2.3 Permutations and Combinations

Permutations and combinations give us quick, algebraic methods of counting. They are used in probability problems for two purposes: to count the number of equally likely possible results for the classical approach to probability, and to count the number of different arrangements of the same items to give a multiplying factor.

(a) Each separate arrangement of all or part of a set of items is called a *permutation*. The number of permutations is the number of different arrangements in which items can be placed. Notice that if the order of the items is changed, the arrangement is different, so we have a different permutation. Say we have a total of n items to be arranged, and we can choose r of those items at a time, where $r \leq n$. The number of permutations of n items chosen r at a time is written ${}_nP_r$. For permutations we consider both the identity of the items and their order.

Let us think for a minute about the number of choices we have at each step along the way. If there are *n distinguishable* items, we have n choices for the first item. Having made that choice, we have $(n–1)$ choices for the second item, then $(n – 2)$ choices for the third item, and so on until we come to the r th item, for which we have $(n – r + 1)$ choices. Then the total number of choices is given by the product $(n)(n – 1)(n – 2)(n – 3)...(n – r + 1)$. But remember that we have a short-hand notation for a related product, $(n)(n – 1)(n – 2)(n – 3)...(3)(2)(1) = n!$, which is called *n factorial* or factorial n. Similarly, $r! = (r)(r – 1)(r – 2)(r – 3)...(3)(2)(1)$, and $(n – r)! = (n – r)(n – r – 1) ((n – r – 2)...(3)(2)(1)$. Then the total number of choices, which is called the *number of permutations of n items taken r at a time*, is

$$ {}_nP_r = \frac{n!}{(n-r)!} = \frac{n(n-1)(n-2)\ldots(2)(1)}{(n-r)(n-r-1)\ldots(3)(2)(1)} \tag{2.6} $$

By definition, $0! = 1$. Then the number of choices of n items taken n at a time is ${}_nP_n = n!$.

Example 2.13

An engineer in technical sales must visit plants in Vancouver, Toronto, and Winnipeg. How many different sequences or orders of visiting these three plants are possible?

Answer: The number of different sequences is equal to ${}_3P_3 = 3! = 6$ different permutations. This can be verified by the following tree diagram:

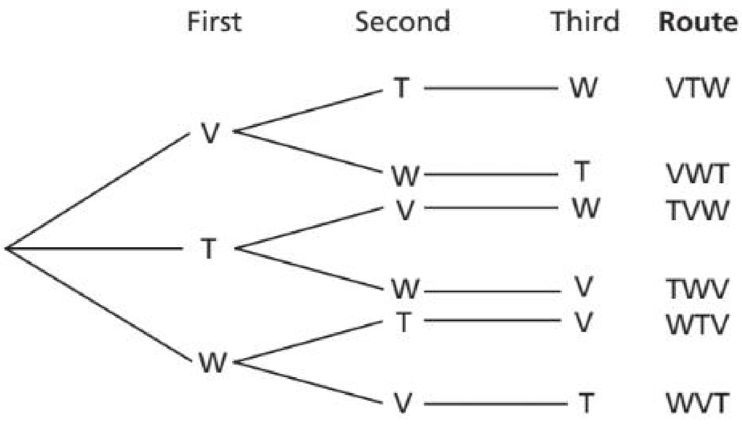

Figure 2.15: Tree Diagram for Visits to Plants

(b) The calculation of permutations is modified if some of the items cannot be distinguished from one another. We speak of this as calculation of the number of *permutations into classes*. We have already seen that if n items are all different, the number of permutations taken n at a time is $n!$. However, if some of them are indistinguishable from one another, the number of possible permutations is reduced. If n_1 items are the same, and the remaining $(n–n_1)$ items are the same of a different class, the number of permutations can be shown to be $\frac{n!}{n_1!(n-n_1)!}$. The numerator, $n!$, would be the number of permutations of n distinguishable items taken n at a time. But n_1 of these items are indistinguishable, so reducing the number of permutations by a factor $\frac{1}{n_1!}$, and another $(n - n_1)$ items are not distinguishable from one another, so reducing the number of permutations by another factor $\frac{1}{(n-n_1)!}$. If we have a total of n items, of which n_1 are the same of one class, n_2 are the same of a second class, and n_3 are the same of a third class, such that $n_1 + n_2 + n_3 = 1$, the number of permutations is $\frac{n!}{n_1!n_2!n_3!}$. This could be extended to further classes.

Example 2.14

A machinist produces 22 items during a shift. Three of the 22 items are defective and the rest are not defective. In how many different orders can the 22 items be arranged if all the defective items are considered identical and all the nondefective items are identical of a different class?

Answer: The number of ways of arranging 3 defective items and

19 nondefective items is $\frac{22!}{(3!)(19!)} = \frac{(22)(21)(20)}{(3)(2)(1)} = 1540.$

Another modification of calculation of permutations gives *circular permutations*. If n items are arranged in a circle, the arrangement doesn't change if every item is moved by one place to the left or to the right. Therefore in this situation one item can be placed at random, and all the other items are placed in relation to the first item. Thus, the number of permutations of n distinct items arranged in a circle is $(n - 1)!$.

The principal use of permutations in probability is as a multiplying factor that gives the number of ways in which a given set of items can be arranged.

(c) *Combinations* are similar to permutations, but with the important difference that combinations take no account of order. Thus, AB and BA are different permutations but the same combination of letters. Then the number of permutations must be larger than the number of combinations, and the ratio between them must be the number of ways the chosen items can be arranged. Say on an examination we have to do any eight questions out of ten. The number of permutations of questions would be ${}_{10}P_8 = \frac{10!}{2!}$. Remember that the number of ways in which eight items can be arranged is 8!, so the number of combinations must be reduced by the factor $\frac{1}{8!}$. Then the number of combinations of 10 distinguishable items taken 8 at a time is $\left(\frac{10!}{2!}\right)\left(\frac{1}{8!}\right)$. In general, the number of combinations of n items taken r at a time is

$$ {}_nC_r = \frac{{}_nP_r}{r!} = \frac{n!}{(n-r)!r!} \tag{2.7} $$

${}_nC_r$ gives the number of equally likely ways of choosing r items from a group of n distinguishable items. That can be used with the classical approach to probability.

Example 2.15

Four card players cut for the deal. That is, each player removes from the top of a well-shuffled 52-card deck as many cards as he or she chooses. He then turns them over to expose the bottom card of his "cut." He or she retains the cut card. The highest card will win, with the ace high. If the first player draws a nine, what is then his probability of winning without a recut for tie?

Answer: For the first player to win, each of the other players must draw an eight or lower. Then Pr [win] = Pr [other three players all get eight or lower].

There are (4)(7) = 28 cards left in the deck below nine after the first player's draw, and there are 52 – 1 = 51 cards left in total. The number of combinations of three cards from 51 cards is ${}_{51}C_3$, all of which are equally likely. Of these, the number of combinations which will result in a win for the first player is the number of combinations of three items from 28 items, which is ${}_{28}C_3$.

The probability that the first player will win is

$$\frac{{}_{28}C_3}{{}_{51}C_3} = \left(\frac{28!}{(25!)(3!)}\right)\left(\frac{(48!)(3!)}{51!}\right) = \left(\frac{(28)(27)(26)}{(3)(2)(1)}\right)\left(\frac{(3)(2)(1)}{(51)(50)(49)}\right) = \left(\frac{(28)(27)(26)}{(51)(50)(49)}\right) = \frac{19{,}656}{124{,}950} = 0.157$$

Like many other problems, this one can be done in more than one way. A solution by the multiplication rule using conditional probability is as follows:

Pr [player #2 gets eight or lower | player #1 drew a nine] = $\frac{28}{51}$

If that happens, Pr [player #3 gets eight or lower]

= Pr [third player gets eight or lower | first player drew a nine and second player drew eight or lower]

$= \frac{27}{50}$

If that happens, Pr [player #4 gets eight or lower]

= Pr [fourth player gets eight or lower | first player drew a nine and both second and third players drew eight or lower]

$= \frac{26}{49}$

The probability that the first player will win is

$$\left(\frac{28}{51}\right)\left(\frac{27}{50}\right)\left(\frac{26}{49}\right) = 0.157.$$

Problems

1. A bench can seat 4 people. How many seating arrangements can be made from a group of 10 people?
2. How many distinct permutations can be formed from all the letters of each of the following words: (a) them, (b) unusual?
3. A student is to answer 7 out of 9 questions on a midterm test.
   i) How many examination selections has he?
   ii) How many if the first 3 questions are compulsory?
   iii) How many if he must answer at least 4 of the first 5 questions?
4. Four light bulbs are selected at random without replacement from 16 bulbs, of which 7 are defective. Find the probability that
   a) none are defective.
   b) exactly one is defective.
   c) at least one is defective.
5. Of 20 light bulbs, 3 are defective. Five bulbs are chosen at random.
   a) Use permutations or combinations to find the probability that none are defective.
   b) What is the probability that at least one is defective?

   (This is a modification of problem 15 of the previous set.)

6. A box contains 18 light bulbs. Of these, four are defective. Five bulbs are chosen at random.
   a) Use permutations or combinations to find the probability that none are defective.
   b) What is the probability that exactly one of the chosen bulbs is defective?
   c) What is the probability that at least one of the chosen bulbs is defective?
7. How many different sums of money can be obtained by choosing two coins from a box containing a nickel, a dime, a quarter, a fifty-cent piece, and a dollar coin? Is this a problem in permutations or in combinations?
8. If three balls are drawn at random from a bag containing 6 red balls, 4 white balls, and 8 blue balls, what is the probability that all three are red? Use permutations or combinations.
9. In a poker hand consisting of five cards, what is the probability of holding:
   a) two aces and two kings?
   b) five spades?
   c) A, K, Q, J, 10 of the same suit?
10. In how many ways can a group of 7 persons arrange themselves
    a) in a row,
    b) around a circular table?
11. In how many ways can a committee of 3 people be selected from 8 people?
12. In playing poker, five cards are dealt to a player. What is the probability of being dealt (i) four-of-a-kind? (ii) a full house (three-of-a-kind and a pair)?
13. A hockey club has 7 forwards, 5 defensemen, and 3 goalies. Each can play only in his designated subgroup. A coach chooses a team of 3 forwards, 2 defense, and 1 goalie.
    a) How many different hockey teams can the coach assemble if position within the subgroup is not considered?
    b) Players A, B and C prefer to play left forward, center, and right defense, respectively. What is the probability that these three players will play on the same team in their preferred positions if the coach assembles the team at random?
14. A shipment of 17 radios includes 5 radios that are defective. The receiver samples 6 radios at random. What is the probability that exactly 3 of the radios selected are defective? Solve the problem
    a) using a probability tree diagram
    b) using permutations and combinations.
15. Three married couples have purchased theater tickets and are seated in a row consisting of just six seats. If they take their seats in a completely random fashion, what is the probability that
    a) Jim and Paula (husband and wife) sit in the two seats on the far left?
    b) Jim and Paula end up sitting next to one another.