SHIFT OPERATOR

Let $y = f(x)$

and let x takes consecutive values $x, x+h$, $x+2h$ etc. Then we define an operator E such that

$$Ef(x) = f(x+h)$$

Here E is called shift operator.

Also

$$E^2 f(x) = E[Ef(x)] = E[f(x+h)] = f(x+2h)$$

In general

$$E^n f(x) = f(x+nh)$$

or

$$E^n y_x = y_{x+nh}$$

for all real values of n.

If $y_0, y_1, y_2, y_3, \ldots$ are consecutive values of the function y_x, then, we can also write

$$Ey_0 = y_1, \quad E^2 y_0 = y_2, \quad E^4 y_0 = y_4, \ldots, \quad E^2 y_2 = y_4$$

and so on.

The inverse operator E^{-1} is defined as

$$E^{-1} f(x) = f(x-h)$$

$$E^{-n} f(x) = f(x-nh)$$

AVERAGE OPERATOR, μ

The average operator is defined as

$$\mu f(x) = \frac{1}{2}\left[f\left(x+\frac{h}{2}\right) + f\left(x-\frac{h}{2}\right)\right]$$

$$= \frac{1}{2}\left[y_{x+(h/2)} + y_{x-(h/2)}\right]$$

DIFFERENTIAL OPERATOR, D

Differential operator D is defined as

$$Df(x) = \frac{d}{dx}f(x) = f'(x)$$

$$D^2 f(x) = \frac{d^2 f(x)}{dx^2} = f''(x)$$

RELATIONSHIP BETWEEN DIFFERENCE OPERATORS

From definition of Δ and E, we have

$$\Delta y_x = y_{x+h} - y_x = E y_x - y_x = (E-1) y_x$$

Therefore

$$\boxed{\Delta = E - 1}$$

From definition of ∇ and E^{-1}

$$\nabla y_x = y_x - y_{x-h} = y_x - E^{-1} y_x = (1-E^{-1}) y_x$$

Therefore

$$\nabla = 1 - E^{-1} = \frac{E-1}{E}$$

$$\boxed{\nabla = \frac{E-1}{E}}$$

From definition of δ and E

$$\delta y_x = y_{x+(h/2)} - y_{x-(h/2)} = E^{1/2} y_x - E^{-1/2} y_x$$

$$\delta y_x = (E^{1/2} - E^{-1/2}) y_x$$

$$\therefore \boxed{\delta = E^{1/2} - E^{-1/2}}$$

From def. of μ and E

$$\mu y_x = \frac{1}{2}\left[y_{x+(h/2)} + y_{x-(h/2)}\right]$$

$$= \frac{1}{2}\left[E^{1/2} + E^{-1/2}\right] y_x$$

$$\boxed{\mu = \frac{1}{2}(E^{1/2} + E^{-1/2})}$$

Since $E y_x = y_{x+h} = f(x+h)$

Using Taylor series expansion, we have

$$E y_x = f(x) + h f'(x) + \frac{h^2}{2!} f''(x) + \cdots$$

$$= f(x) + hDf(x) + \frac{h^2}{2!} D^2 f(x) + \cdots$$

$$= \left(1 + \frac{hD}{1!} + \frac{h^2D^2}{2!} + \cdots\right) f(x)$$

$$E y_x = e^{hD} y_x$$

So $\boxed{hD = \log E}$

Example 6.5 Prove that

$$hD = \log(1+\Delta) = -\log(1-\nabla) = \sinh^{-1}(\mu\delta)$$

SOLUTION

$$hD = \log E = \log(1+\Delta) = -\log E^{-1} = -\log(1-\nabla) \longrightarrow (1)$$

Also

$$\mu\delta = \frac{1}{2}\left(E^{1/2} + E^{-1/2}\right)\left(E^{1/2} - E^{-1/2}\right)$$

$$= \frac{1}{2}\left(E - 1 + 1 - E^{-1}\right)$$

$$= \frac{1}{2}\left(E - E^{-1}\right)$$

$$= \frac{1}{2}\left(e^{hD} - e^{-hD}\right)$$

$$\mu\delta = \sinh(hD)$$

Therefore

$$\boxed{hD = \sinh^{-1}(\mu\delta)} \longrightarrow (2)$$

Eqs. (1) & (2), gives

$$hD = \log E = \log(1+\Delta) = -\log E^{-1} = -\log(1-\nabla) = \sinh^{-1}(\mu\delta)$$

Example 6.6 If Δ, ∇, δ denote forward, backward and central difference operators E and μ are respectively the shift and average operators, in the analysis of data with equal spacing h, show that

(i) $1+\delta^2\mu^2 = \left(1+\frac{\delta^2}{2}\right)^2$ (ii) $E^{1/2} = \mu + \frac{\delta}{2}$

(iii) $\Delta = \frac{\delta^2}{2} + \delta\sqrt{1+(\delta^2/4)}$ (iv) $\mu\delta = \frac{\Delta E^{-1}}{2} + \frac{\Delta}{2}$

(v) $\mu\delta = \frac{\Delta + \nabla}{2}$.

SOLUTION

(iii) $RHS = \frac{\delta^2}{2} + \delta\sqrt{1+\left(\frac{\delta^2}{4}\right)} = \frac{(E^{1/2}-E^{-1/2})^2}{2}$

$+ (E^{1/2}-E^{-1/2})\sqrt{1+\frac{1}{4}(E^{1/2}-E^{-1/2})^2}$

$= \frac{E-2+E^{-1}}{2} + (E^{1/2}-E^{-1/2})\frac{\sqrt{4+E-2+E^{-1}}}{2}$

$= \frac{E-2+E^{-1}}{2} + (E^{1/2}-E^{-1/2})\frac{\sqrt{(E^{1/2}+E^{-1/2})^2}}{2}$

$= \frac{E-2+E^{-1}}{2} + \frac{(E^{1/2}-E^{-1/2})(E^{1/2}+E^{-1/2})}{2}$

$= \frac{E-2+E^{-1}}{2} + \frac{E-E^{-1}}{2}$

$= \frac{E-2+E^{-1}+E-E^{-1}}{2}$

$= \frac{2E-2}{2} = E-1$

$= \Delta = L.H.S$

(v) $$\mu\delta = \tfrac{1}{2}(E^{1/2}+E^{-1/2})(E^{1/2}-E^{-1/2})$$

$$= \tfrac{1}{2}(E-E^{-1})$$

$$= \tfrac{1}{2}(1+\Delta - 1 + \nabla) \qquad \because E = 1+\Delta, \quad E^{-1} = 1-\nabla$$

$$\mu\delta = \tfrac{1}{2}(\Delta+\nabla)$$

Example 6.7 Show that operators μ and E commute

From defs. of μ and E

$$\mu E y_0 = \mu y_1 = \tfrac{1}{2}(y_{3/2} + y_{1/2}) \longrightarrow (1)$$

$$E\mu y_0 = \tfrac{1}{2}E(y_{1/2} + y_{-1/2}) = \tfrac{1}{2}(y_{3/2}+y_{1/2}) \longrightarrow (2)$$

From Eqs. (1) & (2)

$$\mu E = E\mu$$

Therefore operators μE & $E\mu$ commute.

Theorem 6.1 (Differences of a polynomial)
The nth differences of a polynomial of degree n is constant, when the values of the independent variable are given at equal intervals.

Proof

Consider a polynomial of degree n

$$y_x = a_0x^n + a_1x^{n-1} + a_2x^{n-2} + \dots + a_{n-1}x + a_n$$

where $a_0 \neq 0$ and $a_0, a_1, a_2, \ldots a_n$ are constants

Let h be the interval of differencing

Then

$$y_{x+h} = a_0(x+h)^n + a_1(x+h)^{n-1} + a_2(x+h)^{n-2} + \cdots + a_{n-1}(x+h) + a_0$$

The difference of the polynomial is

$$\Delta y_x = y_{x+h} - y_x = a_0\left[(x+h)^n - x^n\right] + a_1\left[(x+h)^{n-1} - x^{n-1}\right] + a_2\left[(x+h)^{n-2} - x^{n-2}\right] + \cdots + a_{n-1}(x+h-x)$$

Using Binomial expansion

$$\Delta y_x = a_0\left(x^n + {}^nC_1 x^{n-1}h + {}^nC_2 x^{n-2}h^2 + \cdots + h^n - x^n\right) + a_1\left[x^{n-1} + {}^{(n-1)}C_1 x^{n-2}h + {}^{(n-1)}C_2 x^{n-3}h^2 + \cdots + h^{n-1} - x^{n-1}\right) + \cdots + a_{n-1}h$$

$$= a_0 n h x^{n-1} + \left[a_0 {}^nC_2 h^2 + a_1 {}^{(n-1)}C_1 h\right] x^{n-2} + \cdots + a_{n-1}h$$

Therefore

$$\Delta y_x = a_0 n h x^{n-1} + b' x^{n-2} + c' x^{n-3} + \cdots + k' x + l'$$

where $b', c', \ldots, k', l'$ are constants involving h but not x.

Thus first difference of polynomial of degree n is another polynomial of degree $(n-1)$

Similarly

$$\Delta^2 y_x = \Delta(\Delta y_x) = \Delta y_{x+h} - \Delta y_x$$

$$= a_0 nh\left[(x+h)^{n-1} - x^{n-1}\right] + b'\left[(x+h)^{n-2} - x^{n-2}\right]$$

$$+ \cdots + k'(x+h-x)$$

$$\Delta^2 y_x = a_0(n-1)h^2 x^{n-2} + b'' x^{n-3} + c'' x^{n-4} + \cdots + q''$$

$\Delta^2 y_x$ is polynomial of degree $(n-2)$ in x

Similarly after differencing n times we get

$$\Delta^n y_x = a_0 n(n-1)(n-2)\cdots(2)(1)h^n$$

$$= a_0(n!)h^n = \text{constant}$$

$$\therefore \Delta^{n+1} y_x = 0$$

Hence $(n+1)$th and heigher order differences of polynomial of degree n are zero.