HERMITE POLYNOMIAL

In Hermite interpolation, we use the expansion involving not only the function values but also its first derivative.

Given a set of data points (x_i, y_i, y_i'), $i = 0, 1, 2, \ldots, n$, we have to determine a polynomial $P(x)$ of degree $(2n+1)$.

Keeping in mind Lagrange interpolation formula, we seek $P(x)$ in the form

$$P(x) = \sum_{i=0}^{n} U_i(x) y_i + \sum_{i=0}^{n} V_i(x) y_i' \longrightarrow (1)$$

where $U_i(x)$ and $V_i(x)$ are polynomials of degree $(2n+1)$ that satisfy the relations

$$\left.\begin{aligned}
& U_i(x_j) = \delta_{ij} \\
& \left.\frac{\partial U_i}{\partial x}\right|_{x=x_j} = 0 \\
\text{and} \quad & \\
& V_i(x_j) = 0 \\
& \left.\frac{\partial V_i}{\partial x}\right|_{x=x_j} = \delta_{ij}
\end{aligned}\right\} \longrightarrow (2)$$

where $\delta_{ij} = \begin{cases} 1, & i = j \\ 0, & i \neq j \end{cases}$

are called <u>Hermite polynomials.</u>

We define

$$U_i = \left\{1 - 2(x-x_i)\left.\frac{dL_i}{dx}\right|_{x=x_i}\right\}[L_i(x)]^2 \longrightarrow (3)$$

and $$V_i = (x-x_i)[L_i(x)]^2 \longrightarrow (4)$$

which satisfy the conditions (2).

and $L_i(x_j) = \delta_{ij}$ is the Lagrange polynomial.

Substituting $x = x_i$ in Eq. (3) & (4), we get

$$U_i(x_i) = [L_i(x_i)]^2 = 1$$

and $$V_i(x_i) = 0$$

Differentiating Eq. (3) & (4), we have

$$U_i'(x) = [1 - L_i'(x_i)(x-x_i)]\,2L_i(x)L_i'(x) - 2L_i'(x_i)[L_i(x)]^2$$

$$V_i'(x_i) = [L_i(x_i)]^2 + (x-x_i)\,2L_i(x)L_i'(x)$$

Now $U_i'(x_j) = 0$, $V_i'(x_j) = 0$ for $i \neq j$.

$L_i(x_i) = 1$, Then

$$U_i'(x_i) = 2L_i'(x_i) - 2L_i'(x_i) = 0$$

and $$V_i'(x_i) = [L_i(x_i)]^2 = 1$$

Hence Hermite interpolation formula is given as

$$P(x) = \sum_{i=0}^{n}[1 - 2L_i'(x_i)(x-x_i)][L_i(x)]^2 y_i + (x-x_i)[L_i(x)]^2 y_i'$$

Example 6.25 Estimate the value of $y(1.05)$ Using Hermite interpolation formula from the following data.

x	y	y'
1.00	1.00000	0.5000
1.10	1.04881	0.47673

Solution

$$L_0(x) = \frac{x - x_1}{x_0 - x_1} = \frac{1.05 - 1.10}{1.00 - 1.10} = 0.5$$

$$L_1(x) = \frac{x - x_0}{x_1 - x_0} = \frac{1.05 - 1.00}{1.10 - 1.00} = 0.5$$

$$L_0'(x) = \frac{1}{x_0 - x_1} = -\frac{1}{0.10}$$

$$L_1'(x) = \frac{1}{x_1 - x_0} = \frac{1}{0.1}$$

Substituting in Hermite formula

$$P(x) = \sum_{i=0}^{n} [1 - 2L_i'(x_i)(x - x_i)][L_i(x)]^2 y_i + (x - x_i)[L_i(x)]^2 y_i'$$

we find

$$y(1.05) = \left[1 - 2\left(-\frac{1}{0.1}\right)(0.05)\right]\left(\frac{1}{2}\right)^2(1) + (0.05)\left(\frac{1}{2}\right)^2(0.5)$$
$$+ \left[1 - 2\left(\frac{1}{0.1}\right)(-0.05)\right]\left(\frac{1}{2}\right)^2(1.04881) + (-0.05)\left(\frac{1}{2}\right)^2(0.47673)$$

$$\boxed{y(1.05) = 1.0247}$$

Numerical Integration

Consider the definite integral

$$I = \int_{x=a}^{b} f(x)\,dx$$

where $f(x)$ is known either explicitely or is given as a table of values corresponding to some known values of x. whether equispaced or not.

Integration of such functions can be carried out using numerical techniques.

Newton-Cotes integration formulae

In this method, we approximate the given tabulated function by a polynomial $P_n(x)$ and then integrate this polynomial.

Let the data (x_i, y_i) $i = 0, 1, \ldots n$ is given at equispaced points with spacing $h = x_{i+1} - x_i$.

Suppose we use Lagrange approximation then

$$f(x) = \sum L_k(x)\, y(x_k) \longrightarrow (1)$$

where

$$L_k(x) = \frac{\Pi(x)}{(x - x_k)\,\Pi'(x_k)} \longrightarrow (2)$$

$$\Pi(x) = (x - x_0)(x - x_1) \cdots (x - x_n) \longrightarrow (3)$$

then we obtain an equivalent integration formula to the definite integral in the form

$$\int_a^b f(x)\,dx \approx \sum_{k=1}^{n} c_k\, y(x_k) \longrightarrow (4)$$

where c_k are weighting coefficients given by

$$c_k = \int_a^b L_k(x)\,dx \longrightarrow (5)$$

which are also called Cotes numbers.
let the equispaced nodes are defined by

$$\left.\begin{aligned} &x_0 = a, \quad x_n = b, \quad h = \frac{b-a}{n}, \quad x_k = x_0 + kh \\ &\text{So that} \\ &x_k - x_1 = (k-1)h \end{aligned}\right\} \longrightarrow (6)$$

Let $x = x_0 + ph$, Then

$\longrightarrow (7)$

then Eqs. (2) & (3) become

$$\Pi(x) = ph \cdot (p-1)h \cdots\cdots h(p-n)$$
$$= h^{n+1} p(p-1) \cdots (p-n)$$

and

$$L_k(x) = \frac{(x-x_0)(x-x_1)\cdots(x-x_{k-1})(x-x_{k+1})\cdots(x-x_n)}{(x_k-x_0)(x_k-x_1)\cdots(x_k-x_{k-1})(x_k-x_{k+1})\cdots(x_k-x_n)}$$

$$= \frac{(ph)(p-1)h\cdots(p-k+1)h\,(p-k-1)h\cdots(p-n)h}{(kh)(k-1)h\cdots(1)h(-1)h\cdots(k-n)h}$$

$$L_k(x) = \frac{(-1)^{(n-k)}\, p(p-1)\cdots(p-k+1)(p-k-1)\cdots(p-n)}{k!\,(n-k)!}$$

Also $dx = h\,dp$.

Therefore

$$C_k = \frac{(-1)^{n-k}\,h}{k!\,(n-k)!}\int_0^n p(p-1)\cdots(p-k+1)(p-k-1)\cdots(p-n)\,dp$$