PROBLEMS

To solve some of these problems it may be necessary to make certain assumptions, such as sample points are equally likely, or trials are independent, etc., when such assumptions are not explicitly stated. Some of the more difficult problems, or those that require special knowledge, are marked with an *.

1 One urn contains one black ball and one gold ball. A second urn contains one white and one gold ball. One ball is selected at random from each urn.

(*a*) Exhibit a sample space for this experiment.
(*b*) Exhibit the event space.
(*c*) What is the probability that both balls will be of the same color?
(*d*) What is the probability that one ball will be green?

2 One urn contains three red balls, two white balls, and one blue ball. A second urn contains one red ball, two white balls, and three blue balls.

(*a*) One ball is selected at random from each urn.
  (i) Describe a sample space for this experiment.
  (ii) Find the probability that both balls will be of the same color.
  (iii) Is the probability that both balls will be red greater than the probability that both will be white?

(*b*) The balls in the two urns are mixed together in a single urn, and then a sample of three is drawn. Find the probability that all three colors are represented, when (i) sampling with replacement and (ii) without replacement.

3 If A and B are disjoint events, $P[A] = .5$, and $P[A \cup B] = .6$, what is $P[B]$?

4 An urn contains five balls numbered 1 to 5 of which the first three are black and the last two are gold. A sample of size 2 is drawn with replacement. Let B_1 denote the event that the first ball drawn is black and B_2 denote the event that the second ball drawn is black.

(*a*) Describe a sample space for the experiment, and exhibit the events B_1, B_2, and B_1B_2.
(*b*) Find $P[B_1]$, $P[B_2]$, and $P[B_1B_2]$.
(*c*) Repeat parts (*a*) and (*b*) for sampling without replacement.

5 A car with six spark plugs is known to have two malfunctioning spark plugs. If two plugs are pulled at random, what is the probability of getting both of the malfunctioning plugs?

6 In an assembly-line operation, $\frac{1}{5}$ of the items being produced are defective. If three items are picked at random and tested, what is the probability:

(*a*) That exactly one of them will be defective?
(*b*) That at least one of them will be defective?

7 In a certain game a participant is allowed three attempts at scoring a hit. In the three attempts he must alternate which hand is used; thus he has two possible strategies: right hand, left hand, right hand; or left hand, right hand, left hand. His chance of scoring a hit with his right hand is .8, while it is only .5 with his left hand. If he is successful at the game provided that he scores at least two hits in a row, what strategy gives the better chance of success? Answer the same question if .8 is replaced by p_1 and .5 by p_2. Does your answer depend on p_1 and p_2?

8 (*a*) Suppose that A and B are two equally strong teams. Is it more probable that A will beat B in three games out of four or in five games out of seven?

(*b*) Suppose now that the probability that A beats B in an individual game is p. Answer part (*a*). Does your answer depend on p?

9 If $P[A] = \frac{1}{3}$ and $P[\bar{B}] = \frac{1}{4}$, can A and B be disjoint? Explain.

10 Prove or disprove: If $P[A] = P[B] = p$, then $P[AB] \leq p^2$.

11 Prove or disprove: If $P[A] = P[\bar{B}]$ then $\bar{A} = B$.

12 Prove or disprove: If $P[A] = 0$, then $A = \phi$.

13 Prove or disprove: If $P[A] = 0$, then $P[AB] = 0$.

14 Prove: If $P[\bar{A}] = \alpha$ and $P[\bar{B}] = \beta$, then $P[AB] \geq 1 - \alpha - \beta$.

15 Prove properties (i) to (iv) of indicator functions.

16 Prove the more general statement in Theorem 19.

17 Exhibit (if such exists) a probability space, denoted by $(\Omega, \mathscr{A}, P[\cdot])$, which satisfies the following. For A_1 and A_2 members of $\mathscr{A}$, if $P[A_1] = P[A_2]$, then $A_1 = A_2$.

18 Four drinkers (say I, II, III, and IV) are to rank three different brands of beer (say A, B, and C) in a blindfold test. Each drinker ranks the three beers as 1 (for the beer he likes best), 2, and 3, and then the assigned ranks of each brand of beer are summed. Assume that the drinkers really cannot discriminate between beers so that each is assigning his rankings at random.

(*a*) What is the probability that beer A will receive a total score of 4?

(*b*) What is the probability that some beer will receive a total score of 4?

(*c*) What is the probability that some beer will receive a total score of 5 or less?

19 The following are three of the classical problems in probability.

(*a*) Compare the probability of a total of 9 with a total of 10 when three fair dice are tossed once (Galileo and Duke of Tuscany).

(*b*) Compare the probability of at least one 6 in 4 tosses of a fair die with the probability of at least one double-6 in 24 tosses of two fair dice (Chevalier de Méré).

(*c*) Compare the probability of at least one 6 when six dice are rolled with the probability of at least two 6s when twelve dice are rolled (Pepys to Newton).

20 A seller has a dozen small electric motors, two of which are faulty. A customer is interested in the dozen motors. The seller can crate the motors with all twelve in one box or with six in each of two boxes; he knows that the customer will inspect two of the twelve motors if they are all crated in one box and one motor from each of the two smaller boxes if they are crated six each to two smaller boxes. He has three strategies in his attempt to sell the faulty motors: (i) crate all twelve in one box; (ii) put one faulty motor in each of the two smaller boxes; or (iii) put both of the faulty motors in one of the smaller boxes and no faulty motors in the other. What is the probability that the customer will not inspect a faulty motor under each of the three strategies?