CHAPTER 5

FRICTION

5.1 Friction

WHEN two bodies, subject to external forces, are in contact with each other, then, in general, each exerts a force on the other (Fig. 5·1). These forces of constraint are called mutual *action* and *reaction*. According to Newton's third law, if a body *B* exerts a force **R** on another body *A*, then the force exerted by *A* on *B* will be —**R**. Either of these forces may be called action. Then the other force will be reaction. **R** is generally inclined at a certain angle to the common normal at a point of contact.

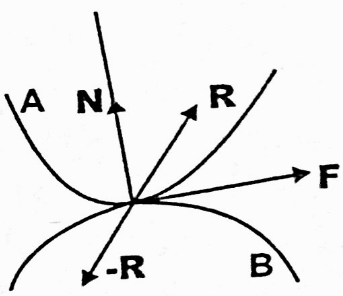


Fig. 5.1

If **N** and **F** be respectively the resolved parts of **R** along the common normal (away from *B*) and in the tangent plane, then **N** is called the *normal reaction* of B on A and **F** is *friction force* or merely the *friction* between them.

The normal reaction prevents the penetration of A into B and the friction force has the tendency of preventing sliding of one body along the other.

In case the friction force **F** between two bodies in contact with each other is zero, the contact is said to be *smooth*. If a body can have only a smooth contact with every other body it is said to be a smooth body.

If the friction force between two bodies in contact is not zero, bodies are said to be *rough*.

In nature no perfectly smooth body exists. Each body is capable of exerting some friction force, although it may be quite small as in the case of glass, steel etc.

Friction is of two types; *static friction* and *dynamic* (or *kinetic* or *sliding*) *friction*.

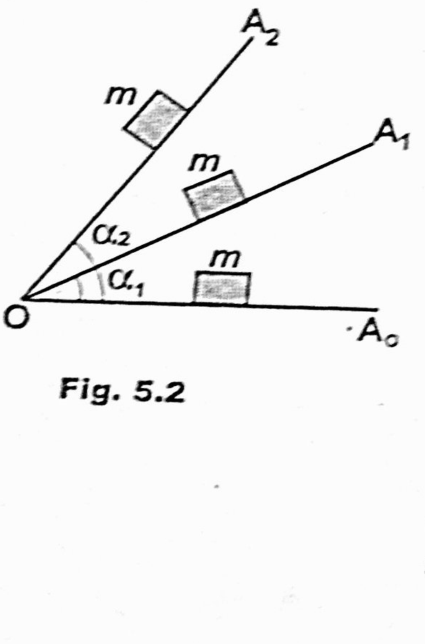


Fig. 5.2

If a particle of mass m is in equilibrium on a horizontal plane OA_0 (Fig. 5·2) the weight of the particle is balanced by the (normal) reaction of the plane and in the absence of any applied force along the horizontal plane no friction comes into play. If the plane is tilted, with the particle on it, to occupy a position OA_1, inclined at an angle of magnitude α_1 with the original position, friction comes into play and in case the particle still remains at rest, the friction force balances the component, $mg \sin \alpha_1$, of the weight down the plane.

If the plane is tilted still further, there will come a stage when the friction force will not be sufficient to prevent motion and the particle will begin to slide down the plane.

When the particle is just on the point of sliding, but actual sliding has not started the friction exerted on the particle is *limiting friction* and the equilibrium of the particle at such a stage is called *limiting equilibrium*. When the motion has actually started, the friction force is called *dynamic* (or *kinetic* or *sliding*) friction.

Thus four possible cases arise when two rough bodies are in contact with each other :

(1) No friction force is acting between them.

(2) Friction force is acting but neither body is on the point of sliding along the other. The friction in this case is *non-limiting*.

(3) One of the bodies is on the point of sliding along the other. The friction in this case is *limiting*.

(4) One body slides along the other. The friction force in such a case which opposes motion is the *kinetic friction*.

5.2 Laws of Friction

Friction force, which as seen above, is a *self-adjusting* force. It behaves according to some definite laws which can be verified experimentally. Laws of both types of friction are stated below :

(1) *The direction of friction is opposite to the direction in which the body moves* (in case of kinetic friction) *or tends to move* (in case of static friction).

(2) *The magnitude of friction is, up to a certain extent, equal to the force tending to produce motion.*

(3) *Only a certain amount of friction can be called into play, and in each particular case it cannot exceed a certain limit.*

As stated earlier, the maximum amount of friction which can be called into play is called *limiting friction*.

(4) *The magnitude of the limiting friction (for given surfaces) bears a constant ratio* μ *to the normal pressure between the surfaces.*

This constant μ depends on the nature of the surfaces and is called the *coefficient of friction*.

(5) *The amount of friction is independent of the areas and shape of the surfaces in contact provided the normal pressure remains unaltered.*

(6) *When a motion takes place, the friction still opposes the motion. It is independent of the velocity and is proportional to the normal reaction, but is slightly less than the limiting friction.*

If F is the magnitude of friction, R that of normal reaction, and μ the coefficient of friction then, in accordance with Law (4),

$$\frac{F}{R}=\mu \quad \text{or} \quad F=\mu R.$$

It may be remembered that this law holds only in case of the limiting friction and not in case of non-limiting friction.

In case of kinetic friction

$$\frac{F}{R}=\mu',$$

where μ' is a constant and is called *the coefficient of kinetic friction.*

In case of the same substances μ' is slightly less than μ.

Values of μ in case of some substances are given below :

Substances	*Approximate Value of* μ
Wood on wood	0·25 to 0·5
Wood on stone	0·4
Wood on metals	0·2 to 0·6
Metals on metals	0·15 to 0·25
Leather on metals	0·56.

slipping of the wheel at the point of contact. The friction force not only prevents slipping, but acting forward, it makes the wheel roll. Thus the tractive force of the engine comes from friction.

5.6 Equilibrium of a Particle on a Rough Inclined Plane

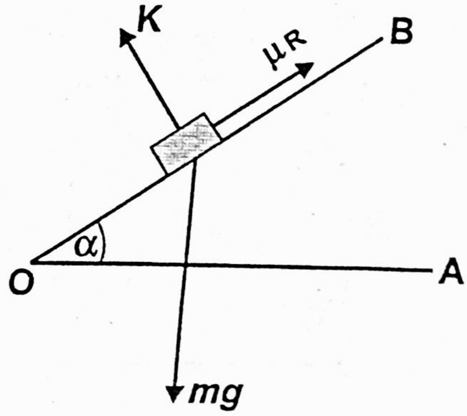


Fig. 5.7

(1) Condition for limiting equilibrium

Let a particle of mass m be in limiting equilibrium on a plane of inclination α under its weight only as shown in Fig. 5·7. Let μ be the coefficient of friction and λ the magnitude of the angle of friction, so that

$$F=\mu R,$$

and $$\mu=\tan \lambda.$$

Resolving along and perpendicular to the plane,

$$mg \sin \alpha=F=\mu R,$$

and $$mg \cos \alpha=R.$$

Dividing the first equation by the second,

$$\tan \alpha=\mu=\tan \lambda.$$

$$\therefore \quad \alpha=\lambda.$$

Hence, *if a particle be in limiting equilibrium on an inclined plane under its own weight, the inclination of the plane equals the magnitude of the angle of friction.*

(2) To find the least force to drag a particle on a rough horizontal plane

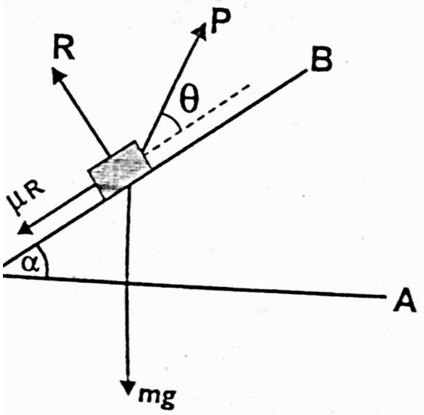


Fig. 5.8

(*i*) Let m be the mass of the particle, P the magnitude of the force necessary to drag the particle *up the plane* and making an angle of measure θ with the plane (Fig. 5·8). The friction is limiting and acts down the plane. Therefore, resolving the forces along and perpendicular to the plane, we have

$$P \cos \theta=\mu R+mg \sin \alpha,$$

$$P \sin \theta+R=mg \cos \alpha.$$

Eliminating the unknown reaction R between these two equations, we get

$$P \cos \theta=\mu (mg \cos \alpha-P \sin \theta)+mg \sin \alpha$$

or $P(\cos\theta+\mu\sin\theta)=mg\,(\mu\cos\alpha+\sin\alpha)$,

i.e.
$$P=mg\,\frac{\mu\cos\alpha+\sin\alpha}{\cos\theta+\mu\sin\theta}.$$

If λ is the measure of the angle of friction,

$$P=mg\,\frac{\tan\lambda\cos\alpha+\sin\alpha}{\cos\theta+\tan\lambda\sin\theta}$$

$$=mg\,\frac{\sin\alpha\cos\lambda+\cos\alpha\sin\lambda}{\cos\theta\cos\lambda+\sin\theta\sin\lambda}$$

$$=mg\,\frac{\sin(\alpha+\lambda)}{\cos(\theta-\lambda)}.$$

$\therefore$ P is least when the denominator of the fraction on the *R.H.S.* is greatest, *i.e.*, when

$$\cos(\theta-\lambda)=1=\cos 0$$

$\therefore$
$$\theta=\lambda.$$

When this condition is satisfied

$$P=mg\sin(\alpha+\lambda).$$

(*ii*) Next suppose the particle is to be dragged down the plane (*see* Fig. 5·9). In this case μR acts up the plane. Therefore, resolving along and perpendicular to the plane, we have

$$\mu R=mg\sin\alpha+P\cos\theta,$$

$$P\sin\theta+R=mg\cos\alpha.$$

Eliminating the unknown reaction R, we get

$\mu(mg\cos\alpha-P\sin\theta)=mg\sin\alpha+P\cos\theta$,

or $P(\cos\theta+\mu\sin\theta)=mg\,\mu\cos\alpha-mg\sin\alpha$,

i.e.,
$$P=mg\,\frac{\mu\cos\alpha-\sin\alpha}{\mu\sin\theta+\cos\theta}$$

$$=mg\,\frac{\tan\lambda\cos\alpha-\sin\alpha}{\tan\lambda\sin\theta+\cos\theta}$$

$$=mg\,\frac{\sin\lambda\cos\alpha-\cos\lambda\sin\alpha}{\sin\lambda\sin\theta+\cos\lambda\cos\theta}$$

$$=mg\,\frac{\sin(\lambda-\alpha)}{\cos(\lambda-\theta)}.$$

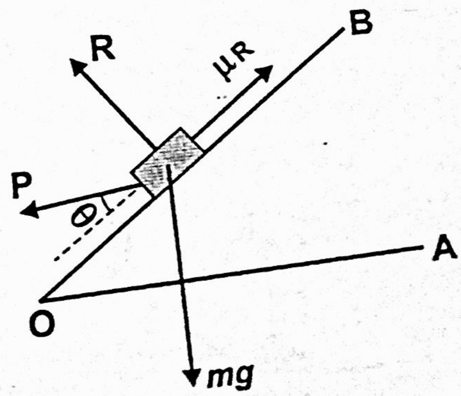


Fig. 5.9

$\therefore$ P is least when $\lambda-\theta=0$, *i.e.*, $\lambda=\theta$, and then

$$P=mg\sin(\lambda-\alpha).$$

P is positive when $\alpha < \lambda$. When $\alpha > \lambda$, $\tan\alpha > \tan\lambda = \mu$ which is not possible in case of equilibrium of the particle.

Therefore, when $\alpha > \lambda$, the particle will itself move down the plane and the question of finding a force to drag it down does not arise.

(3) To find the force necessary just to support a heavy particle on an inclined plane of inclination α ($\alpha > \lambda$).

Let μ be the coefficient of friction and λ the angle of friction.

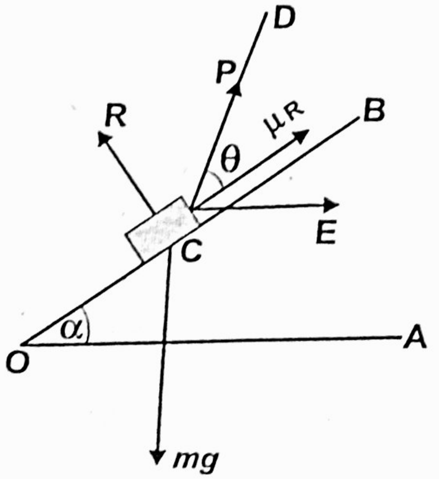


Fig. 5.10

If $\alpha > \lambda$, the friction will not be sufficient to prevent sliding. In such a case an external force will be necessary to prevent motion. Let P be the magnitude of the required force which makes an angle θ with the plane as shown in Fig. 5·10.

Resolving along and perpendicular to the plane, we have

$$P\cos\theta + \mu R = mg\sin\alpha,$$

and $$P\sin\theta + R = mg\cos\alpha.$$

Eliminating R between these equations, we get

$$P(\cos\theta - \mu\sin\theta) = mg(\sin\alpha - \mu\cos\alpha).$$

$$\therefore \quad P = mg\frac{\sin\alpha - \mu\cos\alpha}{\cos\theta - \mu\sin\theta}$$

$$= mg\frac{\sin\alpha - \tan\lambda\cos\alpha}{\cos\theta - \tan\lambda\sin\theta}$$

$$= mg\frac{\sin(\alpha-\lambda)}{\cos(\theta+\lambda)}.$$

P will be minimum when $\cos(\theta+\lambda)=1$ or $\theta+\lambda=0$ or $\theta=-\lambda$. This shows that P acts along a direction CE and not along CD as shown in Fig. 5·10. When $\theta=-\lambda$,

$$P = mg\sin(\alpha-\lambda).$$

5.7 Equilibrium of a Particle on a Rough Horizontal plane

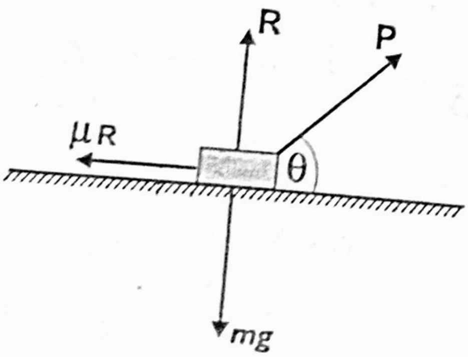


Fig. 5.11

To find the least force which will set into motion a particle at rest on a rough horizontal plane.

Let P be the magnitude of the desired force and let it make an angle of measure θ with the plane (*see* Fig. 5·11).

When the particle is on the point of moving, we have, by resolving along and perpendicular to the plane,

$$P \cos \theta = \mu R,$$

and

$$mg = P \sin \theta + R.$$

Eliminating R, we have

$$P \cos \theta = \mu (mg - P \sin \theta)$$

$$P(\cos \theta + \mu \sin \theta) = mg\,\mu$$

$\therefore$
$$P = mg \frac{\mu}{\cos \theta + \mu \sin \theta}$$

$$= mg \frac{\tan \lambda}{\cos \theta + \sin \theta \tan \lambda}, \text{ where } \lambda \text{ is the angle of friction,}$$

$$= mg \frac{\sin \lambda}{\cos \theta \cos \lambda + \sin \theta \sin \lambda}$$

$$= mg \frac{\sin \lambda}{\cos (\theta - \lambda)}.$$

$\therefore$ P is least when $\theta - \lambda = 0$ or $\theta = \lambda$, and then

$$P = mg \sin \lambda.$$

5.8 Examples

Use of the laws of friction is further illustrated by means of worked examples.

Example 1. A body weighing 40 lb. is resting on a rough horizontal plane and can just be moved by a force of 10 lb. wt. acting horizontally. Find the coefficient of friction.

Sol. Resolving the forces on the body along and perpendicular to the plane, we have

$$40\,g = R,$$

and $$10\,g = \mu R.$$

Eliminating R, we have

$$\mu = \tfrac{1}{4} = 0{\cdot}25.$$

Example 2. The least force which will move a weight up an inclined plane is of magnitude P. Show that the least force, acting parallel to the plane, which will move the weight upwards is

$$P\sqrt{1+\mu^2},$$

where μ is the coefficient of friction.

Sol. Let α be the inclination of the plane.

As in Art. 5·6 (2) (*i*), the least value of P, when inclined at an angle α with the plane, is

i.e., $P = mg \sin(\alpha + \lambda)$, where $\lambda = \tan^{-1}\mu$.

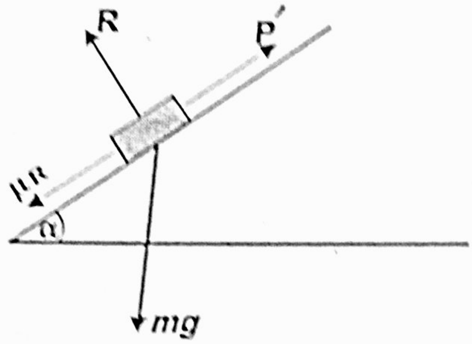


Fig. 5.12

Let P' be the magnitude of the least force which when acting parallel to the plane (Fig. 5·12), sets the weight into motion. Then

$$P' = mg \sin\alpha + \mu R$$

and $$R = mg\cos\alpha.$$

Thus $$P' = mg\sin\alpha + \mu\, mg\cos\alpha$$

$$= mg(\sin\alpha + \mu\cos\alpha).$$

$$= mg(\sin\alpha + \tan\lambda\cos\alpha)$$

$$= mg\,\frac{\sin\alpha \,.\, \cos\lambda + \cos\alpha\sin\lambda}{\cos\lambda}$$

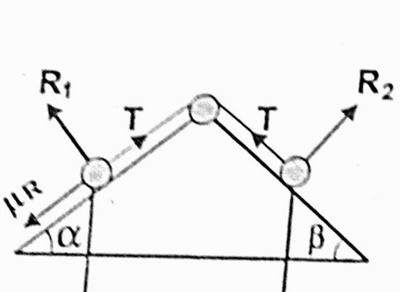
$$= mg\,\frac{\sin(\alpha+\lambda)}{\cos\lambda}$$

$$= \frac{P}{\cos\lambda} \quad \text{(by (1)},$$

$$= P\sec\lambda$$

$$= P\sqrt{1+\tan^2\lambda}$$

$$= P\sqrt{1+\mu^2}$$



6.5 Principle of Virtual Work

We are now in a position to state and prove an important theorem, known as the Principle of Virtual Work.

For the sake of clarity and lucidity we separately prove the principle as applicable in the case of a single particle, a set of particles and a rigid body.

(i) A single particle

A particle subject to workless constraints, is in equilibrium if and only if zero virtual work is done by the applied forces in any arbitrary infinitesimal displacement consistent with the constraints.

Proof. Let the total applied force on the particle be $\mathbf{F}_a$ and the total force of constraint $\mathbf{F}_c$. The particle is in equilibrium if and only if $\mathbf{F}_a+\mathbf{F}_c=0$

(6.1)

If $\delta\mathbf{r}$ is any infinitesimal displacement of the particle consistent with the constraints, the total virtual work will be zero if and only if

$$(\mathbf{F}_a+\mathbf{F}_c)\cdot\delta\mathbf{r}=0 \qquad (6.2)$$

Since the constraints are workless equation (6·2) implies

$$\mathbf{F}_a\cdot\delta\mathbf{r}=0 \qquad (6.3)$$

If equation (6·1) holds, then equation (6·2) and, therefore, equation (6·3) holds. Conversely, if equation (6·3) holds, then equation (6·2) holds, and since $\delta\mathbf{r}$ is arbitrary, equation (6·1) also holds.

This completes the proof of the theorem.

(ii) A set of particles

A set of particles, subject to workless constraints, is in equilibrium if and only if zero virtual work is done by the applied forces in any arbitrary infinitesimal displacement consistent with the constraints.

Proof. Let the particles of the set be $m_1, m_2, \ldots, m_n$. Let the total applied force on m_i be $\mathbf{F}_{ia}$ and the total force of constraint on it be $\mathbf{F}_{ic}$. The set of particles will be in equilibrium if and only if the total force on each particle vanishes, *i.e.*, if and only if

$$\mathbf{F}_{ia}+\mathbf{F}_{ic}=0, \quad i=1, 2, \ldots, n. \qquad (6.4)$$

Internal forces do not occur in this equation because they occur as pairs of equal and opposite forces, so their vector sum vanishes.

Let $\delta\mathbf{r}_i$ be an arbitrary infinitesimal displacement of m_i.

The total virtual work done by all the forces on the particles of the set will vanish if and only if

$$\sum_i (\mathbf{F}_{ia}+\mathbf{F}_{ic})\cdot\delta\mathbf{r}_i=0^* \qquad \ldots (6.5)$$

Since the constraints are assumed to be workless, equation (6·5) implies

$$\sum_i \mathbf{F}_{ia}\cdot\delta\mathbf{r}_i=0 \qquad \ldots (6.6)$$

*In writing down this equation, we have omitted the internal forces whose contribution to the virtual work has been *assumed* to be nil.

If equation (6·4) holds, then equation (6·5) and, therefore, equation (6·6) holds. Conversely, if equation (6·6) holds, then equation (6·5) holds, and since $\delta \mathbf{r}_i$ are arbitrary and mutually independent, equation (6·4) also holds.

Hence the theorem.

(iii) A rigid body or a set of rigid bodies

A rigid body or a set of rigid bodies, subject to workless constraints, is in equilibrium if and only if zero virtual work is done by the applied forces and applied torques in any arbitrary infinitesimal displacement consistent with the constraints.

By an *applied torque* is meant the moment of an applied couple.

Proof. It is sufficient to consider the case of a single rigid body only. The result of the theorem can be extended to a set of bodies by summing for the quantities involved for all of them. A general system of forces acting on a rigid body is equivalent to a force

$$\mathbf{R}=\Sigma\,\mathbf{F}_i$$

passing through any point O, together with a couple

$$\mathbf{G}=\Sigma\,\mathbf{r}_i\times\mathbf{F}_i,$$

where $\mathbf{F}_i$ is the force acting at a point of the body whose position vector relative to O is $\mathbf{r}_i$.

Suppose that in the present case

$$\mathbf{R}=\mathbf{R}_a+\mathbf{R}_c,$$

where $\mathbf{R}_a$ is the (vector) sum of the applied forces and $\mathbf{R}_c$ is the sum of the reactive forces and

$$\mathbf{G}=\mathbf{G}_a+\mathbf{G}_c,$$

where $\mathbf{G}_a$ is the sum of the applied torques and $\mathbf{G}_c$ is the sum of torques of constraints.

The body will be in equilibrium if and only if

$$\mathbf{R}=\mathbf{R}_a+\mathbf{R}_c=O \qquad (6\cdot7)^*$$

$$\mathbf{G}=\mathbf{G}_a+\mathbf{G}_c=O.$$

* We ignore the internal forces for the reasons stated in Part (*ii*) of this article.

An arbitrary virtual displacement of the body will consist of a translation $\delta\mathbf{s}$ and a rotation $\delta\mathbf{\Omega}$ about O. The vanishing of the total virtual work, therefore, implies

$$(\mathbf{R}_a+\mathbf{R}_c).\delta\,\mathbf{s}+(\mathbf{G}_a+\mathbf{G}_c).\,\delta\mathbf{\Omega}=0. \tag{6·8}$$

Since the constraints are workless, the last equation implies

$$\mathbf{R}_a\,.\,\delta\mathbf{s}+\mathbf{G}_a\,.\,\delta\mathbf{\Omega}=0. \tag{6·9}$$

If equations (6·7) hold, then equation (6·8) holds and, therefore, equation (6·9) holds.

Conversely, if equation (6·9) holds, then equation (6·8) holds, and since $\delta\mathbf{s}$ and $\delta\Omega$ are mutually independent, equations (6·7) hold.

Hence the theorem.

The chief advantage of the Principle of Virtual Work lies in the fact that with its help we can find equilibrium position of a body without considering unknown reactions, avoidance of which saves much labour.

6.6 Examples

Use of the Principle is illustrated by means of a number of worked examples. Although the chief function of the Principle is to determine an equilibrium position of a system or a body, it may also be used to find a force of constraint. For this purpose we imagine the constraint to have been removed and the reactive force replaced by an equal applied force. This procedure will be clear from the first two examples that follow.

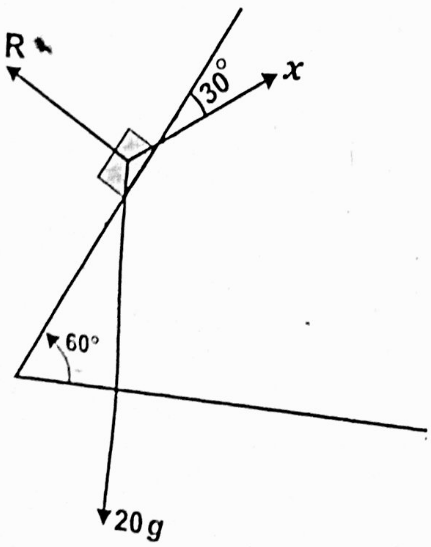


Fig. 6.6

Example 1. A particle of mass 20 lb. is supported on a smooth plane inclined at 60° to the horizontal by a force of magnitude x poundals which makes an angle of 30° with the plane. Find x and also the reaction of the plane on the particle.

Sol. First we consider a virtual displacement δs up the plane (Fig. 6·6). The equation of virtual work is

$$x\cos 30^\circ.\,\delta s-20\,g\sin 60^\circ.\,\delta s=0,$$

whence

$$x=20\,g\,\frac{\sin 60^\circ}{\cos 30^\circ}\text{ poundals}$$
$$=20\text{ lb. wt.}$$

To find the reaction R, we consider a virtual displacement δy in the direction of R and consider R to be an applied force.

The equation of virtual work is

$$R\,\delta y + x\cos 120^\circ\,\delta y + 20\,g\cos 120^\circ\,\delta y = 0,$$

or

$$R = -20\,g\cos 120^\circ - 20\,g\cos 120^\circ$$
$$= -40\,g\cos 120^\circ \text{ poundals}$$
$$= 40\,g\cdot\tfrac{1}{2}\cdot \text{ poundals}$$
$$= 20 \text{ lb. wt.}$$

Example 2. A light thin rod, 12 ft. long, can turn in a vertical plane about one of its points which is attached to a pivot. If weights of 3 lb. and 4 lb. are suspended from its ends, it rests in a horizontal position. Find the position of the pivot and its reaction on the rod.

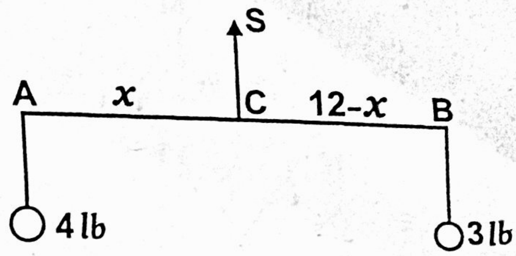


Fig. 6.7

Sol. Let A, B be the ends of the rod, C, the pivot (Fig. 6·7) and let

$$AC = x.$$

Let us consider a small angular displacement $\delta\Omega$ of the rod about the pivot C, as in Fig. 6·8.

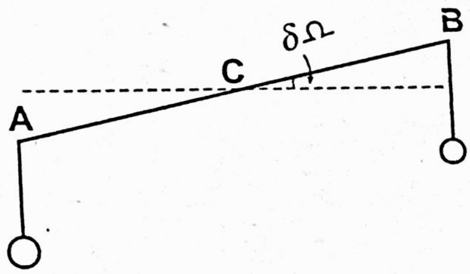


Fig. 6.8

The moments of the weights about the pivot are $4x$ and $-3(12-x)$.

$\therefore$ equation of virtual work is

$$4x\,\delta\Omega - 3(12-x)\,\delta\Omega = 0,$$

whence $\quad 4x - 36 + 3x = 0,$

or $\quad 7x = 36.$

$\therefore$ $\quad x = AC = \dfrac{36}{7} = 5\dfrac{1}{7}$ ft.

and $\quad BC = 12 - 5\dfrac{1}{7} = 6\dfrac{6}{7}$ ft.

To find the reaction S of the pivot, we imagine the pivot to have been removed and an active force S applied at C.

Now we consider an infinitesimal vertical (upward) displacement δy of the system. The equation of virtual work in this case is

$$S\,\delta y - 3\,\delta y - 4\,\delta y = 0,$$

which gives

$$S = 3 + 4$$
$$= 7 \text{ lb. wt.}$$