LANGRANGE INTERPOLATION FORMULA

In this section we find approximating polynomials that are determined simply by specifying certain points on the plane through which they must pass.

The problem of determining polynomial of degree one that passes through the distinct points (x_0, y_0) and (x_1, y_1) is the same as approximating a function f for which

$$f(x_0) = y_0$$

and $f(x_1) = y_1$

by means of first degree polynomial interpolating, or agreeing with the values of f at given points.
We first define the fns.

$$L_0(x) = \frac{x - x_1}{x_0 - x_1}$$

and $$L_1(x) = \frac{x - x_0}{x_1 - x_0}$$

and then define

$$P(x) = L_0(x)\, f(x_0) + L_1(x)\, f(x_1)$$

Since

$$L_0(x_0) = 1,$$

$$L_0(x_1) = 0$$

$$L_1(x_0) = 0$$

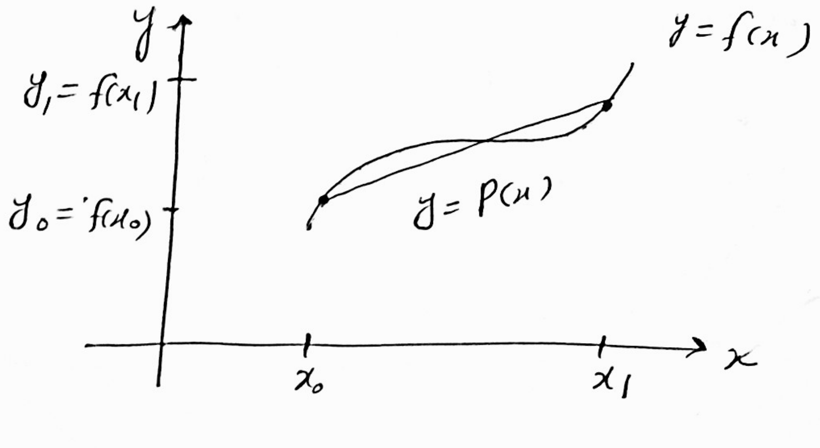
and $\quad L_1(x_1) = 1$

we have

$$P(x_0) = 1 . f(x_0) + 0 \cdot f(x_1) = f(x_0) = y_0$$

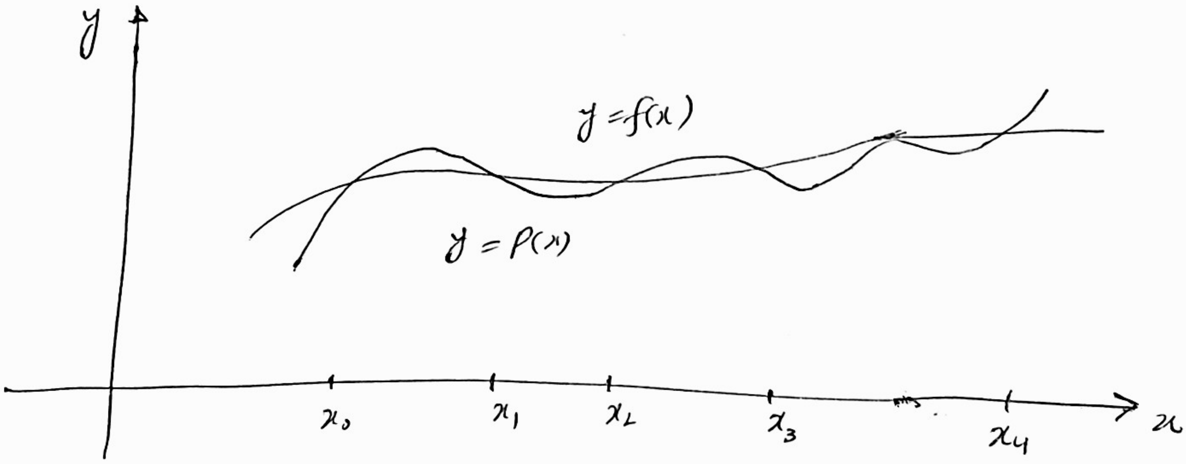
and $\quad P(x_1) = 0 . f(x_0) + 1 \cdot f(x_1) = f(x_1) = y_1$

So P is the unique linear function passing through (x_0, y_0) and (x_1, y_1)



To generalize the concept of linear interpolation, consider the construction of a polynomial of degree at most n that passes through the $(n+1)$ points

$$(x_0, f(x_0)), (x_1, f(x_1)), \ldots, (x_n, f(x_n)).$$



In this case we need to construct for each $K = 0, 1, \ldots n$, a function $L_{n,K}(x)$ with the property that

$$L_{n,K}(x_i) = 0 \quad \text{when} \quad i \neq K$$

and $$L_{n,K}(x_i) = 1$$

To satisfy $L_{n,K}(x_i) = 0$ for each $i \neq K$ requires that the numerator of $L_{n,K}(x)$ contains the terms

$$(x-x_0)(x-x_1)\cdots(x-x_{k-1})(x-x_{k+1})\cdots(x-x_n)$$

To satisfy $L_{n,k}(x_i)=1$, the denominator of $L_{n,k}(x)$ must be equal to this term evaluated at $x=x_k$. Thus

$$L_{n,k}(x)=\frac{(x-x_0)\cdots(x-x_{k-1})(x-x_{k+1})\cdots(x-x_n)}{(x_k-x_0)\cdots(x_k-x_{k-1})(x_k-x_{k+1})\cdots(x_k-x_n)}$$

then nth Lagrange interpolating polynomial is defined in the following Theorem

<u>THEOREM</u>

If $x_0, x_1, \cdots x_n$ are $(n+1)$ distinct numbers ~~of~~ & and f is a fn. whose values are given at these numbers, then a unique polynomial $P(x)$ of degree at most n exists with

$$f(x_k) = P(x_k) \quad \text{for each} \quad k=0,1,\cdots n$$

This polynomial is given by

$$P(x) = P(x_0)L_{n,0}(x)+\cdots+f(x_n)L_{n,n}(x)$$

$$= \sum_{k=0}^{n} f(x_k)L_{n,k}(x)$$

where for each $k = 0, 1, \dots n$

$$L_{n,k}(x) = \frac{(x-x_0)(x-x_1)\cdots(x-x_{k-1})(x-x_{k+1})\cdots(x-x_n)}{(x_k-x_0)(x_k-x_1)\cdots(x_k-x_{k-1})(x_k-x_{k+1})\cdots(x_k-x_n)}$$

$$= \prod_{\substack{i=0 \\ i \neq k}}^{n} \frac{(x-x_i)}{(x_k-x_i)}$$

Example Using the numbers $x_0 = 2$, $x_1 = 2.5$, $x_2 = 4$ find second interpolating polynomial for $f(x) = 1/x$

Solution We determine the coefficient polynomials

$$L_0(x) = \frac{(x-2.5)(x-4)}{(2-2.5)(2-4)} = (x-6.65)x+10$$

$$L_1(x) = \frac{(x-2)(x-4)}{(2.5-2)(2.5-4)} = \frac{(-4x+24)x-32}{3}$$

$$L_2(x) = \frac{(x-2)(x-2.5)}{(4-2)(4-2.5)} = \frac{(x-4.5)x+5}{3}$$

Since $f(x_0) = f(2) = 0.5$, $f(x_1) = f(2.5) = 0.4$
$f(x_1) = f(4) = 0.25$, we have

$$P(x) = \sum_{k=0}^{2} f(x_k) L_k(x)$$

$$= 0.5[(x-6.5)x+10] + 0.4 \frac{[-4x+24]x-32}{3}$$

$$+ 0.25 \frac{(x-4.5)x+5}{3}$$

$$= (0.05x - 0.425)x + 1.15$$

An approximation to $f(3) = \frac{1}{3}$ is

$$f(3) \approx P(3) = 0.325$$