$$= \frac{x^3 - 13x^2 + 54x - 72}{-30}(-3) + \frac{x^3 - 11x^2 + 34x - 24}{6}(0)$$

$$+ \frac{x^3 - 10x^2 + 27x - 18}{-6}(30) + \frac{x^3 - 8x^2 + 19x - 12}{30}(132)$$

On simplification, we get

$$y(x) = \frac{1}{10}\left(-5x^3 + 135x^2 - 460x + 300\right) = \frac{1}{2}(-x^3 + 27x^2 - 92x + 60)$$

which is the required Lagrange's interpolation polynomial. Now, $y(5) = 75$.

Example 6.15 Given the following data, evaluate $f(3)$ using Lagrange's interpolating polynomial.

x	1	2	5
$f(x)$	1	4	10

Solution Using Lagrange's interpolation formula given by Eq. (6.37), we have

$$f(x) = \frac{(x - x_1)(x - x_2)}{(x_0 - x_1)(x_0 - x_2)}f(x_0) + \frac{(x - x_0)(x - x_2)}{(x_1 - x_0)(x_1 - x_2)}f(x_1)$$

$$+ \frac{(x - x_0)(x - x_1)}{(x_2 - x_0)(x_2 - x_1)}f(x_2)$$

Therefore,

$$f(3) = \frac{(3-2)(3-5)}{(1-2)(1-5)}(1) + \frac{(3-1)(3-5)}{(2-1)(2-5)}(4) + \frac{(3-1)(3-2)}{(5-1)(5-2)}(10) = 6.4999$$

6.6 DIVIDED DIFFERENCES

When the function values are given at non-equispaced points, we have already developed the Lagrange's interpolation formula for interpolation in Section 6.5. Now, we shall introduce the concept of divided differences and then develop Newton's divided difference interpolation formula, whose accuracy is same as that of Lagrange's formula, but has the advantage of being computationally economical in the sense that it involves less number of arithmetic operations.

Let us assume that the function $y = f(x)$ is known for several values of x, (x_i, y_i), for $i = 0\,(1)\,n$. The divided differences of orders 0, 1, 2, ..., n are defined recursively as follows:

$$y[x_0] = y(x_0) = y_0$$

is the 0th order divided difference. The first order divided difference is defined as

$$y[x_0, x_1] = \frac{y_1 - y_0}{x_1 - x_0}$$

Similarly, the higher order divided differences are defined in terms of lower order divided differences by the relations (Hildebrand, 1982) of the form

$$y[x_0, x_1, x_2] = \frac{y[x_1, x_2] - y[x_0, x_1]}{x_2 - x_0}$$

while

$$y[x_0, x_1, \cdots, x_n] = \frac{y[x_1, x_2, \cdots, x_n] - y[x_0, x_1, \cdots, x_{n-1}]}{x_n - x_0} \tag{6.46}$$

The standard format of the divided differences are displayed in Table 6.4.

Table 6.4 Divided Differences

x	$y(x)$	1st order	2nd order	3rd order	4th order
x_0	y_0	$y[x_1, x_0]$			
x_1	y_1	$y[x_2, x_1]$	$y[x_0, x_1, x_2]$	$y[x_0, x_1, x_2, x_3]$	
x_2	y_2	$y[x_3, x_2]$	$y[x_1, x_2, x_3]$	$y[x_1, x_2, x_3, x_4]$	$y[x_0, x_1, x_2, x_3, x_4]$
x_3	y_3	$y[x_4, x_3]$	$y[x_2, x_3, x_4]$		
x_4	y_4				

We can easily verify that the divided difference is a symmetric function of its arguments. That is,

$$y[x_1, x_0] = y[x_0, x_1] = \frac{y_0}{x_0 - x_1} + \frac{y_1}{x_1 - x_0}$$

Now,

$$y[x_0, x_1, x_2] = \frac{y[x_1, x_2] - y[x_0, x_1]}{x_2 - x_0} = \frac{1}{x_2 - x_0}\left(\frac{y_2 - y_1}{x_2 - x_1} - \frac{y_1 - y_0}{x_1 - x_0}\right)$$

Therefore,

$$y[x_0, x_1, x_2] = \frac{y_0}{(x_0 - x_1)(x_0 - x_2)} + \frac{y_1}{(x_1 - x_0)(x_1 - x_2)} + \frac{y_2}{(x_2 - x_0)(x_2 - x_1)}$$

which is a symmetric form, hence suggests the general result as

$$y[x_0, \cdots, x_k] = \frac{y_0}{(x_0 - x_1)\cdots(x_0 - x_k)} + \frac{y_1}{(x_1 - x_0)\cdots(x_1 - x_k)} + \cdots$$

$$+ \frac{y_k}{(x_k - x_0)\cdots(x_k - x_{k-1})}$$

$$= \sum_{i=0}^{k} \frac{y_i}{\prod\limits_{\substack{i=0 \\ i \neq j}}^{k} (x_i - x_j)} \tag{6.47}$$

In Eq. (6.47), it can be noted that zero factor $(x_i - x_i)$ is omitted in the denominator of each term of the sum.