11

Stressees Due to Applied Loads

11.1. INTRODUCTION

Stresses are induced in a soil mass due to weight of overlying soil and due to the applied loads. These stresses are required for the stability analysis of the soil mass, the settlement analysis of foundations and the determination of the earth pressures. The stresses due to self weight of soil have been discussed in chapter 10. These stresses are summarised in Section 11.3. The rest of the chapter is devoted to the determination of stresses due to applied loads.

The stresses induced in soil due to applied loads depend upon its stress-strain characteristic. The stress-strain behaviour of soils is extremely complex and it depends upon a large number of factors, such as drainage conditions, water content, void ratio, rate of loading, the load level, and the stress path. However, simplifying assumptions are generally made in the analysis to obtain stresses. It is generally assumed that the soil mass is homogeneous and isotropic. The stress-strain relationship is assumed to be linear. The theory of elasticity is used to determine the stresses in the soil mass. It involves considerable simplification of real soil behaviour and the stresses computed are approximate ones. Fortunately, the results are good enough for soil problems usually encountered in practice. For more accurate results, realistic stress-strain characteristics should be used. However, the procedure becomes complex and numerical techniques and a high speed computer are required.

11.2. STRESS-STRAIN PARAMETERS

The main stress-strain parameters required for the application of elastic theories are modulus of elasticity (E) and Poisson's ratio (ν). The modulus of elasticity can be determined in the laboratory by conducting a triaxial compression test (see chapter 13). The stress-strain curve is plotted between the deviator stress ($\sigma_1 - \sigma_3$), and the axial strain (ε_1). An unconsolidated-undrained (UU) or an unconfined compression test can be performed for saturated, cohesive soils. A consolidated drained (CD) is usually conducted for cohesionless soils. The value of modulus is generally taken as the secant modulus at 1/2 to 1/3 of the peak stress (Fig. 11.1). Sometimes, instead of the secant modulus, the initial tangent modulus or the tangent modulus at 1/2 to 1/3 of the peak stress is also used.

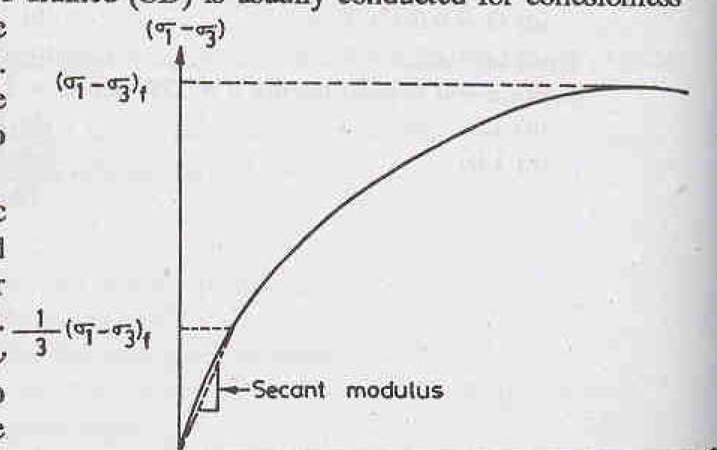


Fig. 11.1. Stress—Strain Curve

The value of Poisson's ratio (ν) for an elastic material varies from 0.0 to 0.50. For undrained conditions, the value of Poisson's ratio is 0.50. For drained conditions, the Poisson's ratio is less than 0.50. As the soil is not a purely elastic material, the value of ν outside the elastic range of 0.0 to 0.50 is also occasionally encountered. It is difficult to ascertain the exact value of Poisson's ratio. Fortunately, the effect of

Poisson's ratio on the computed stresses is not significant and an approximate value can be used without much error.

Tables 11.1 and 11.2 give typical range of values of modulus of elasticity and Poisson's ratio, respectively, for some soils.

Table 11.1. Typical Values of *E*

S. No.	*Type of Soil*	*E*	
		MN/m²	kN/m²
1.	Soft Clay	1.5—4.0	1500—4000
2.	Hard clay	6.0—15.0	6000—15000
3.	Silty Sand	6.0—20.0	6000—20000
4.	Loose Sand	10.0—25.0	10000—25000
5.	Dense Sand	40.0—80.0	40000—80000
6.	Dense gravel	100—200	1×10^5 to 2×10^5

Table 11.2. Typical Values of Poisson's Ratio (*ν*)

S. No.	*Type of Soil*	*ν*
1.	Saturated clay	0.4—0.5
2.	Unsaturated clay	0.1—0.3
3.	Silt	0.3—0.35
4.	Loose sand	0.30—0.50
5.	Dense sand	0.20—0.30

11.3. GEOSTATIC STRESSES

The method for the determination of total vertical stresses due to self weight of the soil have been discussed in chapter 10. The stresses due to self weight of soils are generally large in comparison with those induced due to imposed loads. This is unlike many other civil engineering structures, such as steel bridges, wherein the stresses due to self weight are relatively small. In soil engineering problems, the stresses due to self weight are significant. In many cases the stresses due to self weight are a large proportion of the total stresses and may govern the design.

When the ground surface is horizontal and the properties of the soil do not change along a horizontal plane, the stresses due to self weight are known as *geostatic stresses.* Such a condition generally exists in sedimentary soil deposits. In such a case, the stresses are normal to the horizontal and vertical planes, and there are no shearing stresses on these planes. In other words, these planes are principal planes. The vertical and horizontal stresses can be determined as under.

(*a*) **Vertical stresses.** The vertical stresses are determined using the methods described in chapter 10. Let us consider the horizontal plane *A—A* at a depth *z* below the ground surface [Fig. 11.2 (*a*)]. Let the area of cross-section of the prism be *A*. If the unit weight of soil (γ) is constant, the vertical stress (σ_z) is equal to the weight of soil in prism divided by the area of base. Thus

$$\sigma_z = \frac{\text{Weight of soil in prism}}{\text{Area of base}} = \frac{\gamma (z \times A)}{A}$$

or $$\sigma_z = \gamma z \qquad ...(11.1)$$

If the soil is stratified, having *n* layers of thickness $z_1, z_2, \ldots z_n$, with unit weight $\gamma_1, \gamma_2, \ldots \gamma_n$, the vertical stress is given by

$$\sigma_z = \sum_{i=1}^{n} \gamma_i z_i \qquad ...(11.2)$$

In natural deposits, generally the density of the soil increases with an increase in depth due to the weight of soil above. In such a case, the unit weight of soil cannot be taken as constant. In this case, the weight of soil in the prism is given by

$$W = \int_0^z \gamma A \, dz$$

where dz is the thickness of a small strip of soil at depth z. Therefore, the vertical stress is given by

$$\sigma_z = \frac{W}{A} = \frac{\int_0^z \gamma A \, dz}{A}$$

or

$$\sigma_z = \int_0^z \gamma \, dz \quad \text{...(11.3)}$$

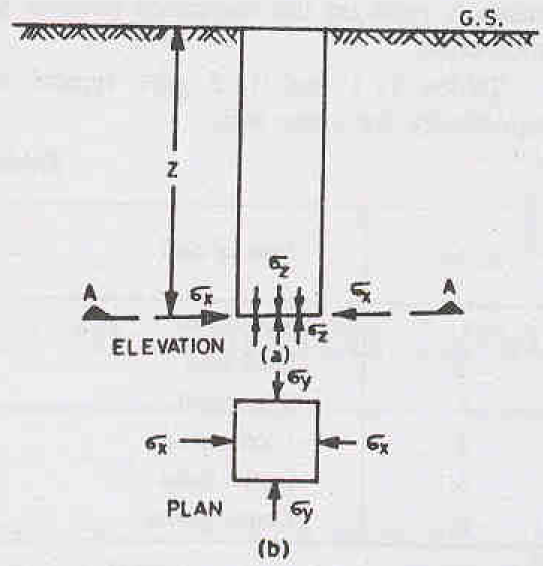


Fig. 11.2. Geostatic Stress.

If the soil is stratified and also has a variable unit weight, the vertical stress is given by

$$\sigma_z = \int_0^{z_1} \gamma_1 \, dz + \int_0^{z_2} \gamma_2 \, dz + \ldots + \int_0^{z_n} \gamma_n \, dz \quad \text{...(11.4)}$$

(*b*) **Horizontal stresses.** The horizontal stresses (σ_x and σ_y) act on vertical planes, as shown in Fig. 11.2 (*b*). The horizontal stresses at a point in a soil mass are highly variable. These depend not only upon the vertical stresses, but also on the type of the soil and on the conditions whether the soil is stretched or compressed laterally. In the treatment that follows it would be assumed that $\sigma_x = \sigma_y$.

The ratio of the horizontal stress (σ_x) to the vertical stress (σ_z) is known as the coefficient of lateral stress or lateral stress ratio (K). Thus

$$K = \frac{\sigma_x}{\sigma_z}$$

or

$$\sigma_x = K \sigma_z \quad \text{...(11.5)}$$

In natural deposits, generally there is no lateral strain. The lateral stress coefficient for this case is known as the *coefficient of lateral pressure at rest* (K_0). The value of its coefficient can be obtained from the theory of elasticity, as explained below. In retaining structures (chapter 19), there is either stretching or contraction of soils and the value of K is different.

The strain in x-direction is given by (see any text on theory of elasticity or mechanics of materials)

$$\varepsilon_x = \frac{1}{E} [\sigma_x - \nu (\sigma_y + \sigma_z)]$$

For conditions of no lateral strain, $\varepsilon_x = 0$.

Thus $$\sigma_x = \nu (\sigma_y + \sigma_z)$$

Taking $\sigma_x = \sigma_y$ and simplifying,

or $$\sigma_x (1 - \nu) = \nu \, \sigma_z$$

or $$\sigma_x = \left(\frac{\nu}{1-\nu}\right) \sigma_z$$

or $$\sigma_x = K_0 \sigma_z \quad \text{...(11.6)}$$

where $$K_0 = \frac{\nu}{1 - \nu} \quad \text{...(11.7)}$$

The value of K_0 can be obtained if the Poisson's ratio ν is known or estimated. Eq. 11.7 is not of much practical use as the soil is not a purely elastic material and it is difficult to estimate the Poisson ratio.

The value of K_0 is determined from actual measurement of soil pressure or from experience. For a sedimentary sand deposit, its value varies from 0.30 to 0.6, and for a normally consolidated clay, its value generally lies between 0.5 and 1.10. Table 11.3 gives the average values of K_0 for different types of soils.

Jaky's formula is commonly used, according to which

$$K_0 = 1 - \sin \varphi'$$

where ϕ' is the angle of shearing resistance.

Table 11.3. Values of Lateral Pressure Coefficient at Rest (K_0)

S. No.	Type of Soil	K_0
1.	*Loose sand*	0.5—0.60
2.	*Dense sand*	0.3—0.50
3.	*Clay (drained)*	0.5—0.60
4.	*Clay (undrained)*	0.80—1.1
5.	*Over-consolidated clay*	1.0—3.0

11.4. VERTICAL STRESSES DUE TO A CONCENTRATED LOAD

Boussinesq (pronounced as Boo-si-nesk) gave the theoretical solutions for the stress distribution in an elastic medium subjected to a concentrated load on its surface. The solutions are commonly used to obtain the stresses in a soil mass due to externally applied loads. The following assumptions are made.

(1) The soil mass is an elastic continuum, having a constant value of modulus of elasticity (E), *i.e.*, the ratio between the stress and strain is constant.

(2) The soil is homogeneous, *i.e.*, it has identical properties at different points.

(3) The soil is isotropic, *i.e.*, it has identical properties in all directions.

(4) The soil mass is semi-infinite, *i.e.*, it extends to infinity in the downward direction and lateral directions. In other words, it is limited on its top by a horizontal plane and extends to infinity in all other directions.

(5) The soil is weightless and is free from residual stresses before the application of the load.

[**Note.** The stresses due to self weight are computed separately as explained in the preceding section].

Fig. 11.3 shows a horizontal surface of the elastic continuum subjected to a point load Q at point O. The origin of the coordinates is taken at O. Using logarithmic stress function for the solution of elasticity problem, Boussinesq proved that the polar stress σ_R at point P (x, y, z) is given by

$$\sigma_R = \frac{3}{2\pi} \frac{Q \cos \beta}{R^2} \qquad \text{...(11.8)}$$

where R = polar distance between the origin O and point P.

β = angle which the line OP makes with the vertical.

Obviously, $R = \sqrt{x^2 + y^2 + z^2}$

or $R = \sqrt{r^2 + z^2}$ where $r^2 = x^2 + y^2$

and $\sin \beta = r/R$ and $\cos \beta = z/R$

The vertical stress (σ_z) at point P is given by

$$\sigma_z = \sigma_R \cos^2 \beta$$

$$= \frac{3}{2\pi} \left(\frac{Q \cos \beta}{R^2} \right) \cos^2 \beta$$

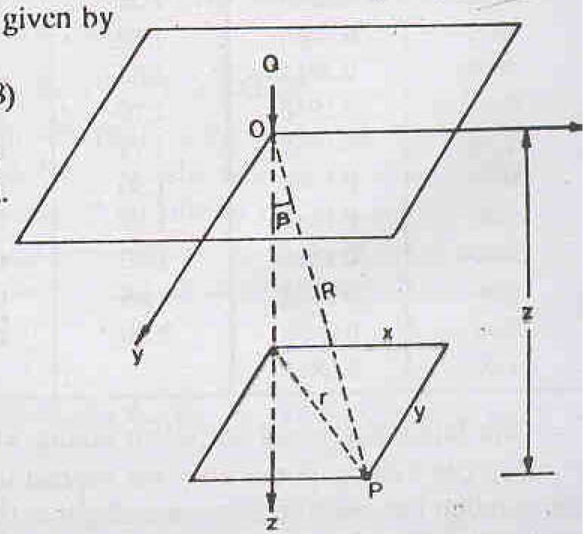


Fig. 11.3. Stresses due to a concentrated load

or $$\sigma_z = \frac{3Q}{2\pi} \cdot \frac{\cos^3 \beta}{R^2}$$

or $$\sigma_z = \frac{3Q}{2\pi} \cdot \frac{(z/R)^3}{R^2} = \frac{3Q}{2\pi} \cdot \frac{z^3}{R^5}$$

or $$\sigma_z = \frac{3Q}{2\pi} \cdot \frac{1}{z^2} \cdot \frac{z^5}{R^5}$$

or $$\sigma_z = \frac{3Q}{2\pi} \cdot \frac{1}{z^2} \cdot \left[\frac{z^5}{(r^2 + z^2)^{5/2}}\right]$$

or $$\sigma_z = \frac{3Q}{2\pi} \cdot \frac{1}{z^2} \cdot \frac{1}{[1 + (r/z)^2]^{5/2}} \quad \text{...(11.9)}$$

or $$\sigma_Z = I_B \cdot \frac{Q}{z^2} \quad \text{...(11.10)}$$

where $$I_B = \frac{3}{2\pi [1 + (r/z)^2]^{5/2}} \quad \text{...(11.11)}$$

The coefficient I_B is known as the Boussinesq influence coefficient for the vertical stress. The value of I_B can be determined for the given value of r/z from Eq. 11.11. The computed values are tabulated in Table 11.4.

Table 11.4. Values of Boussinesq's Coefficient (I_B)

r/z	I_B	r/z	I_B	r/z	I_B	r/z	I_B
0.00	0.4775	1.05	0.0745	2.05	0.0077	3.25	0.0011
0.05	0.4745	1.10	0.0658	2.10	0.0070	3.50	0.0008
0.10	0.4657	1.15	0.0581	2.15	0.0064	3.75	0.0005
0.15	0.4516	1.20	0.0513	2.20	0.0058	4.00	0.0004
0.20	0.4329	1.25	0.0454	2.25	0.0053	4.25	0.0003
0.25	0.4103	1.30	0.0402	2.30	0.0048	4.50	0.0002
0.30	0.3849	1.35	0.0357	2.35	0.0044	4.75	0.0002
0.35	0.3577	1.40	0.0317	2.40	0.0040	5.00	0.0001
0.40	0.3295	1.45	0.0282	2.45	0.0037	10.00	0.0000
0.45	0.3011	1.50	0.0251	2.50	0.0034		
0.50	0.2733	1.55	0.0224	2.55	0.0031		
0.55	0.2466	1.60	0.0200	2.60	0.0029		
0.60	0.2214	1.65	0.0179	2.65	0.0026		
0.65	0.1978	1.70	0.0160	2.70	0.0024		
0.70	0.1762	1.75	0.0144	2.75	0.0022		
0.75	0.1565	1.80	0.0129	2.80	0.0021		
0.80	0.1386	1.85	0.0116	2.85	0.0019		
0.85	0.1226	1.90	0.0105	2.90	0.0018		
0.90	0.1083	1.95	0.0095	2.95	0.0016		
0.95	0.0956	2.00	0.0085	3.00	0.0015		
1.00	0.0844						

The following points are worth noting when using Eq. 11.10.

(1) The vertical stress does not depend upon the modulus of elasticity (E) and the Poisson ratio (ν). But the solution has been derived assuming that the soil is linearly elastic. The stress distribution will be the same in all linearly elastic materials.

(2) The intensity of vertical stress just below the load point is given by

$$\sigma_Z = 0.4775 \frac{Q}{z^2} \qquad ...(11.12)$$

(3) At the surface ($z = 0$), the vertical stress just below the load is theoretically infinite. However, in an actual case, the soil under the load yields due to very high stresses. The load point spreads over a small but finite area and, therefore, only finite stresses develop.

(4) The vertical stress (σ_z) decreases rapidly with an increase in r/z ratio. Theoretically, the vertical stress would be zero only at an infinite distance from the load point. Actually, at $r/z = 5.0$ or more, the vertical stress becomes extremely small and is neglected.

(5) In actual practice, foundation loads are not applied directly on the ground surface. However, it has been established that the Boussinesq solution can be applied conservatively to field problems concerning loads at shallow depths, provided the distance z is measured from the point of application of the load.

(6) Boussinesq's solution can even by used for negative (upward) loads. For example, if the vertical stress decrease due to an excavation is required, the negative load is equal to the weight of the soil removed. However, as the soil is not fully elastic, the stresses determined are necessarily approximate.

(7) The field measurements indicate that the actual stresses are generally smaller than the theoretical values given by Boussinesq's solution, especially at shallow depths. Thus, the Boussinesq solution gives conservative values and is commonly used in soil engineering problems.

Limitations of Boussinesq's Solution. The solution was initially obtained for determination of stresses in elastic solids. Its application to soils may be questioned, as the soils are far from purely elastic solids. However. experience indicates that the results obtained are satisfactory.

The application of Boussinesq's solution can be justified when the stress changes are such that only a stress increase occurs in the soil. The real requirement for use of the solution is not that the soil be elastic (*i.e.*, fully recoverable), but it should have a constant ratio between stress and strain. When the stress decrease occurs, the relation between stress and strain is not linear and, therefore, the solution is not strictly applicable. If the stresses induced in the soil are small in comparison with the shear strength of the soil, the soil behaves somewhat elastically and the Boussinesq solution can be used.

For practical cases, the Boussinesq solution can be safely used for homogeneous deposits of clay, man-made fills and for limited thickness of uniform sand deposits. In deep sand deposits, the modulus of elasticity increases with an increase in depth and, therefore, the Boussinesq solution will not give satisfactory results. In this case, the assumption of proportionality between stress and strain cannot be justified. For such a case, non-linear elastic solutions or elastic-plastic solutions are required.

The point loads applied below ground surface cause somewhat smaller stresses than are caused by surface loads, and, therefore, the Boussinesq solution is not strictly applicable. However, the solution is frequently used for shallow footings, in which z is measured below the base of the footing.

11.5. HORIZONTAL AND SHEAR STRESSES DUE TO A CONCENTRATED LOAD

The method for determination of the vertical stress (σ_z) has been discussed in the preceding section. In most soil engineering problems, only the vertical stresses are required. Occasionally, other stress components (σ_x, σ_y, τ_{xy}, and τ_{yz}) are also required. These components can be determined as follows :

Fig. 11.4 shows an elementary stress block, indicating all the stress components. In all there are 9 stress components. namely, σ_x, σ_y, σ_z, τ_{xy}, τ_{xz}, τ_{yx}, τ_{yz}, τ_{zx} and τ_{zy}. However, the moment equation gives the following relations.

$$\tau_{xy} = \tau_{yx}\,; \quad \tau_{yz} = \tau_{zy}\,; \quad \tau_{zx} = \tau_{xz}.$$

and, therefore independent unknown components are only six $\sigma_x, \sigma_y, \sigma_z, \tau_{xy}, \tau_{yz}, \tau_{zx}$. The equations for determination, of σ_z have already been given. The corresponding equations for other components are :

$$\sigma_x = \frac{3Q}{2\pi}\left[\frac{x^2 z}{R^5} + \frac{(1-2\nu)}{3}\left\{\frac{1}{R\,(R+z)} - \frac{(2R+z)\,x^2}{R^3\,(R+z)^2} - \frac{z}{R^3}\right\}\right]$$

$$\sigma_y = \frac{3Q}{2\pi}\left[\frac{y^2 z}{R^5} + \frac{(1-2\nu)}{2}\left\{\frac{1}{R(R+z)} - \frac{(2R+z)\,y^2}{R^3(R+z)^2} - \frac{z}{R^3}\right\}\right]$$

$$\tau_{xy} = \frac{3Q}{2\pi}\left[\frac{xyz}{R^5} - \frac{(1-2\nu)}{3}\,\frac{(2R+z)xy}{R^3(R+z)^2}\right]$$

$$\tau_{yz} = \frac{3Q}{2\pi}\cdot\frac{yz^2}{R^5} \qquad \text{...(11.13)}$$

$$\tau_{zx} = \frac{3Q}{2\pi}\cdot\frac{xz^2}{R^5}$$

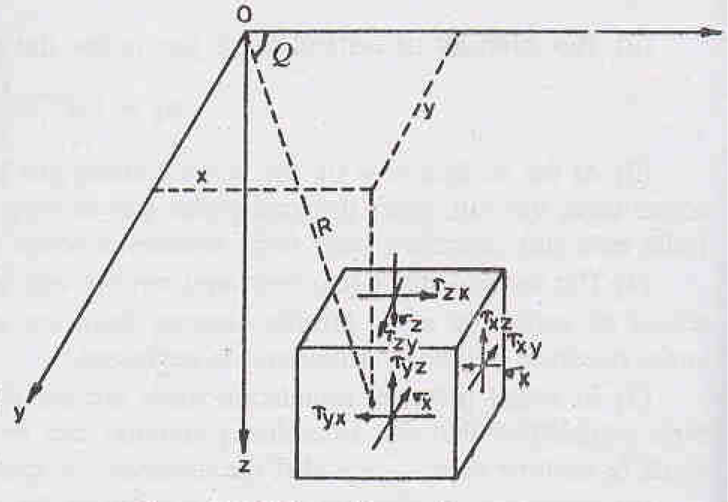


Fig. 11.4. Different Stress Components.

It may be noted that σ_x, σ_y and τ_{xy} depend upon Poisson's ratio.

Cylindrical Coordinates. Sometimes, it is more convenient to use cylindrical coordinates (r, θ, z) instead of cartesian coordinates (x, y, z). The Boussinesq solution in terms of cylindrical coordinates is as under (Fig. 11.5).

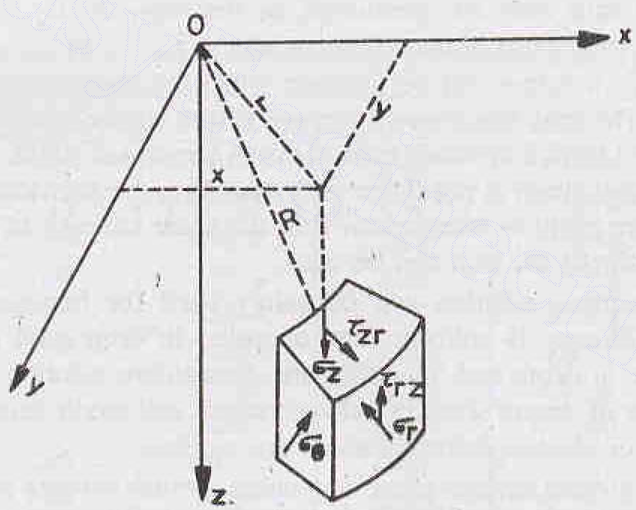


Fig. 11.5. Cylindrical Coordinates.

Vertical stress, $\sigma_z = \dfrac{3Q}{2\pi}\cdot\dfrac{z^3}{R^5}$

Radial stress, $\sigma_r = \dfrac{Q}{2\pi}\left[\dfrac{3zr^2}{R^5} - \dfrac{(1-2\nu)}{R(R+z)}\right]$ ...(11.14)

Tangential stress, $\sigma_\theta = \dfrac{Q}{2\pi}(1-2\nu)\left[\dfrac{1}{R(R+z)} - \dfrac{z}{R^3}\right]$

Shear stress, $\tau_{rz} = \dfrac{3Q}{2\pi}\cdot\dfrac{rz^2}{R^5}$

Shear stresses $\tau_{r\theta} = \tau_{z\theta} = 0$

where $R = \sqrt{r^2 + z^2}$, as before.

11.6. ISOBAR DIAGRAM

An isobar is a curve joining the points of equal stress intensity. In other words, an isobar is a contour of equal stress. An isobar is a spatial curved surface of the shape of an electrical bulb or an onion. The curved surface is symmetrical about the vertical axis passing through the load point.

The isobar of a particular intensity can be obtained using Eq. 11.10. The calculations are shown in a tabular form. Table 11.5 gives calculations for an isobar of intensity 0.1 Q per unit area.

From Eq. 11.10, $\sigma_z = I_B \dfrac{Q}{z^2}$

Taking $\sigma_z = 0.1\,Q$,

$$0.1\,Q = I_B \cdot \frac{Q}{z^2}$$

or $$I_B = 0.1\,z^2 \qquad \ldots(a)$$

For different depths z, the value of I_B is computed from Eq. (a), as shown in the second row of Table 11.5. The values of r/z for computed values of I_B are obtained from Eq. 11.11 or Table 11.4. Once the values of r/z have been determined, the radial distance r can be obtained as shown in table. It may be observed that r is zero at the load point, and it attains a maximum value at $r/z = 0.75$ and again decreases. As the isobar is symmetrical about the load axis, the other half can be drawn from symmetry. The shape of an isobar approaches a leminscate curve (not circle). Fig. 11.6 shows the pressure bulb of intensity 0.1 Q.

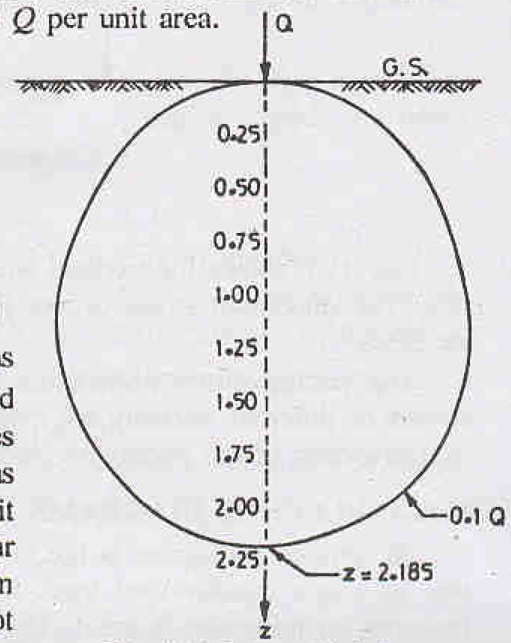


Fig. 11.6. Isobar of 0.1 Q

In the same manner, the isobars of other intensity 0.2 Q, 0.3 Q, etc. can be drawn. Obviously, the isobars of higher intensity shall lie within the isobar of 0.1 Q.

Table 11.5. Calculations for Isobar of 0.1 Q

Depth z	0.25	0.50	0.75	1.0	1.25	1.50	1.75	2.00	2.185
I_B	0.00625	0.25	0.05625	0.10	0.1562	0.225	0.3062	0.400	0.4775
r/z	2.16	1.50	1.16	0.93	0.75	0.59	0.44	0.27	0.000
r	0.54	0.75	0.87	0.93	0.938	0.885	0.770	0.540	0.000

Isobars are useful for determining the effect of the load on the vertical stresses at various points. The zone within which the stresses have a significant effect on the settlement of structures is known as the ***pressure bulb***. It is generally assumed that an isobar of 0.1 Q forms a pressure bulb. The area outside the pressure bulb is assumed to have negligible stresses.

11.7. VERTICAL STRESS DISTRIBUTION ON A HORIZONTAL PLANE

The vertical stresses at various points on a horizontal plane at a particular depth z can be obtained using Eq. 11.10. Let us determine the stresses at a depth of 2 m. Therefore,

$$\sigma_z = I_B \cdot (Q/z^2) = I_B \cdot (Q/4) = 0.25\,I_B\,Q.$$

The value of σ_z are computed (see Table 11.6) for different values of r/z, after obtaining I_B from Eq. 11.11 or Table 11.4.

Table 11.6. Calculation for vertical stress at z = 2m

r	0	0.50	1.00	1.50	2.00	2.50	3.00	4.00
r/z	0	0.25	0.50	0.75	1.00	1.25	1.50	2.00
I_B	0.4775	0.4103	0.2733	0.1565	0.0844	0.0454	0.0251	0.0085
σ_z	0.1194 Q	0.1026 Q	0.0683 Q	0.0390 Q	0.0211 Q	0.0113 Q	0.0063 Q	0.0021 Q

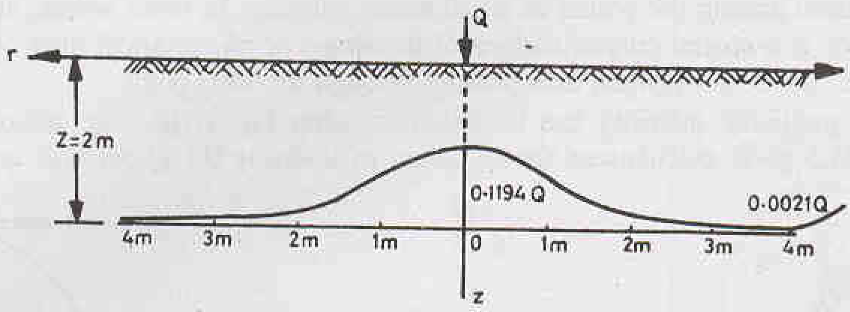


Fig. 11.7. Vertical Stress on a horizontal plane.

Fig. 11.7 shows the vertical stress distribution diagram. The diagram is symmetrical about the vertical axis. The maximum stress occurs just below the load (r = 0), and it decreases rapidly as the distance r increases.

The vertical stress distribution diagram on a horizontal plane can also be obtained graphically if the isobars of different intensity are available. The horizontal plane is drawn on the isobars diagram. The points of intersection of the horizontal plane with the isobar of a particular intensity give that vertical stress.

11.8. INFLUENCE DIAGRAMS

An influence diagram is the vertical stress distribution diagram on a horizontal plane at a given depth, due to a unit concentrated load. In Fig. 11.7, if the concentrated load Q is taken as unity, the diagram becomes an influence diagram. The influence diagrams are useful for determination of the vertical stress at any point on that horizontal plane due to a number of concentrated loads applied at the ground surface.

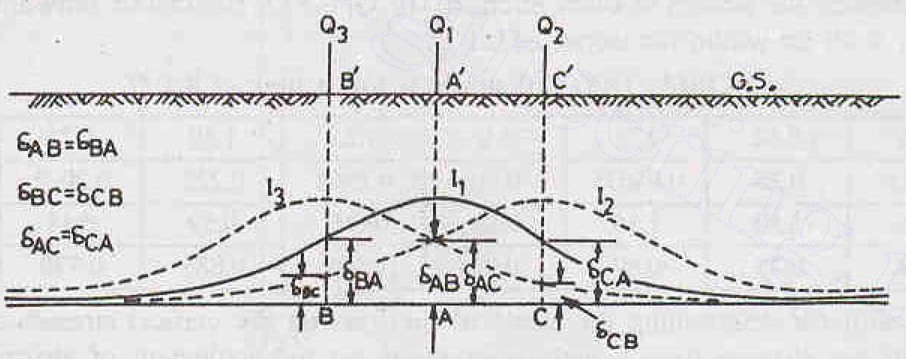


Fig. 11.8. Influence Diagrams.

Fig. 11.8 shows three influence diagrams, marked I_1, I_2 and I_3, due to unit loads applied at three points A', C' and B' on the ground surface. The stress at any point A on the horizontal plane at depth z due to three loads Q_1, Q_2 and Q_3 is given by

$$(\sigma_z)_A = Q_1\,\sigma_{AA} + Q_2\,\sigma_{AB} + Q_3\,\sigma_{AC} \quad \text{...(11.15)}$$

where σ_{AA} = vertical stress at A due to unit load at A'

σ_{AB} = vertical stress at A due to unit load at B'

and σ_{AC} = vertical stress at A due to unit load at C'

The values of σ_{AA}, σ_{AB} and σ_{AC} can be abtained from the influence diagram I_1, I_3 and I_2.

The computation work is considerably simplified using the reciprocal theorem, according to which

$$\sigma_{AB} = \sigma_{BA}, \quad \sigma_{AC} = \sigma_{CA} \quad \text{and} \quad \sigma_{BC} = \sigma_{CB}$$

where the first suffix denotes the point where the stress is required and the second suffix gives the point above which the load is applied. Accordingly, Eq. 11.15 can be written as

$$(\sigma_z)_A = Q_1\,\sigma_{AA} + Q_2\,\sigma_{BA} + Q_3\,\sigma_{CA} \quad \text{...(11.16)}$$

where σ_{AA} = vertical stress at A due to unit load at A',

σ_{BA} = vertical stress at B due to unit load at A',

and σ_{CA} = vertical stress at C due to unit load at A'.

Therefore, there is no need of drawing three influence diagrams in this case. Only one influence diagram (I_1) with unit load at A' is sufficient. The values of σ_{BA} and σ_{CA} are determined from I_1 diagram below the load points B' and C'.

If the stresses at any other point, say point B, are required, then the influence line for load above that point (B' in this case) would be drawn. Alternatively, the influence line diagram I_1 can be traced on a paper and placed in such a way that its axis of symmetry passes through the point B'.

11.9. VERTICAL STRESS DISTRIBUTION ON A VERTICAL PLANE

The vertical stress distribution on a vertical plane at a radial distance of r can be obtained using Eq. 11.10. In this case, the radial distance r is constant and the depth changes. The values of r/z are obtained for different depths z. The values of I_B are obtained from Eq. 11.11 or Table 11.4 and the stresses computed as $\sigma_z = (I_B/z^2)\,Q$. Table 11.7 shows the calculations for vertical stresses on a vertical plane at r = 1 m.

Table 11.7. Calculations of Vertical stresses at r = 1m

z	0.25	0.50	1.00	1.50	2.00	2.50	5.00
r/z	4.0	2.0	1.00	0.667	0.50	0.40	0.20
I_B	0.0004	0.0085	0.0844	0.1904	0.2733	0.3294	0.4329
σ_z	0.0064 Q	0.0340 Q	0.0844 Q	0.0845 Q	0.0683 Q	0.0527 Q	0.017 Q

Fig. 11.9 shows the variation of vertical stresses on a vertical plane at r = 1 m. The vertical stresses are plotted horizontally along r-axis, and the depth, parallel to the z-axis. It may be noted that the vertical stress first increases and then decreases. The maximum vertical stress occurs at r/z = 0.817. This corresponds to the point of intersection of the vertical plane with the line drawn at 39°15′ to the vertical axis of the load.

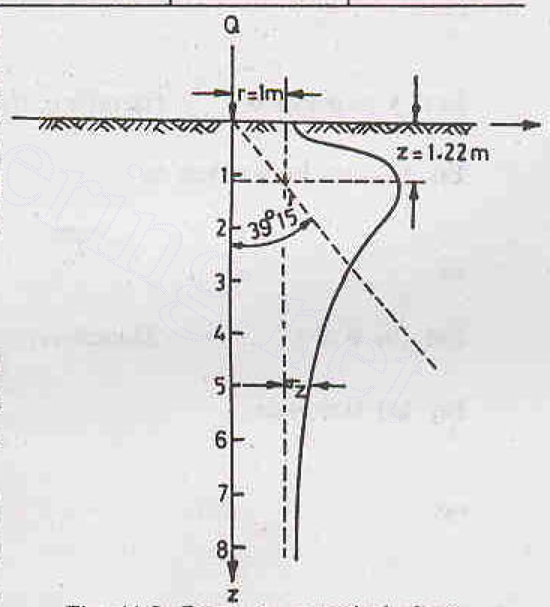


Fig. 11.9. Stress on a vertical plane

11.10. VERTICAL STRESSES DUE TO A LINE LOAD

The vertical stresses in a soil mass due to a vertical line load can be obtained using Boussinesq's solution. Let the vertical line load be of intensity q' per unit length, along the y-axis, acting on the surface of a semi-infinite soil mass, as shown in Fig. 11.10.

Let us consider the load acting on a small length δy. The load can be taken as a point load of $q'\,\delta y$ and Boussinesq's solution can be applied to determine the vertical stress at P (x, y, z). From Eq. 11.9,

$$\Delta\sigma_z = \frac{3(q'\,\delta y)}{2\pi}\cdot\frac{z^3}{(r^2+z^2)^{5/2}} \qquad \ldots(a)$$

The vertical stress at P due to the line load extending from $-\infty$ to $+\infty$ is obtained by integration,

$$\sigma_z = \frac{3q'z^3}{2\pi}\int_{-\infty}^{+\infty}\frac{dy}{(r^2+z^2)^{5/2}}$$

or

$$\sigma_z = \frac{3\,q'\,z^3}{2\,\pi}\int_{-\infty}^{+\infty}\frac{dy}{(x^2+y^2+z^2)^{5/2}} \qquad \ldots(b)$$

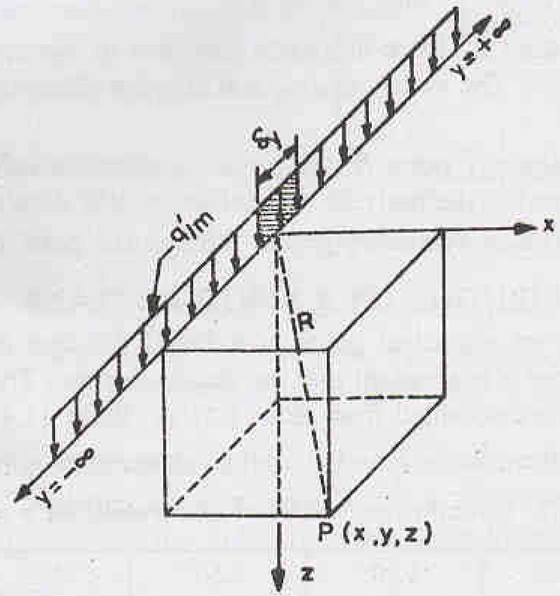


Fig. 11.10

Substituting $x^2 + z^2 = u^2$ in Eq. (*b*),

$$\sigma_z = \frac{3\,q'\,z^3}{2\,\pi} \int_{-\infty}^{+\infty} \frac{dy}{(u^2 + y^2)^{5/2}} \qquad \ldots(c)$$

Let $y = u \tan \theta$. Therefore, $dy = u \sec^2 \theta \, d\theta$.

Eq. (*c*) can be written as $$\sigma_z = \frac{3\,q'\,z^3}{2\,\pi} \int_0^{\pi/2} \frac{u \sec^2 \theta}{u^5 \sec^5 \theta}\, d\theta$$

or $$\sigma_z = \frac{3\,q'\,z^3}{2\,\pi u^4} \int_0^{\pi/2} \cos^3 \theta \, d\theta \qquad \ldots(d)$$

Let $\sin \theta = t$. Therefore, $\cos \theta \, d\theta = dt$

Eq. (*d*) becomes $$\sigma_z = \frac{3\,q'\,z^3}{\pi u^4} \int_0^1 (1 - t^2)\, dt$$

or $$\sigma_z = \frac{3\,q'\,z^3}{\pi u^4} \left[t - \frac{1}{3} t^3 \right]_0^1$$

or $$\sigma_z = \frac{3\,q'\,z^3}{\pi u^4} \times \frac{2}{3} = \frac{2\,q'\,z^3}{\pi\,(x^2 + z^2)^2}$$

or $$\sigma_z = \frac{2\,q'}{\pi z} \left[\frac{1}{1 + (x/z)^2} \right]^2 \qquad \ldots(11.17)$$

Eq. 11.17 can be used to determine the vertical stress at point *P*.

When the point *P* lies vertically below the line load, $x = 0$.

Therefore, $$\sigma_z = \frac{2\,q'}{\pi z} \qquad \ldots(11.18)$$

The expressions for the stresses σ_x and τ_{xz} can be obtained in a similar manner, starting from Eq. 11.13.

$$\sigma_x = \frac{2\,q'}{\pi}\,\frac{x^2 z}{(x^2 + z^2)^2} \qquad \ldots(11.19)$$

and
$$\tau_{xz} = \frac{2\,q'}{\pi}\cdot\frac{xz^2}{(x^2 + z^2)^2} \qquad \ldots(11.20)$$

11.11. VERTICAL STRESS UNDER A STRIP LOAD

The expression for vertical stress at any point P under a strip load can be developed from Eq. 11.17 of the line load. The expression will depend upon whether the pint P lies below the centre of the strip load or not.

Note. The length of the strip is very long. For convenience, unit length is considered.

(1) Point P below the centre of the strip

Fig. 11.11 shows a strip load of width B (= $2b$) and intensity q. Let us consider the load acting on a small elementary width dx at a distance x from the centre of the load. This small load of $q\,dx$ can be considered as a line load of intensity q'. From Eq. 11.17,

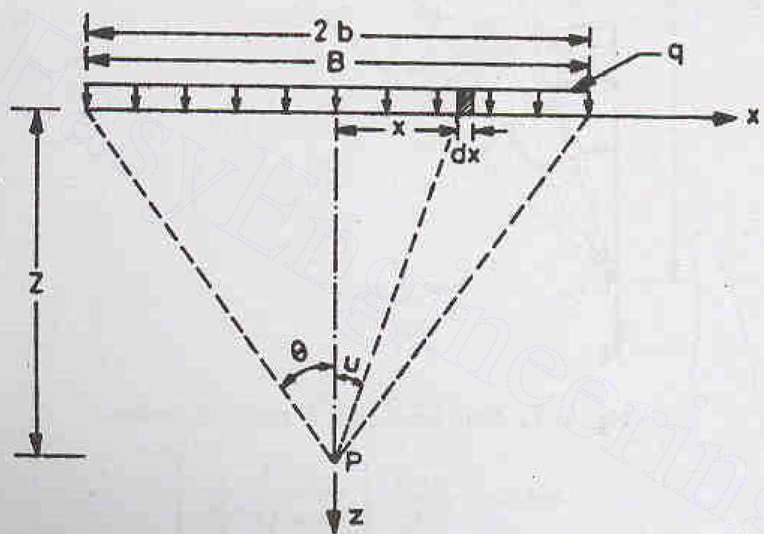


Fig. 11.11. Strip Load, point P below centre.

$$\Delta\sigma_z = \frac{2\,q\,dx}{\pi z}\left[\frac{1}{1 + (x/z)^2}\right]^2$$

The stress due to entire strip load is obtained as

$$\sigma_z = \frac{2\,q}{\pi z}\int_{-b}^{+b}\frac{1}{[1 + (x/z)^2]^2}\,dx \qquad \ldots(a)$$

Let $x/z = \tan u$. Therefore, $dx = z\sec^2 u\,du$

Substituting in Eq. (a),
$$\sigma_z = \frac{2q}{\pi z}\times 2\int_0^\theta \frac{z\sec^2 u}{(1 + \tan^2 u)^2}\,du$$

where $\theta = \tan^{-1}(b/z)$ = angle made by extemities of the strip at P.

or
$$\sigma_z = \frac{4q}{\pi}\int_0^\theta \cos^2 u\;du$$

or
$$\sigma_z = \frac{4q}{\pi}\int_0^\theta \left(\frac{1 + \cos 2u}{2}\right)du$$

or $$\sigma_z = \frac{q}{\pi}\,(2\theta + \sin 2\theta) \qquad ...(11.21)$$

(2) Point P not below the centre of the strip

Fig. 11.12 shows the case when the point P is not below the centre of the strip. The extremities of the strip make angles of β_1 and β_2 at P. As in the previous case, the load $q\,dx$ acting on a small length dx can be considered as a line load. The vertical stress at P is given by Eq. 11.17 as

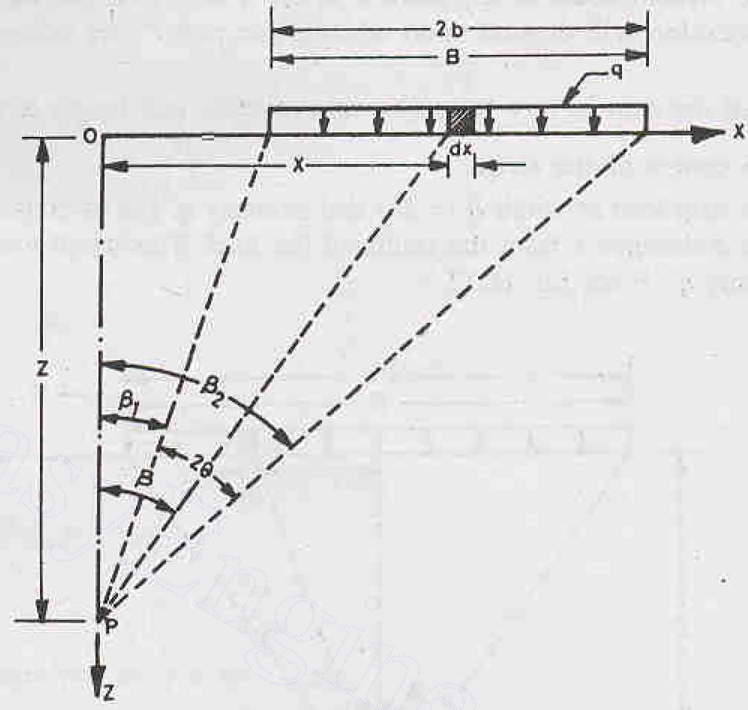


Fig. 11.12. Strip load, point P not at the centre.

$$\Delta\sigma_z = \frac{2q\,dx}{\pi z}\left[\frac{1}{1+(x/z)^2}\right]^2 \qquad ...(a)$$

Eq. (a) is simplified making the following substitution,

$$x = z\tan\beta \quad \text{or} \quad dx = z\sec^2\beta\,d\beta$$

Therefore $$\Delta\sigma_z = \frac{2q\,(z\sec^2\beta)\,d\beta}{\pi z}\left[\frac{1}{1+\tan^2\beta}\right]^2$$

or $$\Delta\sigma_z = \frac{2q}{\pi}\cos^2\beta\,d\beta$$

Integrating, $$\sigma_z = \frac{q}{\pi}\int_{\beta 1}^{\beta 2}(1+\cos 2\beta)\,d\beta$$

$$= \frac{q}{\pi}\left[\beta + \frac{1}{2}\sin 2\beta\right]_{\beta 1}^{\beta 2}$$

or $$\sigma_z = \frac{q}{\pi}\left[(\beta_2-\beta_1) + (\sin\beta_2\cos\beta_2 - \sin\beta_1\cos\beta_1)\right]$$

Substituting $\beta_2 - \beta_1 = 2\theta$,

$$\sigma_z = \frac{q}{\pi}\left[2\theta + (\sin\beta_2\cos\beta_2 - \sin\beta_1\cos\beta_1)\right] \qquad ...(b)$$

If $(\beta_1 + \beta_2) = 2\varphi$, it can be shown that

$$\sin \beta_2 \cos \beta_2 - \sin \beta_1 \cos \beta_1 = \sin 2\theta \cos 2\varphi$$

Therefore, Eq. (*b*) becomes

$$\sigma_z = \frac{q}{\pi}\,[2\theta + \sin 2\theta \cos 2\varphi] \quad \text{...(11.22)}$$

The expressions for σ_x and τ_{xz} can be likewise derived.

$$\sigma_x = \frac{q}{\pi}\,[2\theta - \sin 2\theta \cos 2\varphi] \quad \text{...(11.23)}$$

and
$$\tau_{xz} = \frac{q}{\pi}\,[\sin 2\theta \sin 2\varphi] \quad \text{...(11.24)}$$

It may be mentioned that Eqs. 11.22 to 11.24 are general equations which can be used even for the case when the point *P* is below the centre of the load.

In this case, $\beta_2 = -\beta_1 = \theta$

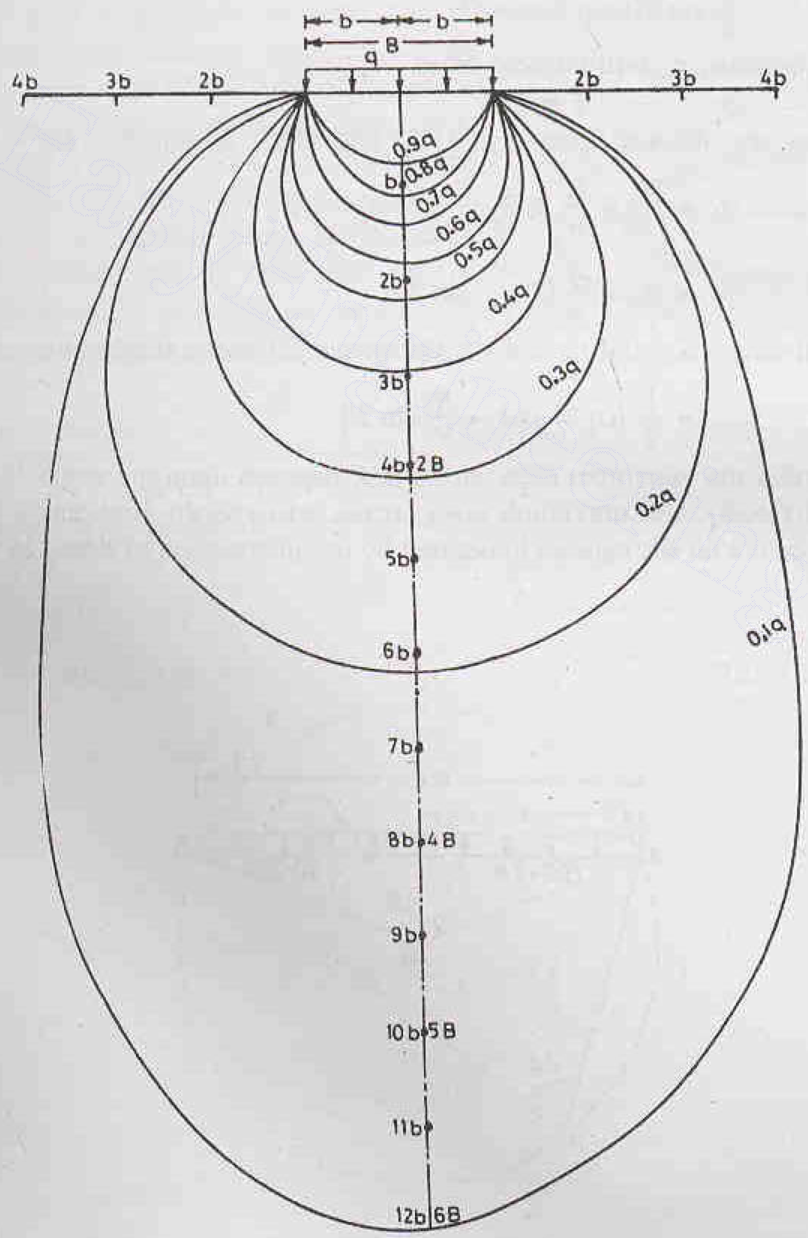


Fig. 11.13. Isobars of strip load.

and $\beta_1 + \beta_2 = 0$ or $\varphi = 0$

Therefore, Eq. 11.22 gives

$$\sigma_z = \frac{q}{\pi}(2\theta + \sin 2\theta) \qquad \text{...(same as Eq. 11.21)}$$

Eq. 11.22 can be used to determine isobars of different intensity due to strip load. Fig. 11.13 shows the isobars. The isobar of load intensity 0.1 q is at a depth of about 6 B below the load. Fig. 11.13 can be used for determination of vertical stresses at various points.

11.12. MAXIMUM SHEAR STRESSES AT POINTS UNDER A STRIP LOAD

The shear stress at any point *P* below a strip load is given by Eq. 11.24 as $\tau_{xz} = \frac{q}{\pi}\sin 2\theta \sin 2\varphi$. The planes on which the shear stresses are zero are known as principal planes. Therefore for principal planes, $\tau_{xz} = 0$.

or $$\frac{q}{\pi}\sin 2\theta \sin 2\varphi = 0$$

As q and θ cannot be zero, τ_{xz} will be zero when

$\sin 2\phi = 0$ or $2\varphi = 0$

The principal stresses are obtained from Eqs. 11.22 and 11.23, substituting $2\varphi = 0$.

$$\sigma_1 = \sigma_z = \frac{q}{\pi}(2\theta + \sin 2\theta) \qquad \text{...(11.25)}$$

$$\sigma_2 = \sigma_x = \frac{q}{\pi}(2\theta - \sin 2\theta) \qquad \text{...(11.26)}$$

The maximum shear stress is equal to half the difference of the principal stresses. Thus

$$\tau_{max} = \frac{1}{2}(\sigma_1 - \sigma_3) = \frac{q}{\pi}\sin 2\theta \qquad \text{...(11.27)}$$

Eq. 11.27 indicates that the maximum shear stress at *P* depends upon the angle 2θ subtended by the strip load at the point *P*. Obviously, the maximum shear stress will remain constant if the angle 2θ does not change. Let us draw a circle with the centre *O* obtained by the intersection of lines *OA* and *OB* making angles

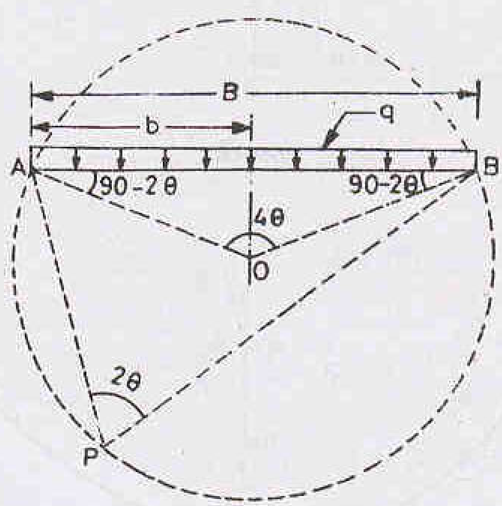


Fig. 11.14. Maximum shear stresses

(90–2θ) with the ends of the load, as shown in Fig. 11.14. As the angle subtended at the centre of circle is twice that at the circumference, the point *P* makes an angle 2θ. All the points on this circle will subtend an angle 2θ.

From Eq. 11.27, as the maximum shear stress depends on the angle 2θ, the circle is the locus of all points with shear stress equal to τ_{max}. The absolute maximum value of shear stress, $(\tau_{max})_{max}$ will occur when sin 2θ = 1 in Eq. 11.27. Thus

$$(\tau_{max})_{max} = \frac{q}{\pi}$$

The locus of $(\tau_{max})_{max}$ is a semi circle, which has the width of the loaded strip, *B*, as its diameter, In this case,

$$\sin 2\theta = 1 \quad \text{or} \quad 2\theta = 90°.$$

11.13. VERTICAL STRESSES UNDER A CIRCULAR AREA

The loads applied to soil surface by footings are not concentrated loads. These are usually spread over a finite area of the footing. It is generally assumed that the footing is flexible and the contact pressure is uniform. In other words, the load is assumed to by uniformly distributed over the area of the base of footings.

Let us determine the vertical stress at the point *P* at depth *z* below the centre of a uniformly loaded circular area (Fig. 11.15). Let the intensity of the load be *q* per unit area and *R* be the radius of the loaded area. Boussinesq's solution can be used to determine σ_z. The load on the elementary ring of radius *r* and width *dr* is equal to *q* (2 πr) *dr*. The load acts at a constant radial distance *r* from the point *P*. From Eq. 11.9,

$$\Delta\sigma_z = \frac{3(q \times 2\pi r dr)}{2\pi} \cdot \frac{1}{z^2} \cdot \frac{1}{[1 + (r/z)^2]^{5/2}}$$

The vertical stress due to entire load is given by

$$\sigma_z = 3qz^3 \int_0^R \frac{r\,dr}{(r^2 + z^2)^{5/2}} \qquad \ldots(a)$$

Let $r^2 + z^2 = u$. Therefore, $2r\,dr = du$

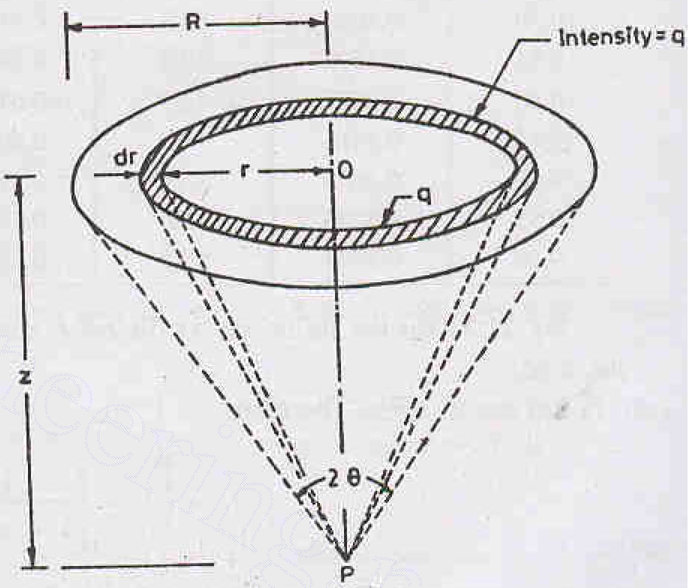


Fig. 11.15. Circular Load.

Eq. (*a*) becomes
$$\sigma_z = 3qz^3 \int_{z^2}^{(R^2+z^2)} \frac{du}{2u^{5/2}}$$

$$= \frac{3}{2} qz^3 \left(-\frac{2}{3}\right) \left[u^{-3/2}\right]_{z^2}^{R^2+z^2}$$

$$= -qz^3 \left[\frac{1}{(R^2+z^2)^{3/2}} - \frac{1}{(z^2)^{3/2}}\right]$$

$$= qz^3 \left[\frac{1}{z^3} - \frac{1}{(R^2+z^2)^{3/2}}\right]$$

or
$$\sigma_z = q\left[1 - \left\{\frac{1}{1+(R/z)^2}\right\}^{3/2}\right] \qquad \ldots(11.29)$$

or
$$\sigma_z = I_c \cdot q \qquad \ldots(11.30)$$

where I_c is the influence coefficient for the circular area, and is given by

$$I_c = \left[1 - \left\{\frac{1}{1 + (R/z)^2}\right\}^{3/2}\right] \quad \text{...(11.31)}$$

Table 11.8 gives the value of the influence coefficient I_c for different values of R/z.

Table 11.8. Influence Coefficients I_c for the Circular Area

R/z	I_c	R/z	I_c	R/z	I_c	R/z	I_c
0.00	0.0000	0.65	0.4109	1.30	0.7734	1.95	0.9050
0.05	0.0037	0.70	0.4502	1.35	0.7891	2.00	0.9106
0.10	0.0148	0.75	0.4880	1.40	0.8036	2.50	0.9488
0.15	0.0328	0.80	0.5239	1.45	0.8170	3.00	0.9684
0.20	0.0571	0.85	0.5577	1.50	0.8293	3.50	0.9793
0.25	0.0869	0.90	0.5893	1.55	0.8407	4.00	0.9857
0.30	0.1286	0.95	0.6189	1.60	0.8511	5.00	0.9925
0.35	0.1592	1.00	0.6465	1.65	0.8608	6.00	0.9956
0.40	0.2079	1.05	0.6720	1.70	0.8697	7.00	0.9972
0.45	0.2416	1.10	0.6956	1.75	0.8779	8.00	0.9981
0.50	0.2845	1.15	0.7175	1.80	0.8855	9.00	0.9987
0.55	0.3273	1.20	0.7376	1.85	0.8925	10.00	0.9990
0.60	0.3695	1.25	0.7562	1.90	0.8990	∞	1.0000

Eq. 11.31 for the influence coefficient I_c can be written in terms of the angle 2θ subtended at point P by the load.

Let $\tan\theta = R/z$. Therefore,

$$I_c = \left[1 - \left\{\frac{1}{1 + \tan^2\theta}\right\}^{3/2}\right]$$

or

$$I_c = 1 - (\cos^2\theta)^{3/2} = 1 - \cos^3\theta \quad \text{...(11.32)}$$

Eq. 11.32 indicates that as θ tends to 90°, the value of I_c approaches unity. In other words, when a uniformly loaded area tends to by very large in comparison with the depth z, the vertical stress at the point P is approximately equal to q.

When the point P is not below the centre of the load, analysis becomes complicated and is outside the scope of this text. In that case, the isobars shown in Fig. 11.16 can be used to determine the vertical stress at any point. It may be noted that the isobar of $0.1q$ cuts the axis of the load at a depth of about $4R$ (= $2D$) below the loaded area. The zone within which the stresses is indicated by this isobar, as mentioned above, is known as the *bulb of pressure*. The reader should compare this pressure bulb with that below the strip load, which is much deeper.

11.14. VERTICAL STRESS UNDER A CORNER OF A RECTANGULAR AREA

The vertical stress under a corner of a rectangular area (Fig. 11.17) with a uniformly distributed load of intensity q can be obtained from Boussinesq's solution. From Eq. 11.9, the stress at depth z is given by, taking $dQ = q\ dA = q\ dx\ dy$,

$$\Delta\sigma_z = \frac{3\,(q\,dx\,dy)\,z^3}{2\pi}\ \frac{1}{(x^2 + y^2 + z^2)^{5/2}}$$

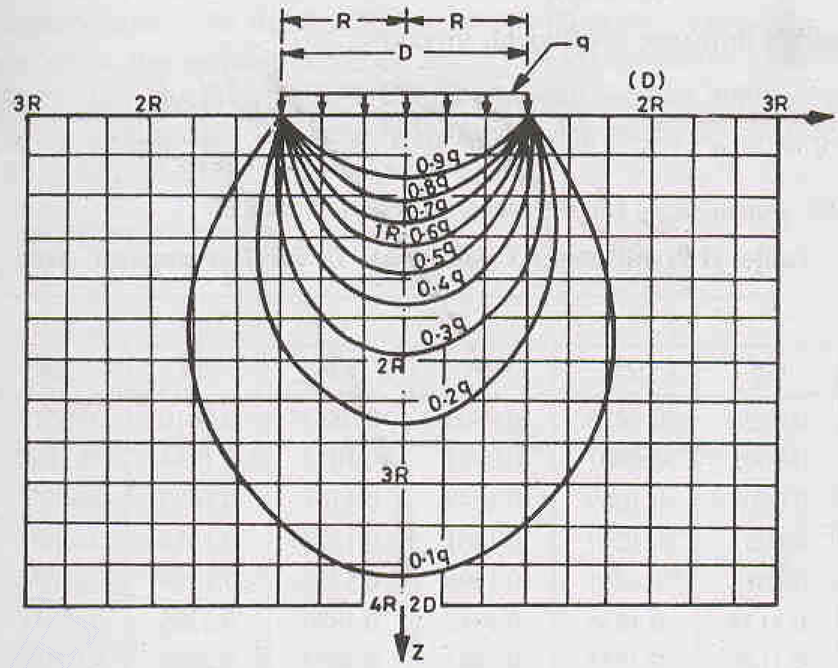


Fig. 11.16. Isobars for circular loaded area.

By integration, $$\sigma_z = \frac{3\,qz^3}{2\pi}\int_0^L\int_0^B \frac{q\,dx\,dy}{(x^2+y^2+z^2)^{5/2}}$$

Although the integral is quite complicated, Newmark was able to perform it The results were presented as follows:

$$\sigma_z = \frac{q}{2\pi}\left[\frac{mn}{\sqrt{m^2+n^2+1}}\cdot\frac{m^2+n^2+2}{m^2+n^2+m^2n^2+1}+\sin^{-1}\left(\frac{mn}{\sqrt{m^2+n^2+m^2n^2+1}}\right)\right] \quad \ldots(11.33)$$

where $m = B/z$ and $n = L/z$

The values of m and n can be interchanged without any effect on the values of σ_z. Eq. 11.33 can be expressed as

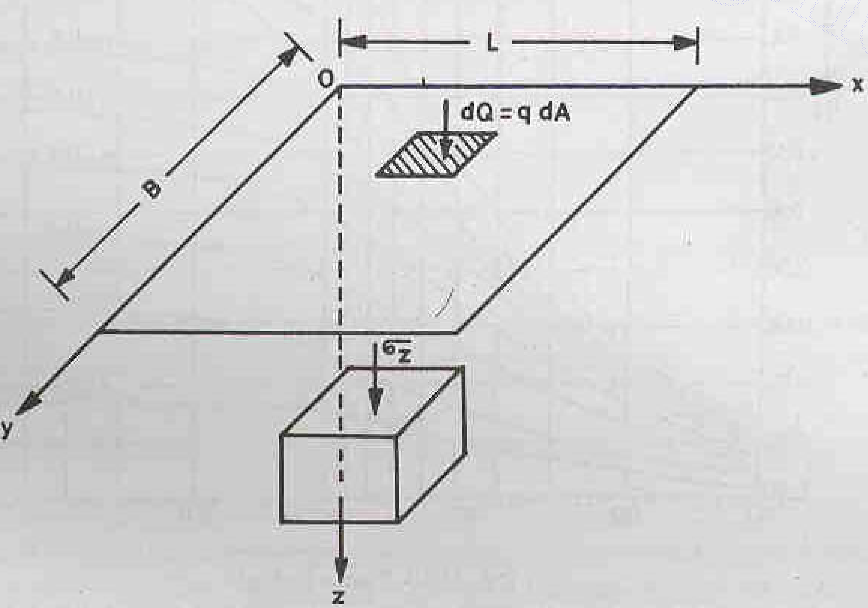


Fig. 11.17. Vertical stress under corner.

$$\sigma_z = I_N q \qquad \ldots(11.34)$$

where I_N is Newmark's influence coefficient, given by

$$I_N = \frac{1}{2\pi}\left[\frac{mn}{\sqrt{m^2+n^2+1}}\cdot\frac{m^2+n^2+2}{m^2+n^2+m^2n^2+1}+\sin^{-1}\left(\frac{mn}{\sqrt{m^2+n^2+m^2n^2+1}}\right)\right]$$

Table 11.9 gives the values of I_N for different values of m and n.

Table 11.9. Influence Coefficients I_N for Rectangular Area

m	*n*								
	0.2	*0.4*	*0.6*	*0.8*	*1.0*	*2.0*	*3.0*	*5.0*	*10.0*
0.2	0.0179	0.0328	0.0435	0.0504	0.0547	0.0610	0.0619	0.0620	0.0620
0.4	0.0328	0.0602	0.0801	0.0931	0.1013	0.1134	0.1150	0.1154	0.1154
0.6	0.0435	0.0801	0.1069	0.1247	0.1361	0.1533	0.1555	0.1561	0.1562
0.8	0.0504	0.0931	0.1247	0.1461	0.1598	0.1812	0.1841	0.1849	0.1850
1.0	0.0547	0.1013	0.1361	0.1598	0.1752	0.1999	0.2034	0.2044	0.2046
2.0	0.0610	0.1134	0.1533	0.1812	0.1999	0.2325	0.2378	0.2395	0.2399
3.0	0.0618	0.1150	0.1555	0.1841	0.2034	0.2378	0.2439	0.2461	0.2465
5.0	0.0620	0.1154	0.1561	0.1849	0.2044	0.2395	0.2461	0.2486	0.2491
10.0	0.0620	0.1154	0.1562	0.1850	0.2046	0.2399	0.2465	0.2491	0.2498

Fadum gave charts for determination of the influence factor I_N (Fig. 11.18). These charts can be used in a design office. The charts can also be used for determination of the vertical stress under a strip load, in which case the length tends to infinity and the curve for $n = \infty$ can be used.

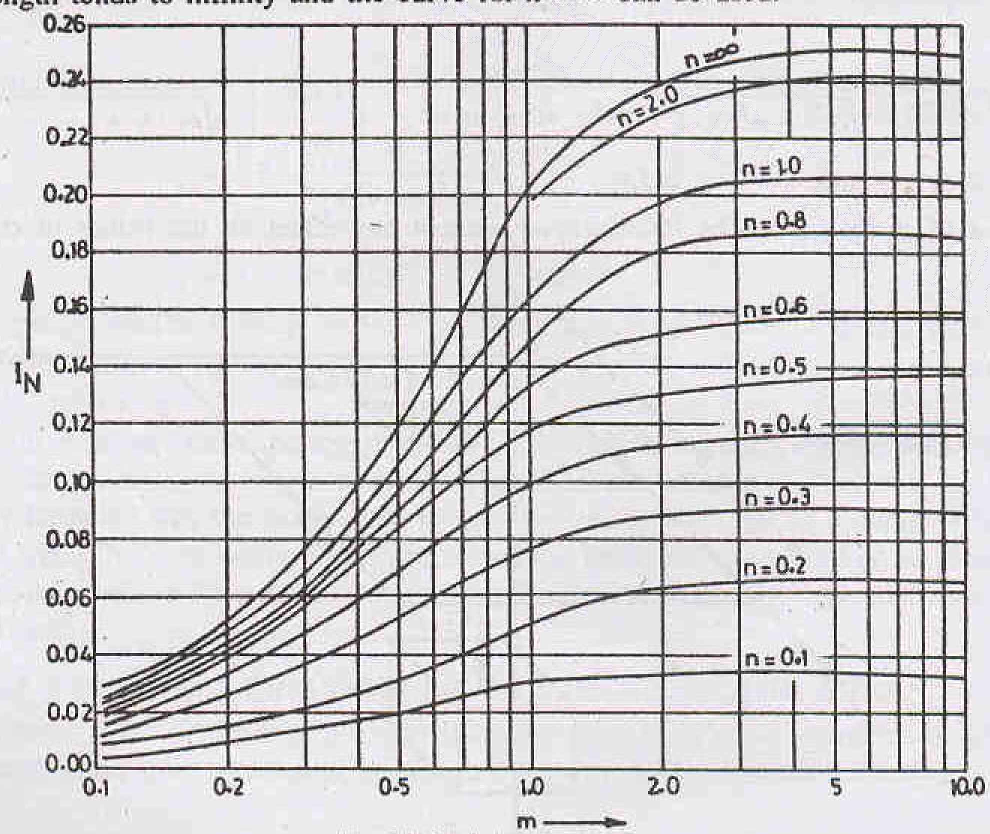


Fig. 11.18. Fadum's chart.

11.15. VERTICAL STRESS AT ANY POINT UNDER A RECTANGULAR AREA

The equations developed in the preceding section can also be used for finding the vertical stress at a point which is not located below the corner. The rectangular area is subdivided into rectangles such that each rectangles has a corner at the point where the vertical stress is required. The vertical stress is determined using the principle of superposition. The following three cases can occur.

(1) Point anywhere below the rectangular area. Fig. 11.19 (*a*) shows the location of the point *P* below the rectangular area *ABCD*. The given rectangle is subdivided into 4 small rectangles *AEPH, EBFP, HPGD* and *PFCG*, each having one corner at *P*. The vertical stress at *P* due to the given rectangular load is equal to that from the four small rectangles. Therefore, using Eq. 11.34,

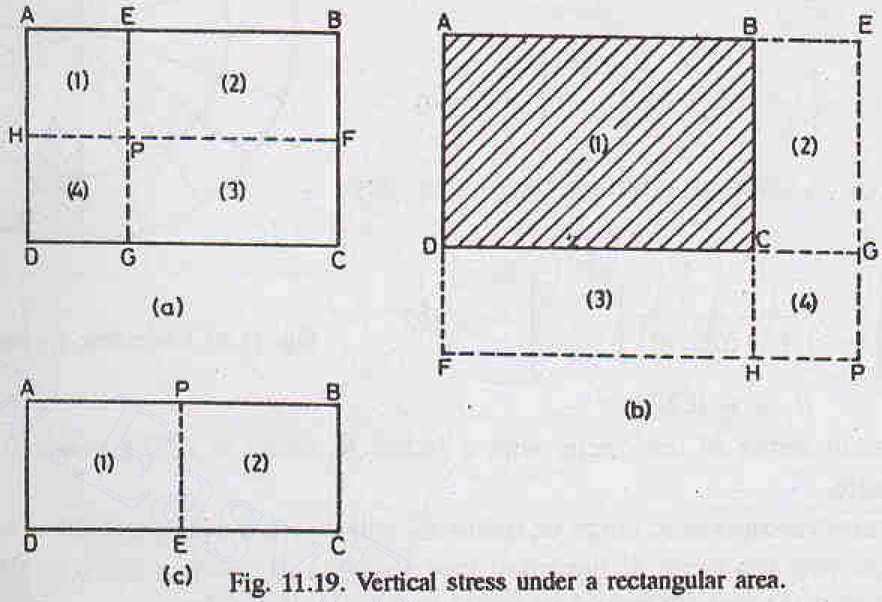


Fig. 11.19. Vertical stress under a rectangular area.

$$\sigma_z = q\,[(I_N)_1 + (I_N)_2 + (I_N)_3 + (I_N)_4] \qquad ...(11.35)$$

where $(I_N)_1$, $(I_N)_2$, $(I_N)_3$ and $(I_N)_4$ are Newmark's influence factors obtained from Table 11.9 for the four rectangles marked (1), (2), (3) and (4).

For the special case, when the point *P* is at the centre of the rectangle *ABCD*, all the four small rectangles are equal, and Eq. 11.35 becomes

$$\sigma_z = 4\,I_N \qquad ...(11.36)$$

where I_N is the influence factor for the small rectangle.

(2) Point outside the loaded area. Fig. 11.19 (*b*) shows the point *P* outside the loaded area *ABCD*. In this case, a large rectangle *AEPF* is drawn with its one corner at *P*.

Now rectangle *ABCD* = rectangle *AEPF* – rectangle *BEPH* – rectangle *DGPF* + rectangle *CGPH*

The last rectangle *CGPH* is given plus sign because this area has been deducted twice, once in rectangle *BEPH* and once in *DGPF*.

Therefore, the stress at *P* due to a load on rectangle *ABCD* is given by

$$\sigma_z = q\,[(I_N)_1 - (I_N)_2 - (I_N)_3 + (I_N)_4] \qquad ...(11.37)$$

where $(I_N)_1$, $(I_N)_2$, $(I_N)_3$ and $(I_N)_4$ are the influence coefficients for the rectangles *AEPF, BEPH, DGPF* and *CGPH*, respectively.

(3) Point below the edge of the loaded area. If the point *P* is below the edge of the loaded area *ABCD* (Fig. 11.19 *c*), the given rectangle is divided into two small rectangles *APED* and PBCE. In this case,

$$\sigma_z = q\,[(I_{N1}) + (I_N)_2] \qquad ...(11.38)$$

where $(I_N)_1$ and $(I_N)_2$ are influence coefficients for rectangles *APED* and *PBCE*, respectively.

11.16. NEWMARK'S INFLUENCE CHARTS

The methods for the determination of vertical stresses under a strip, a circular and a rectangular area have been discussed in the preceding sections. In practice, sometimes one has to find the vertical stresses under a uniformly loaded areas of other shapes. For such cases, Newmark's influence charts are extremely useful.

Newmark's chart is based on the concept of the vertical stress below the centre of the circular area, discussed in Sect. 11.13. Let us consider a uniformly loaded circular area of radius R_1, divided into 20 equal sectors (Fig. 11.20). The vertical stress at point P at depth z just below the centre of the loaded area due to load on one sector (hatched area (1)) will be (1/20) of that due to load on full circle. From Eq. 11.29,

$$\sigma_z = \frac{1}{20} q \left[1 - \left\{ \frac{1}{1 + (R_1/z)^2} \right\}^{3/2} \right] \quad \ldots(a)$$

If the vertical stress (σ_z) is given an arbitrary fixed value, say $0.005q$, Eq. (a) becomes

$$0.005q = \frac{q}{20} \left[1 - \left\{ \frac{1}{1 + (R_1/z)^2} \right\}^{3/2} \right] \quad \ldots(b)$$

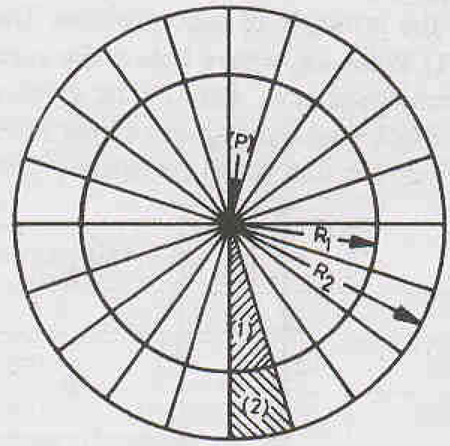


Fig. 11.20. Concentric circles for R_1 and R_2.

Solving Eq. (b), $\quad R_1/z = 0.270 \quad \ldots(11.39)$

Thus every one-twentieth sector of the circle, with a radius R_1 equal to 0.270 z, would give a vertical stress of 0.005 q at its centre.

Let us now consider another concentric circle of radius R_2 and divide it again into 20 equal sectors. Each larger sector is divided into two sub-areas. If the small area (marked 2) exerts a stress of 0.005 q at P, the vertical stress due to both area (1) and (2) would be equal to 2 × 0.005 q. Thus,

$$2 \times 0.005\, q = \frac{q}{20} \left[1 - \left\{ \frac{1}{1 + (R_2/z)^2} \right\}^{3/2} \right] \quad \ldots(c)$$

Solving, $R_2/z = 0.40$.

In other words, the radius of the second circle should be equal to 0.40 z.

Likewise, the radii of the third to the ninth circles can be determined. The values obtained are 0.52 z, 0.64 z, 0.77 z, 0.92 z, 1.11 z, 1.39 z and 1.91 z. The radius of $9\frac{1}{2}$ circle is 2.54 z. The radius for the tenth circle R_{10} is given by

$$10 \times 0.005\, q = \frac{q}{20} \left[1 - \left\{ \frac{1}{1 + (R_{10}/z)^2} \right\}^{3/2} \right]$$

or $\quad R_{10} = \infty$

Therefore, the tenth circle cannot be drawn.

Fig. 11.21 shows the complete Newmark's influence chart, in which only $9\frac{1}{2}$ circles have been drawn for z equal to the distance AB marked on the chart.

Use of Newmark's Chart. The chart can be used to determine the vertical stress at point P below the loaded area. A plan of the loaded area is drawn on a tracing paper to a scale such that the length AB(= 2 cm in this case) is equal to the depth (z) of the point P below the surface. For example, if the pressure is required at a depth of 1 m, the plan should be drawn to a scale of 2 cm = 1 m or R.F. = 1/50. The traced plan of the loaded area is placed over the Newmark chart such that the point P at which the pressure is required coincides with the centre of the chart. The vertical stress at point P is given by

$$\sigma_z = I \times n \times q \quad \ldots(11.40)$$

where I = influence coefficient (= 0.005 in this case),

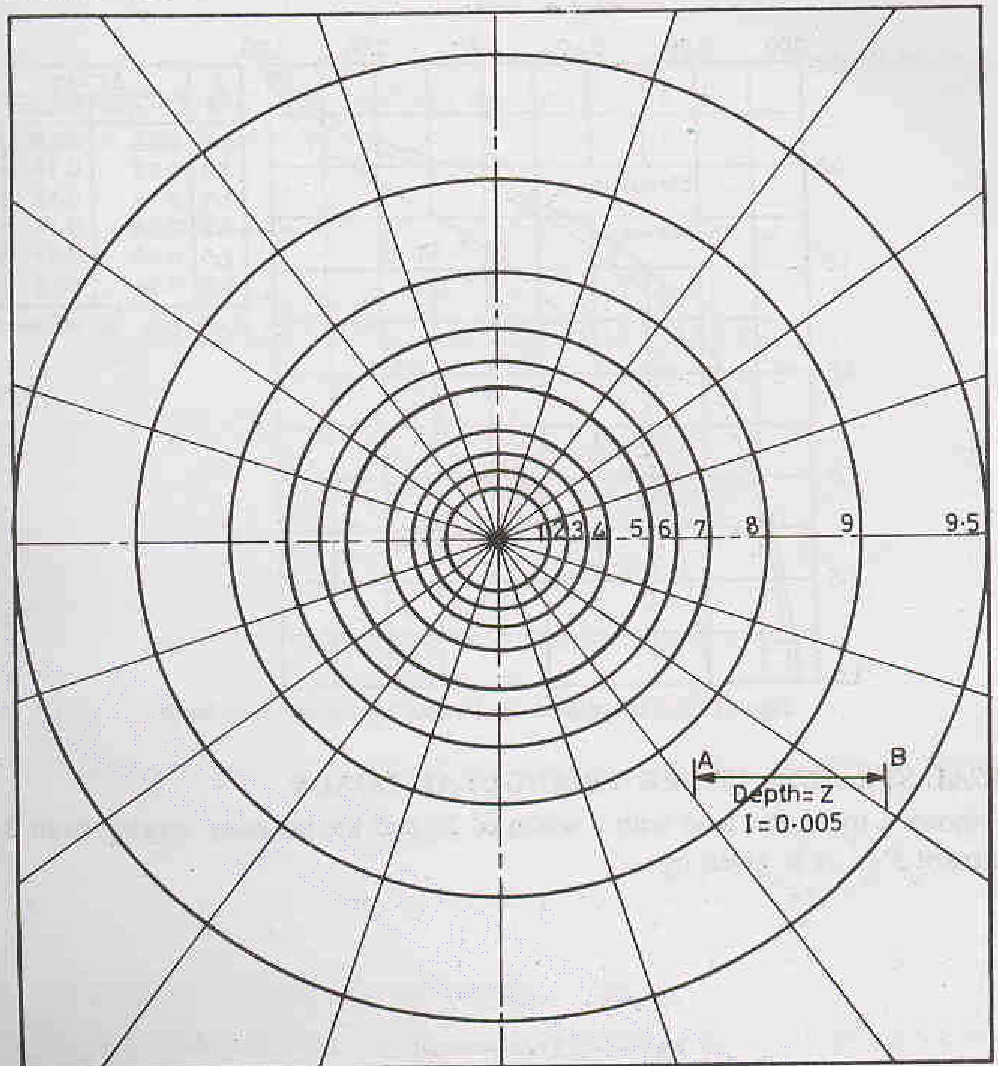


Fig. 11.21. Newmark's Charts.

n = number of small area units covered by the plan. Each area between two successive radial lines and two successive concentric circle is taken as one unit.

q = intensity of load.

The following points are worth noting:

(1) The fractions of the unit areas should also be counted and properly accounted for.

(2) If the plan of the loaded area extends beyonds the $9\frac{1}{2}$th circle. it may be assumed to approach the 10th circle for the purpose of counting the unit areas.

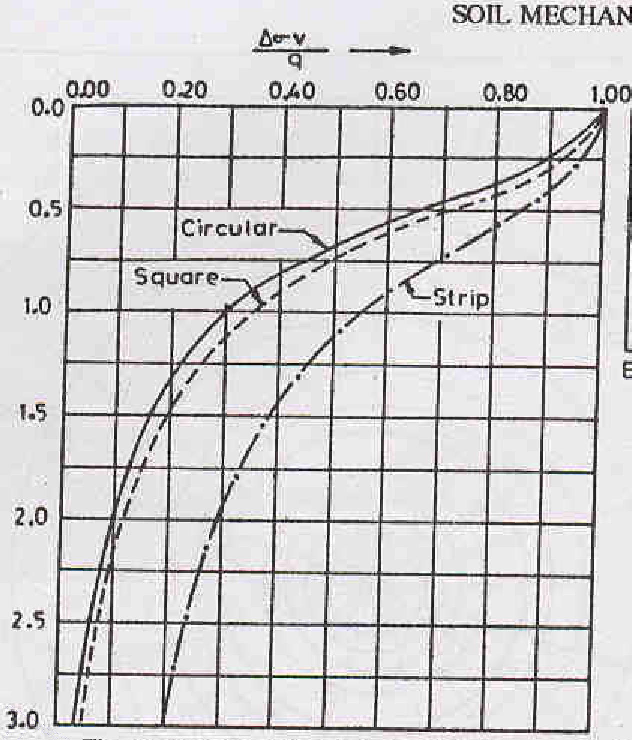
(3) The point P at which the vertical stress is required may be anywhere within or outside the loaded area.

(4) If the depth at which the stress is required is changed, a fresh plan is required such that the new depth is equal to the distance AB on the chart.

11.17. COMPARISON OF STRESSES DUE TO LOADS ON AREAS OF DIFFERENT SHAPES

The variation of vertical stress with depth depends upon the shape and size of the loaded area. Fig. 11.22 shows the variation of the vertical stress with depth below the centre of circular, square and strip loads.

The vertical stresses are equal to the load intensity at the surface and decrease rapidly with an increase in depth (z). In the case of circular and square loads, the vertical stress is about 10% of the surface load (q) at a depth of about $2B$. However, in the case of strip loads, the stresses are much greater. Even at $z = 3B$, the vertical stress is about 20% of the surface load (q). In other words, the pressure bulb in this case is much deeper.



Z/B	Δσv/q Circular	Square	Strip
0.5	0.65	0.70	0.85
1.0	0.28	0.34	0.55
1.5	0.14	0.18	0.39
2.0	0.08	0.14	0.31
2.5	0.06	0.07	0.25
3.0	0.04	0.05	0.20

B = Width or diameter

Fig. 11.22. Comparison of circular, square and strip loads.

11.18. VERTICAL STRESSES UNDER TRIANGULAR LOADS

Fig. 11.23 shows a triangular load with a width of $2b$ and the intensity varying from 0 to q. The vertical stress (σ_z) at a point $P(x, z)$ is given by

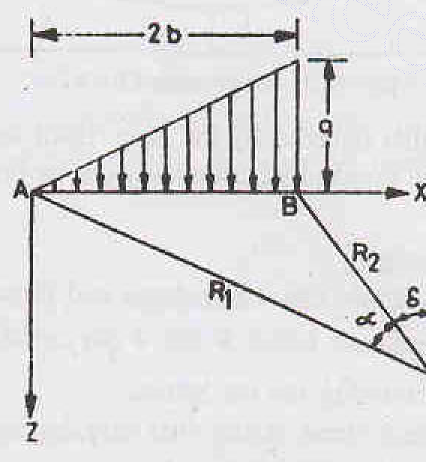


Fig. 11.23. Triangular load.

$$\sigma_z = \frac{q}{2\pi}\left[\frac{x}{b}\,\alpha - \sin 2\delta\right] \qquad \ldots(11.41)$$

where δ is the angle which the line PB marks with vertical, and α is the angle subtended by PA and PB at P.

If the point P is exactly below the end B, $x = 2b$ and $\delta = 0$. Therefore,

$$\sigma_z = \frac{q}{2\pi}\left(\frac{2b}{b}\,\alpha\right) = \frac{q\alpha}{\pi} \qquad \ldots(11.42)$$

The above equations can also be applied to the case when the intensity of the load increases linearly from zero at one end to a maximum q and then decreases to zero (Fig. 11.24).

For the load shown in Fig. 11.24 (*a*),

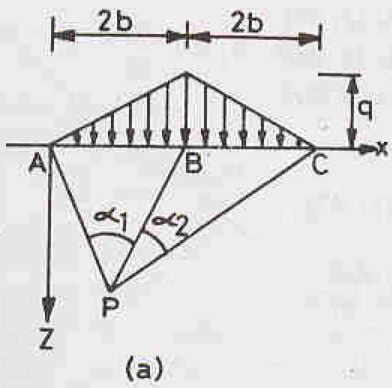


Fig. 11.24. Triangular load with maximum intensity at centre.

$$\sigma_z = \frac{q}{2\,\pi b}\,[2b\,(\alpha_1 + \alpha_2) + x\,(\alpha_1 - \alpha_2)] \qquad \text{...(11.43)}$$

When the point P is exactly below the point B, $\alpha_1 = \alpha_2 = \alpha$ and $x = 2b$. [Fig. 11.24 (*b*)]. Therefore,

$$\sigma_z = \frac{q}{2\,b\pi}\,[2b \times 2\alpha + 2b(\alpha - \alpha)]$$

or
$$\sigma_z = \frac{2\,q\alpha}{\pi} \qquad \text{...(11.44)}$$

11.19. VERTICAL STRESSES UNDER TRAPEZOIDAL LOADS

Fig. 11.25 (*a*) shows a trapezoidal load due to an embankment, which consists of a triangular load over width a and a uniform load of intensity q over width b. The vertical stress at point P is given by

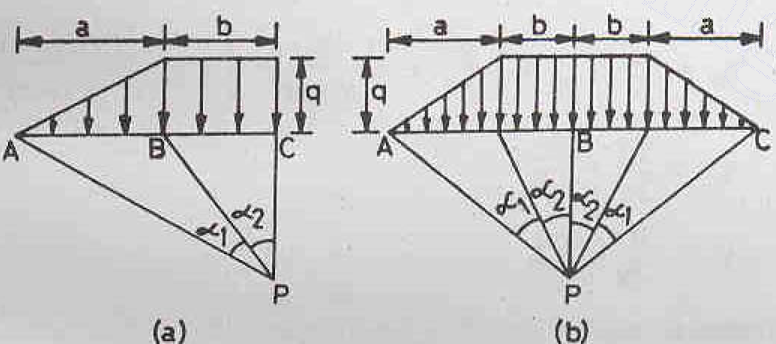


Fig. 11.25. Trapezoidal load.

$$\sigma_z = \frac{q}{\pi}\left[\left(\frac{a+b}{a}\right)(\alpha_1 + \alpha_2) - \frac{b}{a}\,\alpha_2\right]$$

or
$$\sigma_z = \frac{q}{\pi}\left[(\alpha_1 + \alpha_2) + \frac{b}{a}\,\alpha_1\right]$$

or
$$\sigma_z = \frac{q}{\pi a}\,[a\,(\alpha_1 + \alpha_2) + b\alpha_1] \qquad \text{...(11.45)}$$

Obviously, for the trapezoidal load shown in Fig. 11.25 (*b*), the vertical stress at P,

$$\sigma_z = \frac{2q}{\pi a}\left[a(\alpha_1 + \alpha_2) + b\alpha_1\right] \qquad \ldots(11.46)$$

11.20. STRESSES DUE TO HORIZONTAL LOAD

(*a*) **Line Load Q_1** Fig. 11.26 shows a horizontal load Q_1 per unit run acting on the soil surface shown by solid lines. In this case, it is more convenient to determine the radial stress (σ_r) and the tangential stress (σ_θ). The radial stress is given by

$$\sigma_r = \frac{2\,Q_1 \sin\theta}{r\,(2\alpha - \sin 2\alpha)} \qquad \ldots(11.47)$$

where θ = angle made by radial line r with the vertical

α = angle made by soil surface with the vertical

For horizontal ground surface, $\alpha = \pi/2$. Therefore,

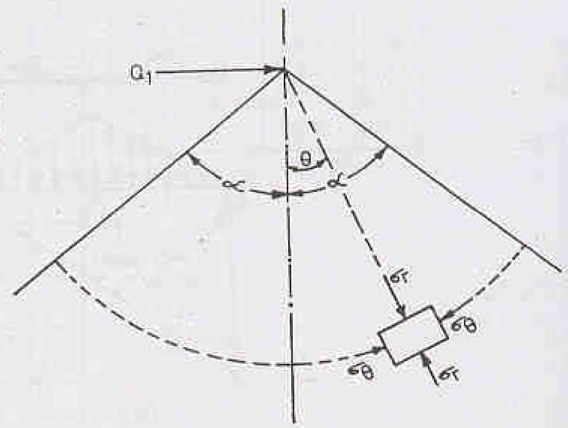


Fig. 11.26. Horizontal load.

$$\sigma_r = \frac{2Q_1 \sin\theta}{r\,(\pi - 0)}$$

$$\text{or} \qquad \sigma_x = \frac{2\,Q_1 \sin\theta}{\pi r} \qquad \ldots(11.48)$$

(*b*) **Concentrated Load Q.** In cartesian coordinates, the stresses due to a horizontal load Q can be written as

$$\sigma_z = \frac{3\,Q\,xz^2}{2\,\pi\,R^5}$$

$$\sigma_x = \frac{Q}{2\pi}\cdot\frac{x}{R^3}\times\left[\frac{3x^2}{R^2} - (1-2\nu) + \frac{(1-2\nu)R^2}{(R+z)^2}\left\{3 - \frac{x^2(3R+z)}{R^2(R+z)}\right\}\right]$$

$$\sigma_y = \frac{Q}{2\pi}\cdot\frac{x}{R^3}\times\left[\frac{3y^2}{R^2} - (1-2\nu) + \frac{(1-2\nu)R^2}{(R+z)^2}\left\{3 - \frac{y^2(3R+z)}{R^2(R+z)}\right\}\right]$$

$$\tau_{xy} = \frac{Q}{2\pi}\cdot\frac{y}{R^3}\times\left[\frac{3x^2}{R^2} + \frac{(1-2\nu)R^2}{(R+z)^2}\left\{1 - \frac{x^2(3R+z)}{R^2(R+z)}\right\}\right]$$

$$\tau_{xz} = \frac{3Q}{2\pi}\cdot\frac{x^2 z}{R^5}$$

$$\tau_{yz} = \frac{3Q}{2\pi}\cdot\frac{xyz}{R^5} \qquad \ldots(11.49)$$

where x,y,z are coordinates of point P and R is the distance OP, as shown in Fig. 11.4.

This case is generally referred to as *Cerutti's problem.*

11.21. STRESSES DUE TO INCLINED LOAD

Fig. 11.27 shows an inclined load Q_2 per unit run acting on the soil surface. The radial stress at a point at an angle θ is given by

$$\sigma_r = \frac{2Q_2}{r}\left(\frac{\cos\beta\cos\theta}{2\alpha + \sin 2\alpha} + \frac{\sin\beta\sin\theta}{2\alpha - \sin 2\alpha}\right) \qquad \ldots(11.50)$$

where β = angle which the load Q_2 makes with vertical,

α = angle the soil surface makes with the vertical.

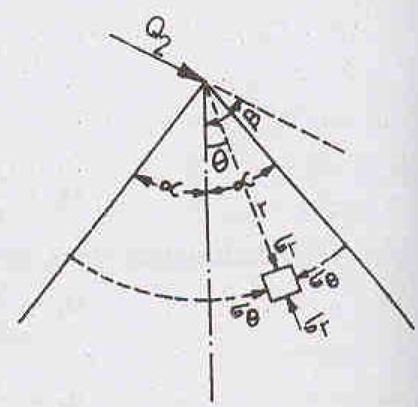


Fig. 11.27. Inclined load.

For the horizontal ground surface, $\alpha = \pi/2$. Thus

$$\sigma_r = \frac{2Q_2}{r}\left[\frac{\cos\beta\cos\theta}{\pi} + \frac{\sin\beta\sin\theta}{\pi}\right]$$

or
$$\sigma_r = \frac{2Q_2}{\pi r}\cos(\theta - \beta) \qquad \text{...(11.51)}$$

When the load is vertical, $\beta = 0$.

$$\sigma_r = \frac{2Q_2}{\pi r}\cos\theta \qquad \text{...[(11.51 (a)]}$$

11.22. WESTERGAARD'S SOLUTION

Boussinesq' solution assumes that the soil deposit is isotropic. Actual sedimentary deposits are generally anisotropic. There are generally thin layers of sand embedded in homogeneous clay strata. Westergaard's solution assumes that there are thin sheets of rigid materials sand-wiched in a homogeneous soil mass. These thin sheets are closely spaced and are of infinite rigidity and are, therefore, incompressible. These permit only downward displacement of the soil mass as a whole without any lateral displacement. Therefore, Westergaard's solution represents more closely the actual sedimentary deposits.

According to Westergaard, the vertical stress at a point P at a depth z below the concentrated load Q is given by

$$\sigma_z = \frac{c/2\pi}{[c^2 + (r/z)^2]^{3/2}} \cdot \frac{Q}{z^2} \qquad \text{...(11.52)}$$

where c depends upon the Poisson ratio (ν) and is given by

$$c = \sqrt{(1-2\nu)/(2-2\nu)}$$

For an elastic material, the value of ν varies between 0.0 to 0.50. For the case when ν is zero, Eq. 11.52 is simplified considerably, taking $c = 1/\sqrt{2}$,

$$\sigma_z = \frac{1}{\pi[1 + 2(r/z)^2]^{3/2}} \cdot \frac{Q}{z^2} \qquad \text{...(11.53)}$$

or
$$\sigma_z = I_w \frac{Q}{z^2} \qquad \text{...(11.54)}$$

where I_w is known as *Westergaard influence coefficient.*

$$I_w = \frac{1}{\pi[1 + 2(r/z)^2]^{3/2}} \qquad \text{...(11.55)}$$

The values of I_w are considerably smaller than the Boussinesq influence factor (I_B). Table 11.10 gives the values of I_w. The values of I_B are also given for comparison.

Table 11.10. Comparison of I_W and I_B

r/z	0.0	0.1	0.2	0.3	0.4	0.5	0.6
I_B	0.4775	0.4657	0.4329	0.3849	0.3295	0.2733	0.2214
I_W	0.3183	0.3090	0.2836	0.2483	0.2099	0.1733	0.1411

r/z	0.7	0.8	0.9	1.0	2.0	3.0	6.0
I_B	0.1762	0.1386	0.1083	0.0844	0.0085	0.0015	0.0001
I_W	0.1142	0.0925	0.0751	0.0613	0.0118	0.0038	0.0005

Fig. 11.28 shows the variation of I_B and I_W with r/z. The Westergaard influence factor is about 2/3

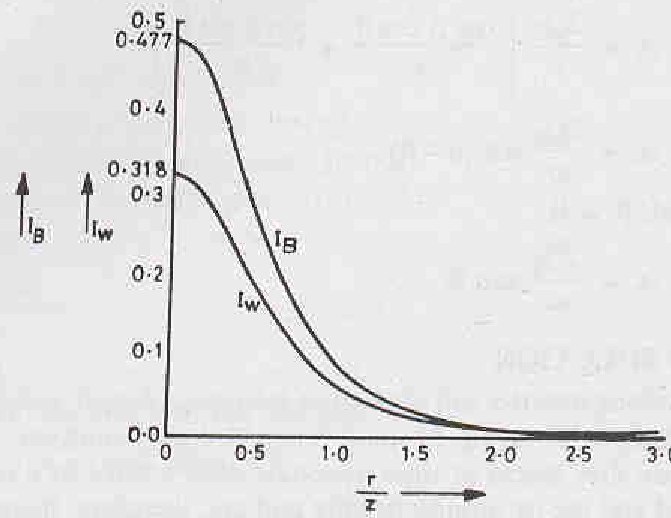


Fig. 11.28. Comparison of I_B and I_W.

of the Boussinesq values for small values of r/z. But for r/z more than 2.0, the Westergaard values are slightly greater. The effect of the load is negligibly small in both the cases when r/z is greater than about 2.0.

11.23. FENSKE'S CHARTS

Just like Newmark's Charts which are based on Boussinesq's solution, Fenske's Charts are based on Westergaard's solution. The Fenske chart can be prepared using Eq. 11.52.

$$\sigma_z = \frac{Q}{2\pi} \cdot \frac{1}{(cz)^2 [1 + (r/cz)^2]^{3/2}}$$

The above equation can be integrated to obtain the vertical stress (σ_z) below the centre of a uniform circular load of intensity q and radius R as was done for the Boussinesq solution for derivation of Eq. 11.29. In this case,

$$\sigma_z = q \left[1 - \left\{ \frac{1}{1 + (R/cz)^2} \right\}^{1/2} \right] \qquad \ldots(11.56)$$

If instead of the full circle, only 1/8th sector of the circle is considered, the stress is given by

$$\sigma_z = \frac{q}{8} \left[1 - \left\{ \frac{1}{1 + (R/cz)^2} \right\}^{1/2} \right] \qquad \ldots(11.57)$$

Eq. 11.57 is similar to the equation used for Newmark's chart, with one difference that the depth used here is the modified depth cz.

The radius R_1 of the first circle can be determined for a constant value of σ_z (say, 0.001 q). Thus

$$0.001\, q = \frac{q}{8} \left[1 - \left\{ \frac{1}{1 + (R_1/cz)^2} \right\}^{1/2} \right]$$

or $$\frac{R_1}{cz} = 0.127$$

or $$R_1 = 0.127\,(cz)$$

The modified depth cz is marked as the distance AB in Fig. 11.29.

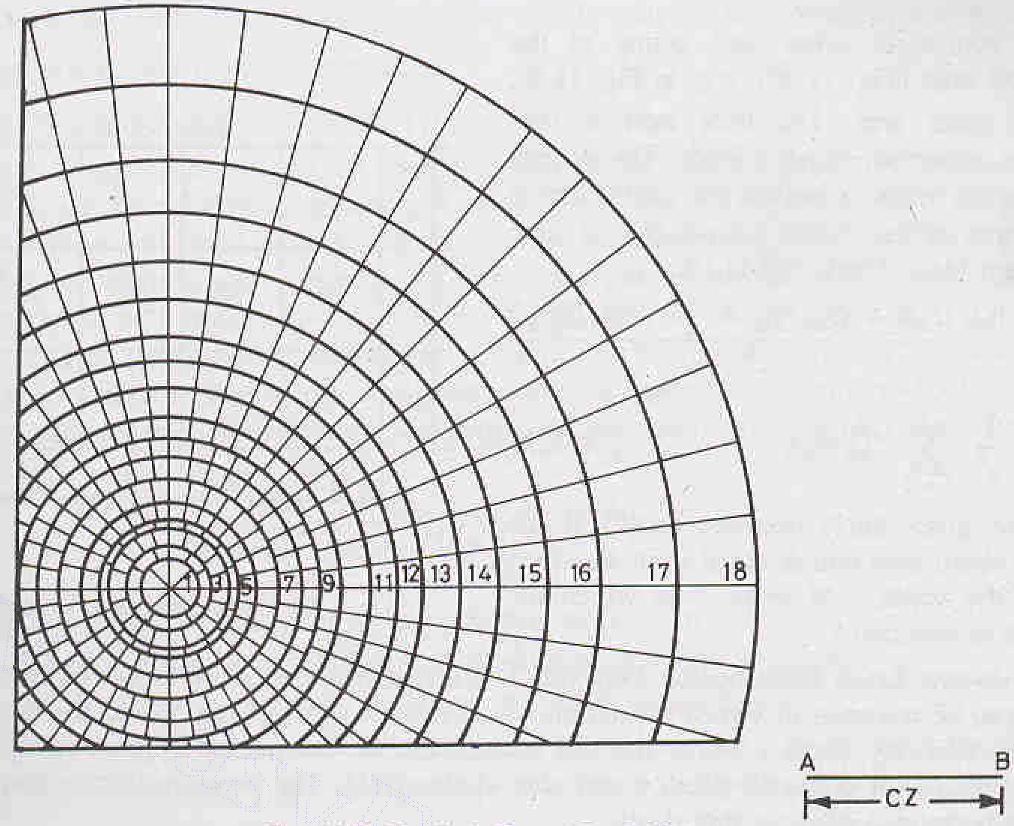


Fig. 11.29. Fenske's Chart (I = 0.001).

Likewise, the radii of other circles are determined. Unlike the Newmark chart, the radial divisions are also changed in Fenske's chart. There are 8 radial divisions for the first circle and 48 radial divisions for the 18th circle. The radii of the circular arcs and the number of radial divisions are so chosen that each influence area unit is approximately a square. Table 11.11 gives the values of $R/(cz)$ for different circles and their corresponding number of division.

The method of using the Fenske chart is similar to that for the Newmark chart. However, in this case the distance AB represents the modified depth cz. The plan of the loaded area is drawn on a tracing paper to a scale such that the distance AB is equal to c times the depth z of the point P at which the stress is required. For Poisson's ratio of zero, the value of c is equal to 0.707.

Table 11.11. Values of R/cz for Fenske's Chart

Circle No.	1	2	3	4	5	6	7	8	9
R/ (cz)	0.127	0.204	0.292	0.376	0.472	0.560	0.664	0.772	0.900
Divisions	8	12	20	24	32	32	40	40	48

Circle No.	10	11	12	13	14	15	16	17	18	19
R/(cz)	1.032	1.176	1.332	1.512	1.712	1.952	2.236	2.592	3.044	4.420
Divisions	48	48	48	48	48	48	48	48	48	48

11.24. APPROXIMATE METHODS

The methods discussed in the preceding sections are relatively more accurate, but are time- consuming. Sometimes, the engineer is interested to estimate the vertical stresses approximately for preliminary designs. The following methods can be used.

(1) Equivalent Point-Load Method. The vertical stress at a point under a loaded area of any shape can be determined by dividing the loaded area into small areas and replacing the distributed load on each small area by an equivalent point load acting at the centroid of the area (Fig. 11.30); *e.g.* in Fig. 11.30, $Q = qa^2$ for each area. The total load is thus converted into a number of point loads. The vertical stress at any point below or outside the loaded area is equal to the sum of the vertical stresses due to these equivalent point loads. Using Eq. 11.10,

$$\sigma_z = \frac{[Q_1 (I_B)_1 + Q_2 (I_B)_2 + \ldots + Q_n (I_B)_n]}{z^2}$$

$$\text{or } \sigma_z = \frac{1}{z^2} \sum_{i=1}^{n} Q_i (I_B)_i \qquad \ldots(11.58)$$

Eq. 11.58 gives fairly accurate results if the side a of the small area unit is equal to or less than one-third of the depth z of point P at which the vertical stress is required.

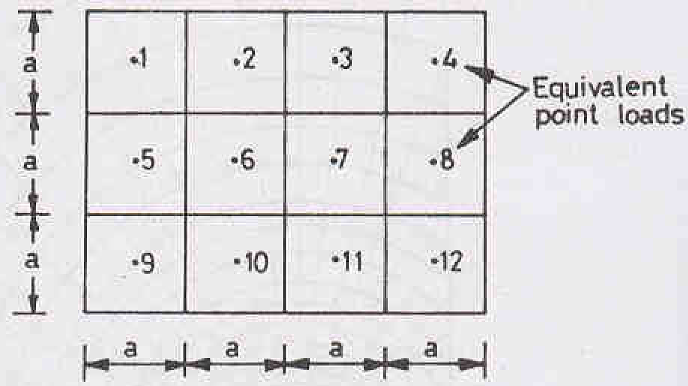


Fig. 11.30. Equivalent Point loads.

(2) Two-to-one Load Distribution Method. The actual distribution of load with the depth is complex. However, it can be assumed to spread approximately at a slope of two (vertical) to one (horizontal). Thus the vertical pressure at any depth z below the soil surface can be determined approximately by constructing a frustum of pyramid (or cone) of depth z and side slopes (2:1). The pressure distribution is assumed to be uniform on a horizontal plane at that depth.

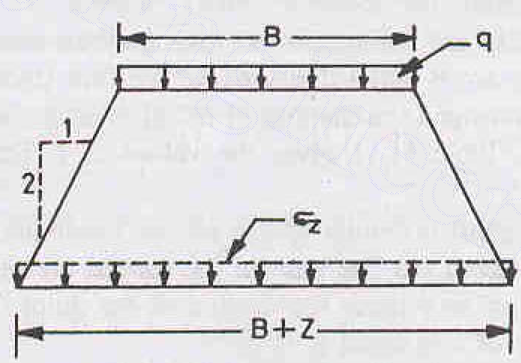


Fig. 11.31. Two-to-One Distribution.

The average vertical stress σ_z depends upon the shape of the loaded area, as given below (see Fig. 11.31).

(1) Square Area (B × B), $$\sigma_z = \frac{q B^2}{(B + z)^2} \qquad \ldots(11.59)$$

(2) Rectangular Area (B × L), $$\sigma_z = \frac{q (B \times L)}{(B + z)(L + z)} \qquad \ldots(11.60)$$

(3) Strip Area (width B, unit length), $$q_z = \frac{q \times (B \times 1)}{(B + z) \times 1} \qquad \ldots(11.61)$$

(4) Circular Area (diameter D), $$\sigma_z = \frac{q D^2}{(D + z)^2} \qquad \ldots(11.62)$$

The above method gives fairly accurate values of the average vertical stress if the depth z is less than 2.5 times the width of the loaded area. The maximum stress is generally taken as 1.5 times the average stress determined above.

(3) Sixty Degree Distribution. This method is similar to the preceding method. In this case, the pressure distribution is assumed along lines making an angle of 60° with the horizontal instead of $63\frac{1}{2}°$ (2 : 1). The method gives approximately the same results.

11.25. CONTACT PRESSURE DISTRIBUTION

The upward pressure due to soil on the underside of the footing is termed contact pressure. In the derivations of the preceding sections, it has been assumed that the footing is flexible and the contact pressure distribution is uniform and equal to q. Actual footings are not flexible as assumed. The actual distribution of the contact pressure depends upon a number of factors such as the elastic properties of the footing material and soil, the thickness of footings. In fact, it is a *soil- structure interaction* problem.

Borowicka (1936, 1938) studied the contact pressure distribution of uniformly loaded strips and circular footings resting on a semi-infinite elastic mass, assuming the base of the footing as frictionless. The analysis showed that the contact pressure distribution depends upon the relative rigidity (K_r) of the footing-soil system. The relative rigidity is defined as

$$K_r = \frac{1}{6}\,\frac{(1-\nu_s^2)}{(1-\nu_f^2)}\left(\frac{E_f}{E_s}\right)\cdot\left(\frac{t}{b}\right)^3 \qquad \text{...(11.63)}$$

where ν_s, ν_f = Poisson's ratios for soil and footing material, respectively,

E_s, E_f = Moduli of elasticity for soil and footing material, respectively.

$2b$ = width (or diameter) of footing, t = thickness of footing.

Fig. 11.32 shows the contact pressure distribution of circular and strip footings for different values of relative stiffness. For a perfectly rigid footing ($K_r = \infty$), the contact pressure is minimum at the centre, with

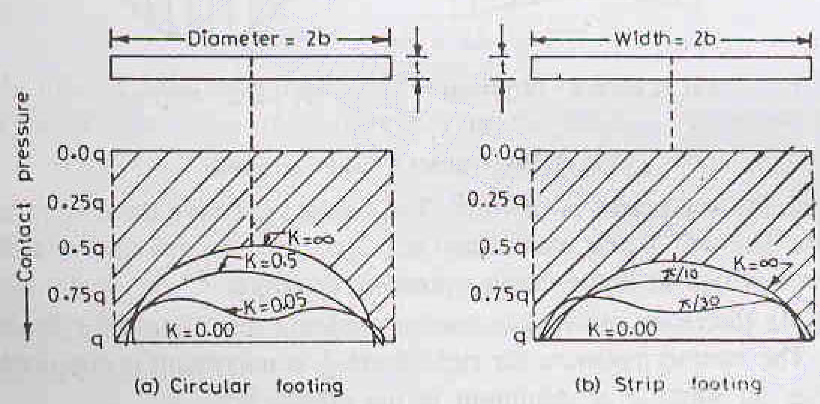


Fig. 11.32. Contact Pressure (Borowicka).

a value of about 0.5 q for the circular footing and 0.67 q for the strip footing. The contact pressure is very large at the edges. In fact, it tends to infinity. For purely flexible footings ($K_r = 0$), the contact pressure is uniform and equal to q.

Borowicka's results can be used to determine the contact pressure on a cohesive soil which behaves like an elastic soil mass. In a cohesionless soil, modulus of elasticity increases with depth due to an increase in confining pressure. Such soils are non-homogeneous.

Contact pressure on saturated clay. Fig. 11.33 shows the qualitative contact pressure distribution under flexible and rigid footings resting on a saturated clay and subjected to a uniformly distributed load q. When the footing is flexible, it deforms into the shape of a bowel, with the maximum deflection at the centre. The contact pressure distribution is uniform.

If the footing is rigid, the settlement is uniform. The contact pressure distribution is minimum at the centre and the maximum at the edges. The stresses at the edges in real soils cannot be infinite as theoretically determined for an elastic mass. In real soils, beyond a certain limiting value of stress, the plastic flow occurs and the pressure becomes finite.

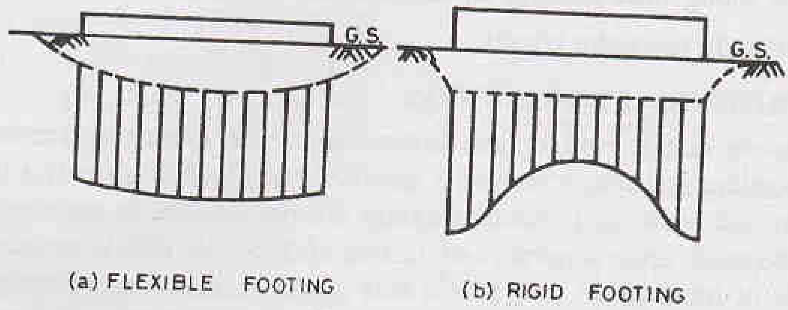


Fig. 11.33. Contact pressure on saturated clay.

Contact Pressure on sand. Fig. 11.34 shows the qualitative contact pressure distribution under flexible and rigid footings resting on a sandy soil and subjected to a uniformly distributed load q. In this case, the edges of the flexible footing undergo a larger settlement than at the centre. The soil at the centre is confined and, therefore, has a high modulus of elasticity and deflects less for the same contact pressure. The contact pressure is uniform.

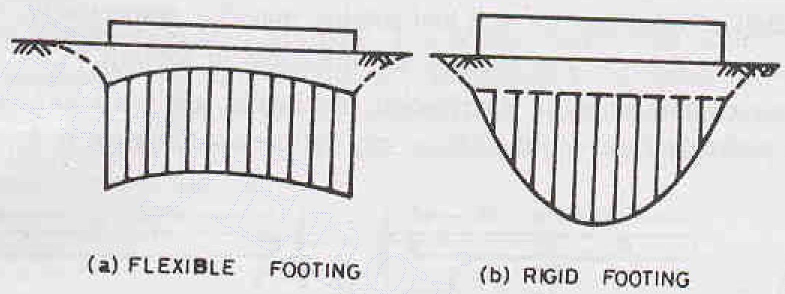


Fig. 11.34. Contact Pressure on Sand.

If the footing is rigid, the settlement is uniform. The contact pressure increases from zero at the edges to a maximum at the centre. The soil, being unconfined at edges, has low modulus of elasticity. However, if the footing is embedded, there would be finite contact pressure at edges.

Usual Assumption. As discussed above, the contact pressure distribution for flexible footings is uniform for both clay and sand. The contact pressure for rigid footing is maximum at the edges for footings on clay, but for the rigid footings on sand, it is minimum at the edges. For convenience, the contact pressure is assumed to be uniform for all types of footings and all types of soils (Fig. 11.35) if load is symmetric.

The above assumption of uniform pressure distribution will result in a slightly unsafe design for rigid footing on clays, as the maximum bending moment at the centre is underestimated. It will give a conservative design for rigid footings on sandy (cohesionless) soils, as the maximum bending moment is overestimated. However, at the ultimate stage just before the failure, the soil behaves as an elasto-plastic material (and not an elastic material) and the contact pressure is uniform, and the assumption is justified at the ultimate stage.

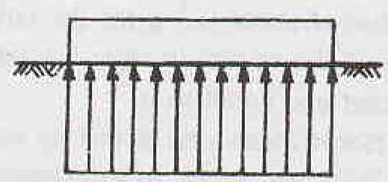


Fig. 11.35. Uniform contact Pressure.

11.26. LIMITATIONS OF ELASTIC THEORIES

Both Boussinesq's and Westergaard's theories are applicable to elastic materials. Actual soils do not behave in the manner as assumed in the analysis. The results obtained are necessarily approximate. The theories have the following limitations.

(1) The soil mass is never truly isotropic and homogeneous.

(2) The soil mass is not elastic as the particles do not return to the original position when the load is removed.

(3) The stress-strain ratio for most soils is not constant.

However, for most soils the stress-strain ratio is approximately constant provided the stresses are well below the failure stresses, and no unloading occurs.

Although the applicability of elastic theories to soil problems is questionable, yet the results are generally not much different from the observed values. A difference of 20 to 30% between the theoretical and the measured values may occur. This difference is generally ignored considering many complexities of the problem. The elastic theories are used as better theories are not yet available which can be used in a design office.

ILLUSTRATIVE EXAMPLES

Illustrative Example 11.1. *A concentrated load of 2000 kN is applied at the ground surface. Determine the vertical stress at a point P which is 6 m directly below the load. Also calculate the vertical stress at a point R which is at a depth of 6 m but at a horizontal distance of 5 m form the axis of the load.*

Solution. From Eq. 11.9,
$$\sigma_z = \frac{3Q}{2\pi z^2} \cdot \frac{1}{[1 + (r/z)^2]^{5/2}}$$

Point P, $r/z = 0$,
$$\sigma_z = \frac{3 \times 2000}{2\pi(6)^2} \cdot \frac{1}{[1 + 0]^{5/2}} = 26.53 \text{ kN/m}^2$$

Point R, $r/z = 5/6$,
$$\sigma_z = \frac{3 \times 2000}{2\pi(6)^2} \cdot \frac{1}{[1 + (5/6)^2]^{5/2}} = 7.1 \text{ kN/m}^2$$

Illustrative Example 11.2. *A long strip footing of width 2 m carries a load of 400 kN/m. Calculate the maximum stress at a depth of 5 m below the centre line of the footing. Compare the results with 2 : 1 distribution method.*

Solution. From Eq. 11.21,
$$\sigma_z = \frac{q}{\pi}(2\theta + \sin 2\theta)$$

In this case, $b = 1$ m and $z = 5$ m.

$\tan\theta = 1/5 = 0.2$ and $2\theta = 0.395$ radians

Taking $q = 400/2 = 200$ kN/m^2,
$$\sigma_z = \frac{200}{\pi}(0.395 + 0.385) = 49.6 \text{ kN/m}^2$$

2 : 1 Distribution method. From Eq. 11.61,
$$\sigma_z = \frac{q \times B}{B + z} = \frac{200 \times 2}{2 + 5} = \mathbf{57.1 \text{ kN/m}^2}$$

$$\text{Percentage error} = \frac{57.1 - 49.6}{49.6} \times 100 = \mathbf{15.2\%}$$

Illustrative Example 11.3. *There is a line load of 120 kN/m acting on the ground surface along y-axis. Determine the vertical stress at a point P which has x and z coordinates as 2 m and 3.5 m, respectively.*

Solution. From Eq. 11.17,
$$\sigma_z = \frac{2q'}{\pi z}\left[\frac{1}{1 + (x/z)^2}\right]^2$$

At point P,
$$\sigma_z = \frac{2 \times 120}{\pi \times 3.5}\left[\frac{1}{1 + (2/3.5)^2}\right]^2 = \mathbf{12.40 \text{ kN/m}^2}$$

Illustrative Example 11.4. *The unit weight of the soil in a uniform deposit of loose sand (K_0 = 0.50) is 16.5 kN/m³. Determine the geostatic stresses at a depth of 2 m.*

Solution. From Eq. 11.1,

$$\sigma_z = \gamma z = 16.5 \times 2.0 = \mathbf{33.0\ kN/m^2}$$

From Eq. 11.6, $$\sigma_x = K_0 \sigma_z = 0.5 \times 33.0 = \mathbf{16.5\ kN/m^2}$$

Illustrative Example 11.5. *Determine the vertical stress at a point P which is 3 m below and at a radial distance of 3 m from the vertical load of 100 kN. Use Westergaard's solution (ν = 0.0).*

Solution. From Eq. 11.53, $$\sigma_z = \frac{1}{\pi[1 + 2(r/z)^2]^{3/2}} \cdot \frac{Q}{z^2}$$

or $$\sigma_z = \frac{1}{\pi[1 + 2(3/3)^2]^{3/2}} \cdot \frac{100}{(3)^2} = 0.681\ \text{kN/m}^2$$

Alternatively Using Eq. 11.54, $\sigma_z = I_w \cdot \dfrac{Q}{z^2}$

Taking I_w from Table 11.10 as 0.0613,

$$\sigma_z = 0.0613 \times \frac{100}{(3)^2} = \mathbf{0.681\ kN/m^2}$$

Illustrative Example 11.6. *Calculate the vertical stress at a point P at a depth of 2.5 m directly under the centre of the circular area of radius 2 m and subjected to a load 100 kN/m². Also calculate the vertical stress at a point Q which is at the same depth of 2.5 m but 2.5 m away from the centre of the loaded area.*

Solution. From Eq. 11.29 $$\sigma_z = q\left[1 - \left\{\frac{1}{1 + (R/z)^2}\right\}^{3/2}\right]$$

$$\sigma_z \text{ at } P = 100\left[1 - \left\{\frac{1}{1 + (2.0/2.5)^2}\right\}^{3/2}\right] = 52.39\ \text{kN/m}^2$$

From Fig. 11.16, the vertical stress at $z = 1.25\ R$ and $r = 1.25\ R$ is about $0.2\ q$.

Therefore, $$\sigma_z \text{ at } Q = 0.2 \times 100 = 20\ \text{kN/m}^2$$

Illustrative Example 11.7. *An L—shaped building in plan (Fig. E 11.7) exerts a pressure of 75 kN/m² on the soil. Determine the vertical stress increment at a depth of 5 m below the interior corner P.*

Solution. The loaded area is subdivided into three small areas such that each small area has one corner at P.

From Eq. 11.35, $\sigma_z = q[(I_N)_1 + (I_N)_2 + (I_N)_3]$

For area A_1 $m = n = 10/5 = 2.0$

From Table 11.9, $(I_N)_1 = 0.2325$

For area A_2 $m = 15/5 = 3, \quad n = 10/5 = 2.0$

From Table 11.9, $(I_N)_2 = 0.2378$

For area A_3 $m = 20/5 = 4, \quad n = 15/5 = 3$

From Table 11.9, $(I_N)_3 = 0.2450$

Therefore, $\sigma_z = 75\ [0.2325 + 0.2378 + 0.2450] = \mathbf{53.65\ kN/m^2}$.

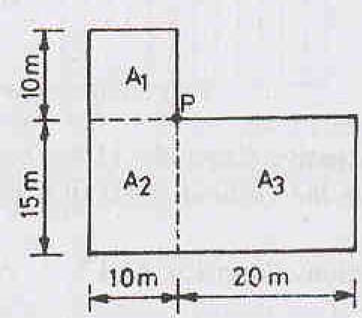


Fig. E 11.7.

Illustrative Example 11.8. *A rectangular foundation 4 m by 5 m carries a uniformly distributed load of*

200 kN/m². Determine the vertical stress at a point P located as shown in Fig. E 11.8 and at a depth of 2.5 m.

Solution. From Eq. 11.35, $\sigma_z = q[(I_N)_1 + (I_N)_2 + (I_N)_3 + (I_N)_4]$

In this case For A_1 and A_2, $m = 2/2.5 = 0.80$, $n = 2/2.5 = 0.80$

$(I_N)_1 = 0.1461$,

For A_3 and A_4, $m = 3/2.5 = 1.20$, $n = 2/2.5 = 0.80$

$(I_N)_3 = 0.1684$

Therefore, $\sigma_z = 200\,[0.1461 + 0.1461 + 0.1684 + 0.1684]$

$= \mathbf{125.8\ kN/m^2}$

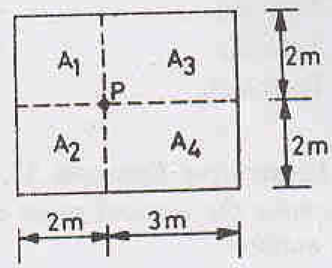


Fig. Ex. 11.8

Illustrative Example 11.9. *A T-shaped foundation (Fig. E 11.9) is loaded with a uniform load of 120 kN/m². Determine the vertical stress at point P at a depth of 5.0 m. Use Newmark's influence chart. Compare the answer by exact method.*

Solution. The foundation plan is drawn on a tracing paper with a scale such that the distance *AB* in Fig. 11.21 represents 5.0 m. The plan is placed on the Newmark chart such that point *P* is at the centre of the chart.

Number of area units occupied by plan = 63

From Eq. 11.40, $\sigma_z = I \times n \times q$

or $\sigma_z = 0.005 \times 63 \times 120 = 37.8\ \text{kN/m}^2$

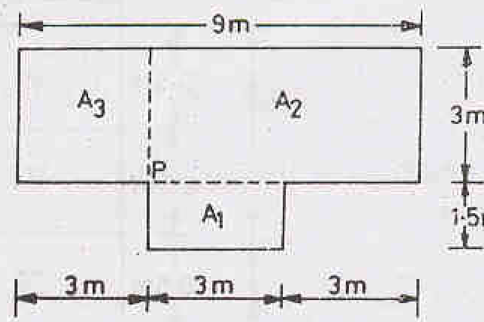


Fig. E 11.9

Exact method. The loaded area is divided into 3 areas, such that they have one corner at *P*.

Area A_1 $m = 3/5.00 = 0.60$; $n = 1.5/5.00 = 0.30$, $(I_N)_1 = 0.0629$

Area A_2 $m = 3/5.00 = 0.60$; $n = 6/5.00 = 1.20$, $(I_N)_2 = 0.1431$

Area A_3 $m = 3/5.00 = 0.60$; $n = 3/5.00 = 0.60$, $(I_N)_2 = 0.1069$

From Eq. 11.35, $\sigma_z = q\,[(I_N)_1 + (I_N)_2 + (I_N)_3\,]$

or $\sigma_z = 120\,[0.0629 + 0.1431 + 0.1069] = 37.55\ \text{kN/m}^2$

Illustrative Example 11.10. *A rectangular loaded area 2 m × 2.5 m carries a load of 80 kN/m² (Fig. E 11.10). Determine the vertical stress at point P located outside the loaded area at a depth of 2.5 m.*

Solution. From Eq. 11.37, $\sigma_z = [(I_N)_1 - (I_N)_2 - (I_N)_3 + (I_N)_4)]$

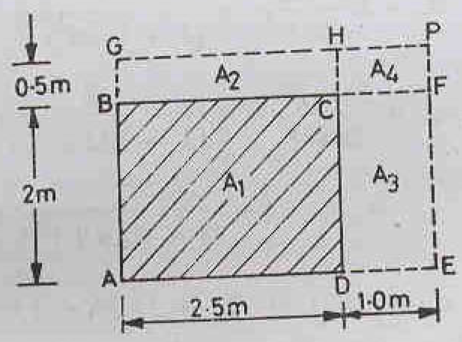


Fig. E 11.10.

For large rectangle *AEPG*, $m = 3.50/2.50 = 1.40$, $n = 2.50/2.50 = 1.00$

and $(I_N)_1 = 0.1914$

Area A_2 $m = 3.5/2.5 = 1.40$, $n = 0.5/2.5 = 0.20$, $(I_N)_2 = 0.0589$

Area A_3 $m = 2.5/2.5 = 1.0$, $n = 1.0/2.50 = 0.40$, $(I_N)_3 = 0.1013$

Area A_4 $m = 0.5/2.5 = 0.2$, $n = 1.0/2.5 = 0.40$, $(I_N)_4 = 0.0328$

Therefore, $\sigma_z = 80\,[0.1914 - 0.059 - 0.1013 + 0.0328]$

$= \mathbf{5.12\ kN/m^2}$

Illustrative Example 11.11. *A rectangular foundation 3.0 × 1.50 m carries a uniform load of 40 kN/m². Determine the vertical stress at P which is 3 m below the ground surface (Fig. E 11.11). Use equivalent point load method.*

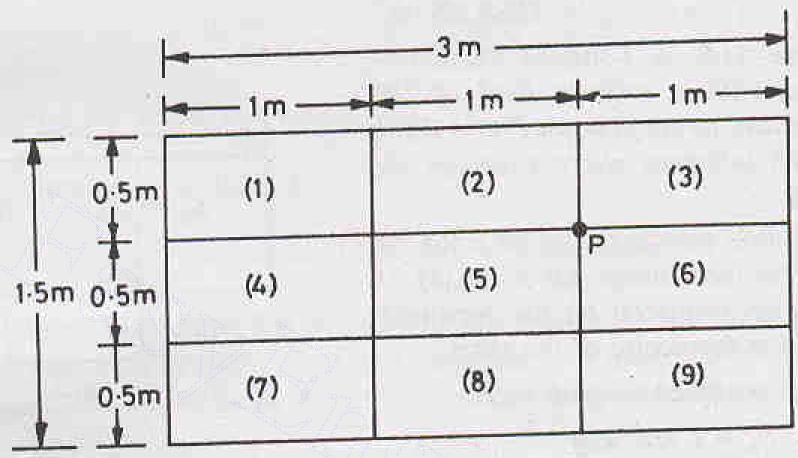


Fig. E 11.11.

Solution. Let us divide the loaded area into 9 small areas of size 0.5 m × 1.0 m.

Load on each area $= 40 \times (1.0 \times 0.5) = 20\text{kN}$

The stresses at point *P* are determined due to 9 point loads, using Boussinesq's solution (Eq. 11.9).

For loads (1) and (4), $r = \sqrt{1.5^2 + (0.25)^2} = 1.521$, $r/z = 0.507$

For loads (2), (3), (5), (6), $r = \sqrt{0.5^2 + 0.25^2} = 0.559$, $r/z = 0.186$

For loads (8) and (9), $r = \sqrt{(0.75)^2 + (0.5)^2} = 0.901$; $r/z = 0.300$

For load (7), $r = \sqrt{(1.5)^2 + (0.75)^2} = 1.677$; $r/z = 0.559$

Therefore,
$$\sigma_z = \Sigma \frac{3Q}{2\pi (z^2)} \times \frac{1}{[1 + (r/z)^2]^{5/2}}$$

In this case,
$$\sigma_z = \frac{3 \times 20}{2\pi (3)^2} \times \left[\frac{2}{[1 + (0.507)^2]^{5/2}} + \frac{4}{[1 + (0.186)^2]^{5/2}} + \frac{2}{[1 + (0.30)^2]^{5/2}} + \frac{1}{[1 + (0.559)^2]^{5/2}} \right]$$

or $\sigma_z = 1.061\ [1.129 + 3.674 + 1.612 + 0.507] = \mathbf{7.34\ kN/m^2}$

Illustrative Example 11.12. *Determine the vertical stress at a point P which is 3 m below the ground surface and is on the centre line of the embankment shown in Fig. E 11.12. Take $\gamma = 18$ kN/m³.*

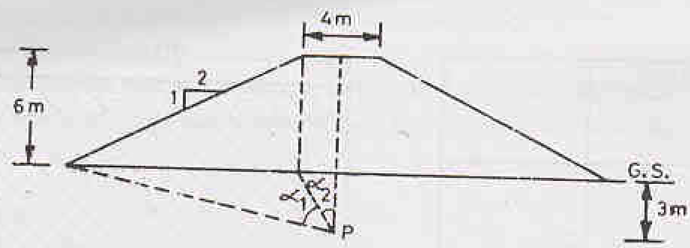


Fig. E 11.12.

Solution. From Eq. 11.46,

$$\sigma_z = \frac{2q}{\pi a}\,[a(\alpha_1 + \alpha_2) + b\,\alpha_1]$$

In this case, $a = 12\ m$ and $b = 2\ m$, $q = 6 \times 18 = 108$ kN/m^2

$$\tan\alpha_2 = 2/3.0 = 0.667; \qquad \alpha_2 = 0.588 \text{ radians}$$

$$\tan(\alpha_1 + \alpha_2) = \frac{14}{3.00} = 4.667; \quad (\alpha_1 + \alpha_2) = 1.359 \text{ radians}$$

Therefore,
$$\sigma_z = \frac{2 \times 108}{\pi \times 12}\,[12\,(1.359) + 2 \times (1.359 - 0.588)] = \mathbf{102.3\ kN/m^2}$$

PROBLEMS

A. Numerical

11.1. A monument weighing 15 MN is erected on the ground surface. Considering the load as a concentrated one, determine the vertical pressure directly under the monument at a depth of 8 m below the ground surface.
[Ans. 111.9 kN/m^2]

11.2. A concentrated load of 50 kN acts on the surface of a homogeneous soil mass of large extent. Determine the stress intensity at a depth of 5 m, directly under the load, and at a horizontal distance of 2.5 m.
[Ans. 0.955 kN/m^2; 0.55 kN/m^2]

11.3. Two columns *A* and *B* are situated 6 m apart. Column *A* transfers a load of 500 kN and column *B*, a load of 250 kN. Determine the resultant vertical stress on a horizontal plane 20 m below the ground surface at points vertically below the points *A* and *B*. [Ans. 59.8 kN/m^2 ; 29.9 kN/m^2]

11.4. An excavation 3 m × 6 m for foundation is to be made to a depth of 2.5 m below ground level in a soil of bulk unit weight 20 kN/m^3. What effect this excavation will have on the vertical pressure at a depth of 6 m measured from the ground surface vertically below the centre of foundation ? I_N for $m = 0.43$ and $n = 0.86$ is 0.10.
[Ans. decrease 20 kN/m^2]

11.5. A square foundation (5 m × 5 m) is to carry a load of 4000 kN. Calculate the vertical stress at a depth of 5 m below the centre of the foundation. $I_N = 0.084$ for $m = n = 0.50$.
(*ii*) Also determine the vertical stress using 1 : 2 distribution. [Ans. 53.76 kN/m^2 ; 40 kN/m^2]

11.6. A water tower has a circular foundation of 10 m diameter. If the total weight of the tower, including the foundation, is 2×10^4 kN, calculate the vertical stress at a depth of 2.5 m below the foundation level.
[Ans. 231.9 kN/m^2]

11.7. A rectangular foundation, 3 m × 2.1 m, is perfectly flexible and carries a load of 300 kN/m^2. Determine the vertical pressure at a depth of 5 m below a point *P* shown in Fig. P 11.7. [Ans. 31.8 kN/m^2]

11.8. The contact pressure for a square footing 2 m × 2 m is 400 kN/m^2. Using 1 : 2 distribution, determine the depth at which the contact pressure is 100 kN/m^2.
[Ans. 2 m]

11.9. A rectangular foundation 20 m × 10 m subjects the subgrade to a contact pressure of 250 kN/m^2. Determine the vertical stress at a point *P* located at a depth of 5 m (Fig. P 11.9).
(Use Table for I_N values) [Ans. 3.375 kN/m^2]

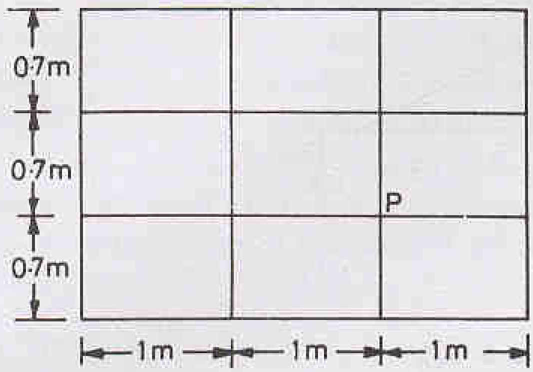


Fig. P 11.7.

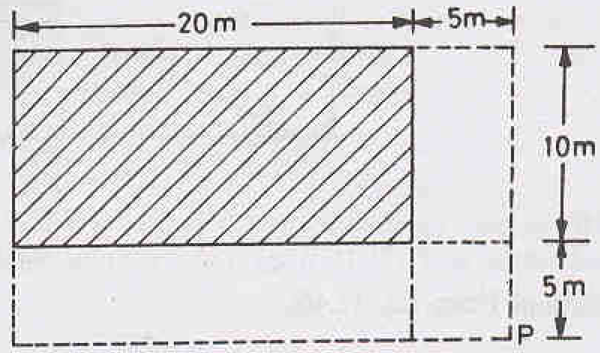


Fig. P 11.9.

11.10. A 1000 kN load is uniformly distributed on a surface area of 3 m × 2.5 m. Find the approximate value of vertical stress at a depth of 2 m, using,

(*i*) 2 : 1 distribution

(*ii*) 60° distribution. [Ans. 44.4 kN/m^2 ; 39.2 kN/m^2]

11.11. A concentrated load of 1000 kN acts vertically at the ground surface. Determine the vertical stress at a point which is at

(*i*) a depth of 2.5 m and a horizontal distance of 4.0 m

(*ii*) at a depth of 5.0 and a radial distance of 2.5 m. [Ans. 3.2 kN/m^2; 10.93 kN/m^2]

B. Descriptive and Objective Type

11.12. State the assumptions made in computing stresses below the ground surface due to a point load acting on it. Discuss their validity in practice.

11.13. Derive an expression for the vertical stress at a point due to a point load, using Boussinesq's theory.

11.14. What do you understand by geostatic stresses ? How are these determined ?

11.15. What is an influence diagram ? What is its use in practice ?

11.16. Derive an expression for the vertical stress at a point due to a line load. Give examples of a line load.

11.17. How would you determine the stresses at a point due to a (*a*) strip load (*b*) circular load. Compare the zones of influence due to the two types of loads.

11.18. Describe the method of calculating the stress at a point below the corner of a rectangular load. How is this method used for finding the stresses at points other than that below the corner ?

11.19. Discuss the basis of the construction of Newmark's influence chart. How is it used ?

11.20. Explain Westergaard's theory for the determination of the vertical stress at a point. How is it different from Boussinesq's solution ?

11.21. What is Fenske's chart ? Explain its construction and use.

11.22. Discuss various approximate methods for the determination of the vertical stress at a point What are their limitations ?

11.23. What do you understand by contact pressure ? What are the factors that affect the contact pressure distributions? Draw the contact pressure distribution diagram for flexible and rigid footings on sand and clayey soils.

11.24. Mention whether the following statements are true or false.

(*a*) The vertical stress due to a point load depends upon modulus of elasticity.

(*b*) For determination of the deformation, the secant modulus at the peak stress is used.

(*c*) The Poisson ratio for most of the soils is zero.

(*d*) The horizontal stress can be more than the vertical stress.

(*e*) The Boussinesq influence coefficient just below the point load is zero.

(*f*) The maximum shear stress due to a strip load is constant at all points.

(*g*) The zone of influence due to a circular load is deeper than that due to a strip load.

(*h*) While determining Newmark's influence coefficient, the constant m and n can be interchanged.

(*i*) The Boussinesq solution always gives stresses greater than the Westergaard solution.
(*j*) The equivalent point load gives reliable results if the dimension of the area is greater than three times the depth.
(*k*) Two-to-one load distribution and sixty-degree distribution give approximately the same stresses.
(*l*) In actual design, the contact pressure distribution is generally taken as uniform.

[**Ans.** True, (*d*), (*h*), (*k*), (*l*)]

C. Multiple-Choice Questions

1. The stress developed at a point in the soil exactly below a point load at the surface is
(*a*) proportional to the depth of point.
(*b*) proportional to the square of the depth of point.
(*c*) inversely proportional to the depth of point.
(*d*) inversely proportional to the square of the depth of point.

2. An isobar is a curve which
(*a*) joins points of equal horizontal stress.
(*b*) joins points of equal vertical stress.
(*c*) joins points of zero vertical stress.
(*d*) joins points of maximum vertical stress.

3. If the entire semi-infinite soil mass is loaded with a load intensity of q at the surface, the vertical stress at any depth is equal to
(*a*) q (*b*) 0.5 q
(*c*) zero (*d*) infinity

4. For a strip of width B subjected to a load intensity of q at the surface, the pressure bulb of intensity 0.2 q extends to a depth of
(*a*) 3B (*b*) 6 B
(*c*) 1.5 B (*d*) B

5. Newmark's influence chart can be used for the determination of vertical stress under
(*a*) circular load area only (*b*) rectangular loaded area only
(*c*) strip load only (*d*) Any shape of loaded area

6. The Westergaard analysis is used for
(*a*) homogeneous soils (*b*) cohesive soils
(*c*) sandy soils (*d*) stratified soils

7. A concentrated load of 1000 kN acts vertically at a point on the soil surface. According to Boussinesq's equation the ratio of the vertical stresses at depths of 3m and 5m is
(*a*) 0.35 (*b*) 0.70
(*c*) 1.75 (*d*) 2.78

8. A load of 2000 kN is uniformly distributed over an area of 3 m × 2m. The average vertical stress at a depth of 2m using 2 : 1 distribution is
(*a*) 160 kN/m^2 (*b*) 100 kN/m^2
(*c*) 48 kN/m^2 (*d*) 37 kN/m^2

[**Ans.** 1. (*d*), 2. (*b*), 3. (*a*), 4. (*a*), 5. (*d*), 6. (*d*), 7. (*d*), 8. (*b*)]