conditional Distribution

We have already defined the conditional probability of event A given event B has occurred as

$P(A/B) = \frac{P(A \cap B)}{P(B)}$

Provided P(B) ≠ 0. If X and Y are discrete random variables and A and B are the events X=x and Y=y, then above equation become

$P(X=x\ /\ Y=y) = \frac{P(X=x, Y=y)}{P(y=y)} = \frac{f(x,y)}{h(y)}$

Where f(x,y) is the joint probability function of X and Y And h(y) is the marginal probability of Y .Thus the conditional probability function of X given Y is Defined as

$f(x/y) = \frac{f(x,y)}{h(y)}, h(y) \neq 0$

similarly the conditional probability function of Y given X is

$f(y/x) = \frac{f(x,y)}{g(x)}, g(x) \neq 0$

Question#1

Find the conditional probability function of f(x/1)

Solution

Taking the data of Question#1

The conditional distribution of X given Y=y is given by

f(x|y)=f(x,y) / h(y) where as h(y)>0

$f(x/1) = P(X=x|Y=1) = \frac{P(X=x, Y=1}{P(y=1)} = \frac{f(x,1)}{h(1)}\ whereas\ x = 0,1,2$

For X =0,1,2, we get from the above tables

$f(0|1) = \frac{f(0,1)}{h(1)} = \frac{6/28}{12/28} = 1/2$

$f(1|1) = \frac{f(1,1)}{h(1)} = \frac{6/28}{12/28} = 1/2$

$f(2|1) = \frac{f(2,1)}{h(1)} = \frac{0}{12/28} = 0$

Hence the conditional Probability function of X given Y =1 is given by

X	0	1	2
f (x|1)	½	1/2	0

P(X=0|Y=1)=f(0|1)=1/2

Question#2

The conditional Density Function of X and Y is given

Solution

Taking the data of Question#2 of joint density function and marginal distribution

(i)

f(x/y) = $\frac{f(x,y)}{h(y)}$,h(y) > 0

$=\frac{(2x+y)/210}{(2y+16)/105} = \frac{2x+y}{2(2y+16)}$ $where\ as\ 2 < x < 6$

(ii)

f(y/x)=$\frac{f(x,y)}{g(x)}, g(x) > 0$

$= \frac{(2x+y)/210}{(4x+5)/84} = \frac{2(2x+y)}{5(4x+5)}$ $where\ as\ 0 < y < 5$

Independence of two random variable

Previously We noted that two events A and B are independent if P(A∩B)

=P(A)P(B) Using the same responsing, we can say that two random variables X and Y are independent if the event that X assumes a specific value x_i is independent of the event that Y assumes a specific value y_j no matter what specific values are selected .

By Definition of independent events, we have P(X=x_i, Y=y_j)=P(X=x_i)P(Y=y_j)

More precisely the variables X and Y of the joint probability function f(x,y) are statistically independent random variables if and only if the joint joint probability function can be expressed as the product of the marginal probability functions that is if and only if f(x,y)=f(x)h(y)

Question#1

Variable X and Y are independent ?

Solution

Taking the data of Question#1

The variables X and Y independents if f(x,y)=g(x)h(y)

Let us consider the pair of (0,1)

f(0,1)=g(0)h(1)

From Part (i) of Question#1

f(0,1)=6/28

From Part (iii) of Question#1

g(0)=10/28

h(1)=12/28

g(0)h(1)=10/28+12/28

=15/28

So

f(0,1) ≠g(0)h(1) So X and Y are not independent

Few Questions Related to Continuous

Question#1

Given the following joint p.d.f

f(x,y) = $\frac{1}{8}(6 - x - y), 0 \leq x \leq 2, 2 \leq y \leq 4$

(a) Verify that f(x,y) is a density function
(b) Calculate (i) P $(X \leq \frac{3}{2}, Y \leq \frac{5}{2})$
(c) Find the marginal p.d.f g(x) and h(y)
(d) Find conditional p.d.f f(x|y) and f(y|x)

Question#2

Given the following pdf

f(x,y) = $10xy^2, 0 < x < y < 1$

Find marginal densities g(x), h(y) and the conditional density f(y|x)

Question#3

Given that joint density function

f(x, y) = f(x,y) = $\begin{cases} x(1 + 3y^2) & 0 \leq x \leq 2, 0 \leq y \leq 1 \\ 0 & \text{otherwise} \end{cases}$

Find g(x), h(y), f(x|y)