Probability Theory

Continuous Probability Distribution

We have seen that a random variable which can assume all possible values within a given interval is called a continuous random variable.

Or

A variable that can assume any possible value between two points is called a continuous random variable.

With a given interval of values, there is an infinite number of values. Between any two values, say, 70.5 kg and 71.5kg. or even between 70.99 kg and 70.01 kg there are infinite number of weights, one of which is 71 kg. Therefore the probability that an item chosen at random will weight exactly 71 kg is extremely remote and thus we assign a probability of zero to the event. However if we talk about the probability that the item weights at least 70 kg but not more than 72 kg. We dealing with an interval rather than a point value of our random variable.

So in Simple A r.v X is defined to be continuous if it can take assume every possible value in an interval [a,b]

Examples:

The height of a person, the temperature at a place, the amount of a rain fall, time to failure for an electric system etc.

Probability Density Function

The probability function of the continuous random variable is called probability density function(p.d.f) or simply the density function. It is denoted by f(x), where f(x) is the probability that a r.v X takes the values between "a" and "b"

$$P[a \leq x \leq b] = \int_{-\infty}^{+\infty} f(x)\, dx$$

Properties of Probability Density Function

I. It is non-negative $f(x) \geq 0 \quad for\ all\ x$
II. Total area = $\int_{-\infty}^{+\infty} f(x)\, dx = 1$
III. $P(c \leq x \leq d) = \int_{c}^{d} f(x)\, dx$
IV. $P(X = k) = \int_{k}^{k} f(x)\, dx = 0 \quad where\ k\ is\ constant$,We see that the probability that a continuous r.v assumes a particular point value is zero weather or not particular value is within the range of the variable
V. $P(a \leq x \leq b) = P(a < X < b)$. This means that weather we include an end point of the interval or not when the random variable is continuous

What is P (X = k)?

Let's now revisit this question that we can interpret probabilities as integrals. It is now clear that for a continuous random variable X, we will always have Pr(X = x) = 0, since the area under a single point of a curve is always zero. In other words, if X is a continuous random variable, the probability that X is equal to a particular value will always be zero. We again note this important difference between continuous and discrete random variables.

Example#1

Let X be a random variable having the function

$$f(x) = \begin{cases} cx & 0 \le x \le 2 \\ 0 & otherise \end{cases}$$

Find

(i) the value of the constant c so that function f(x) may be density function

(ii) $\boldsymbol{P(1/2) \le X \le 3/2)}$

(iii) $\boldsymbol{P(X > 1)}$

Solution

(i)

The function f(x) will be density function if $f(x) \ge 0$ and $\int_{-\infty}^{+\infty} f(x)\,dx = 1$

So

$$\int_0^2 cx\,dx = c\frac{x^2}{2}\Big|_0^2 = \frac{c}{2}(4-0) = 1$$

$$c = \frac{1}{2}$$

Since c$> 0, the\ first\ condition\ is\ satisfied\ .thus\ the\ denisty\ function\ of\ \ X\ is$

f(x) = $x/2$, $0 \le x \le 2$

(ii)

$$P(1/2 \le x \le 3/2) = \frac{1}{2}\int_{1/2}^{3/2} x\ dx = \frac{1}{2}.\frac{x^2}{2}dx\Big|_{1/2}^{3/2} = \frac{1}{4}(\frac{9}{4} - \frac{1}{4}) = \frac{1}{2}$$

(iii)

$$P(X > 1) = \frac{1}{2}\int_1^2 x\,dx = \frac{1}{2}.\frac{x^2}{2}\Big|_1^2 = \frac{1}{4}(4-1) = \frac{3}{4}$$

Example#2

A continuous random variable X that can assume values between x=0 and x=2 has a density function given by f(x)=x/2.Show that the area under the curve is equal to 1.

Solution

Do it Yourself.

Example#3

A continuous random variable X has a density Function f(x)= c(4-x) for x=1 to x=3, zero otherwise find c.

Solution

Do it Yourself.

Example 4#

A continuous random variable X has a density function $f(x) = \frac{x+1}{8}$ for x=2 to x=4.

Find

(i) $P(X < 3.5)$
(ii) $P(2.4 \leq X \leq 3.5)$
(iii) $P(X = 1.5)$

Solution

Do it Yourself.

Example#5

A continuous Random Variable has the function

$$f(x)\begin{cases} kx & 0 \leq x \leq 2 \\ k(4-x) & 2 \leq x \leq 4 \\ 0 & otherwise \end{cases}$$

Find the value of the constant k and also find out P$(0.5 \leq X \leq 2.5)$.

Solution

Do it Yourself.

Distribution Function For Continuous Random Variable

The Cumulative Distribution Function In the continuous Case

F(x) = $P(X \leq x) = P(-\infty \leq X \leq x) = \int_{-\infty}^{x} f(t)\, dt$

$$\frac{d\,F(x)}{dx} = \frac{d}{dx} \int_{-\infty}^{x} f(t)\, dt = f(x)$$

The distribution function can be used to find

$$P[a \leq x \leq b] = P(\mathrm{X} \leq \mathrm{b}) - \mathrm{P}(\mathrm{X} \leq \mathrm{a}) = \mathrm{F(b)} - \mathrm{F(a)}$$

Example#1

Obtain the distribution function for the density function

F(x) = $\frac{x}{2}$ $0 \leq x \leq 2$

Solution

F(x) = $P(X \leq x)$ = c

For any x such that $-\infty < x < 0$, $F(x) = \int_{-\infty}^{x} 0\, dx = 0$

If $0 < x \leq 2$ we have , $F(x) = \int_{-\infty}^{0} 0\, dx + \int_{0}^{x} \frac{x}{2} dx = \frac{x^2}{4}$

And finally for x>2 we have $F(x) = \int_{-\infty}^{0} 0\, dx + \int_{0}^{2} \frac{x}{2} dx + \int_{2}^{\infty} 0\, dx = 1$

Hence

$$F(x) = \begin{cases} 0 & \text{for } x < 0 \\ \frac{x^2}{2} & \text{for } 0 \leq x \leq 2 \\ 1 & \text{for } x > 2 \end{cases}$$

Example#2

Find the value of k so that the function the function f(x) defined as follows,may be a density function

$$f(x) = \begin{matrix} kx & 0 \le x \le 2 \\ 0 & \text{elsewhere} \end{matrix}$$

find (i) the value of constant (ii) P(X>1) (iii) Compute the distribution function F(x)

Solution

Do it Yourself

Example#3

A Continuous r.v X Has The d.f F(x) as Follows

$$F(x) = \begin{cases} 0 & \text{for } x < 0 \\ \frac{2x^2}{5} & \text{for } 0 \le x \le 1 \\ -\frac{3}{5} + \frac{2}{5}\left(3x - \frac{x^2}{2}\right) & \text{for } 1 \le x \le 2 \end{cases}$$

And 1 for $x > 2$

Find the p.d And $P(|x| < 1.5)$

By the definition

$f(x) = \frac{d}{dx} F(x)$

$f(x) = \frac{4x}{5} \quad For\ 0 \le x \le 1$

$= \frac{2}{5}(3 - x)$ for $1 \le x \le 2$

= else where

Now $P(|x| < 1.5) = (-1.5 < X < 1.5)$

$$= \int_{-\infty}^{-1.5} 0\, dx + \int_{-1.5}^{0} 0\, dx + \int_{0}^{1} \frac{4x}{5}\, dx + \int_{1}^{1.5} \frac{2}{5}(3 - x)\, dx$$

$= 0.75$

Example#4

A r.v X is of Continuous type with p.d.f

f(x)= 2x , $0<x<1$

(iv) Find (i) $P(X = \frac{1}{2})$

(v) $P\left(X \leq \frac{1}{2}\right)$

(vi) $P(X > \frac{1}{4})$

(vii) $P\left(\frac{1}{4} \leq X < \frac{1}{2}\right)$

Solution

Example#5

If X is a continuous random variable with density function

$$f(x) = \begin{cases} \frac{x}{8} & 0 \leq x \leq 2 \\ 0 & \text{elsewhere} \end{cases}$$

Find (i) the distribution function F(x)

(ii) find P(2≤X≤3) using the distribution function.

Solution

(i)

Do it Yourself

(ii)

The distribution function can be used to find

$$P[a \leq x \leq b] = P(X \leq b) - P(X \leq a) = F(b) - F(a)$$

$$P[2 \leq X \leq 3] = P(X \leq 3) - P(X \leq 2) = F(3) - F(2)$$

= 5/16 Ans

Example#6

A continuous r.v has the density function function

$f(x)=1/a \quad -a/2<x<a/2$

Find the cumulative distribution function of X.

Solution

Do it Yourself

Example#7

Consider the density function

$$f(x)\begin{cases} k\sqrt{x} & \text{for } 0 \le x \le 1 \\ 0 & \text{elsewhere} \end{cases}$$

(i) Evaluate k
(ii) Find F(x) and use it to evaluate $P(0.3< X < 0.6)$

Solution

Do it Yourself