NEWTON'S FORWARD DIFFERENCE INTERPOLATION

FORMULA

Let $y = f(x)$ be a function which takes values $f(x_0)$, $f(x_0+h)$, $f(x_0+2h)$, ... corresponding to various equispaced values of x with spacing h, say x_0, x_0+h, x_0+2h, ...

Suppose we want to evaluate $f(x_0+ph)$ where p is any real number.

We have

$$E^p f(x) = f(x+ph)$$

$$E^p f(x_0) = f(x_0+ph)$$

$$= (1+\Delta)^p f(x_0) \qquad \because E = 1+\Delta$$

$$= \left[1 + p\Delta + \frac{p(p-1)}{2!}\Delta^2 + \frac{p(p-1)(p-2)}{3!}\Delta^3 + \cdots\right] f(x_0)$$

i.e

$$f(x_0+ph) = f(x_0) + p\,\Delta f(x_0) + \frac{p(p-1)}{2!}\Delta^2 f(x_0) + \frac{p(p-1)(p-2)}{3!}\Delta^3 f(x_0) + \cdots + \frac{p(p-1)\cdots(p-(n-1))}{n!}\Delta^n f(x_0) + \text{Error} \longrightarrow (1)$$

This is called Newton's forward difference formula for interpolation, which gives values of $f(x_0+ph)$ in terms of $f(x_0)$ and its leading differences.

Here $p = x - x_0 / h$

In alternate form formula (1) can be written as

$$y_x = y_0 + p\Delta y_0 + \frac{p(p-1)}{2!}\Delta^2 y_0 + \frac{p(p-1)(p-2)}{3!}\Delta^3 y_0 + \cdots$$

$$+ \frac{p(p-1)(p-n+1)}{n!}\Delta^n y_0 + \text{Error}.$$

* This formula is mainly used for interpolating the values of y near the beginning of a set of tabular values and entrapolating values of y, a short distance backward from y_0.

<u>Example 6.8</u> Evaluate $f(15)$, given the following table of values

x	10	20	30	40	50
$y = f(x)$	46	66	81	93	101

<u>SOLUTION</u>

$x = 15$ is very near to the beginning of the table, so we used Newton's forward difference interpolation formula.

x	$y=f(x)$	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$
10	46				
20	66	20	-5		
30	81	15		2	
40	93	12	-3		-3
50	101	8	-4	-1	

Newton's forward difference interpolation formula is

$$y = y_0 + p\Delta y_0 + \frac{p(p-1)}{2!}\Delta^2 y_0 + \frac{p(p-1)(p-2)}{3!}\Delta^3 y_0$$

$$+ \frac{p(p-1)(p-2)(p-3)}{4!}\Delta^4 y_0 \qquad \rightarrow (1)$$

From above table, we have

$x_0 = 10$, $y_0 = 46$, $\Delta y_0 = 20$, $\Delta^2 y_0 = -5$, $\Delta^3 y_0 = 2$

$\Delta^4 y_0 = -3$

Let y_{15} be the value of y when $x = 15$

Then $p = \frac{x - x_0}{h} = \frac{15-10}{10} = 0.5$

Substituting in (1)

$$f(15) = y_{15} = 46 + (0.5)(20) + \frac{(0.5)(0.5-1)}{2}(-5)$$

$$+ \frac{(0.5)(0.5-1)(0.5-2)}{6}(2) + \frac{(0.5)(0.5-1)(0.5-2)(0.5-3)}{24}(-3)$$

$$= 46 + 10 + 0.625 + 0.125 + 0.1172$$

$$= 56.8672$$

Example 6.10

Estimate the missing figure in the following table.

x	1	2	3	4	5
$y = f(x)$	2	5	7	–	32

SOLUTION

Since we are given four entries in the table, the fn. $y = f(x)$ can be represented by a polynomial of degree three.

Using Theorem 6.1, we have

$$\Delta^3 f(x) = \text{constant} \quad \text{and} \quad \Delta^4 f(x) = 0 \quad \forall x.$$

In particular

$$\Delta^4 f(x_0) = 0$$

or $$(E-1)^4 f(x_0) = 0$$

or $$(E^4 - 4E^3 + 6E^2 - 4E + 1) f(x_0) = 0$$

i.e $$f(x_4) - 4f(x_3) + 6f(x_2) - 4f(x_1) + f(x_0) = 0$$

Using the values from table

$$32 - 4f(x_3) + 6 \times 7 - 4 \times 5 + 2 = 0$$

$$f(x_3) = 14$$

Example 6.11 Find a cubic polynomial in x which takes on the values $-3, 3, 11, 27, 57$ and 107; when $x = 0, 1, 2, 3, 4, 5$ respectively.

SOLUTION

Difference table is as follows

x	$f(x)$	$\Delta f(x)$	$\Delta^2 f(x)$	$\Delta^3 f(x)$
0	-3			
		6		
1	3		2	
2	11	8		6
			8	
3	27	16		
				6
4	57	30	14	
5	107	50	20	
				6

Since fourth and higher order differences are zero, we have Newton's interpolation formula as follows

$$f(x_0 + ph) = f(x_0) + p\,\Delta f(x_0) + \frac{p(p-1)}{2}\Delta^2 f(x_0) + \frac{p(p-1)(p-2)}{6}\Delta^3 f(x_0) \longrightarrow (1)$$

Here $p = \dfrac{x - x_0}{h} = \dfrac{x - 0}{1} = x$

$\Delta f(x_0) = 6, \quad \Delta^2 f(x_0) = 2, \quad \Delta^3 f(x_0) = 6$

Substituting these values in (1)

$$f(x) = -3 + 6x + \frac{x(x-1)}{2}(2) + \frac{x(x-1)(x-2)}{6}(6)$$

i.e. $$\boxed{f(x) = x^3 - 2x^2 + 7x - 3}$$