Chapter 6

Interpolation

Def-1

For a given table of values (x_k, y_k), $k=0,1,2,\dots n$ the process of estimating the value of y, for any intermediate value of x, is called interpolation.

Def. 2

The method of computing the value of y, for a given value of x, lying outside the table of values of x is known as extrapolation.

Finite difference operators

In general, for interpolation of tabulated function, the concept of finite differences is important. The knowledge about various finite difference operators and their symbolic relations are very much needed to establish various interpolation formulae.

FORWARD DIFFERENCES

For a given table of values (x_k, y_k), $k=0,1,2,\dots n$ with equally-spaced abscissas of a fn. $y=f(x)$, we define the forward difference operator Δ as follows

$$\Delta y_i = y_{i+1} - y_i \quad , \quad i=0,1,\dots,(n-1)$$

$$\Delta y_0 = y_1 - y_0$$
$$\Delta y_1 = y_2 - y_1$$
$$\vdots$$
$$\Delta y_{n-1} = y_n - y_{n-1}$$

These differences are called <u>first differences</u> of the fn. y and are denoted by Δy_i.
Here Δ is called <u>forward difference operator</u>.

The differences of the first differences i.e

$$\Delta^2 y_0 = \Delta y_1 - \Delta y_0 \quad , \quad \Delta^2 y_1 = \Delta y_2 - \Delta y_1$$

are called <u>second differences</u>.

In general

$$\Delta^2 y_i = \Delta y_{i+1} - \Delta y_i$$

Here Δ^2 is called <u>second difference operator</u>.

Similarly r^{th} difference of y is defined as

$$\Delta^r y_i = \Delta^{r-1} y_{i+1} - \Delta^{r-1} y_i$$

Above defined differences can be written Systematically by Constructing a difference table for values (x_K, y_K), $K = 0, 1, 2, \ldots, 6$

Table: Forward difference table (Diagonal diff. Table)

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$	$\Delta^5 y$	$\Delta^6 y$
x_0	y_0						
		Δy_0					
x_1	y_1		$\Delta^2 y_0$				
		Δy_1		$\Delta^3 y_0$			
x_2	y_2		$\Delta^2 y_1$		$\Delta^4 y_0$		
		Δy_2		$\Delta^3 y_1$		$\Delta^5 y_0$	
x_3	y_3		$\Delta^2 y_2$		$\Delta^4 y_1$		$\Delta^6 y_0$
		Δy_3		$\Delta^3 y_2$		$\Delta^5 y_1$	
x_4	y_4		$\Delta^2 y_3$		$\Delta^4 y_2$		
		Δy_4		$\Delta^3 y_3$			
x_5	y_5		$\Delta^2 y_4$				
		Δy_5					
x_6	y_6						

The first term in the table, i.e y_0 is called <u>leading term</u> and the differences $\Delta y_0, \Delta^2 y_0, \Delta^3 y_0, \ldots$ are called <u>leading differences</u>.

<u>Example 6.1</u> Construct a forward difference table for the following values of x and y.

x	0.1	0.3	0.5	0.7	0.9	1.1	1.3
y	0.003	0.067	0.148	0.248	0.370	0.518	0.697

Solution

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$	$\Delta^5 y$
0.1	0.003					
		0.064				
0.3	0.067		0.017			
		0.081		0.002		
0.5	0.148		0.019		0.001	
		0.100		0.003		0.000
0.7	0.248		0.022		0.001	
		0.122		0.004		0.000
0.9	0.370		0.026		0.001	
		0.148		0.005		
1.1	0.518		0.031			
		0.179				
1.3	0.697					

Example 6.2 Express $\Delta^2 y_0$ and $\Delta^3 y_0$ in terms of the values of the function y.

Solution

Since
$$\Delta^2 y_0 = \Delta y_1 - \Delta y_0$$
$$= (y_2 - y_1) - (y_1 - y_0)$$
$$\boxed{\Delta^2 y_0 = y_2 - 2y_1 + y_0} \longrightarrow (1)$$

and
$$\Delta^3 y_0 = \Delta^2 y_1 - \Delta^2 y_0$$
$$= (\Delta y_2 - \Delta y_1) - (\Delta y_1 - \Delta y_0)$$
$$= [(y_3 - y_2) - (y_2 - y_1)] - [(y_2 - y_1) - (y_1 - y_0)]$$
$$= y_3 - 2y_2 + y_1 - y_2 + 2y_1 - y_0$$
$$\boxed{\Delta^3 y_0 = y_3 - 3y_2 + 3y_1 - y_0} \longrightarrow (2)$$

From (1) & (2), we see that coefficients of values of y, in expansion of $\Delta^2 y_0$ and $\Delta^3 y_0$ are binomial coefficients. Thus in general, we can write

$$\boxed{\Delta^n y_0 = y_n - {}^nC_1\, y_{n-1} + {}^nC_2\, y_{n-2} - {}^nC_3\, y_{n-3} + \cdots + (-1)^n y_0}$$

Example 6.3 Show that the value of y_n can be expressed in terms of the leading value y_0 and the leading differences $\Delta y_0, \Delta^2 y_0, \ldots \Delta^n y_0$.

Solution

From forward difference table, we have

$$\left.\begin{aligned} y_1 - y_0 &= \Delta y_0 \quad \text{or} \quad y_1 = y_0 + \Delta y_0 \\ y_2 - y_1 &= \Delta y_1 \quad \text{or} \quad y_2 = y_1 + \Delta y_1 \\ y_3 - y_2 &= \Delta y_2 \quad \text{or} \quad y_3 = y_2 + \Delta y_2 \end{aligned}\right\} \rightarrow (1)$$

and so on

Similarly

$$\left.\begin{aligned} \Delta y_1 - \Delta y_0 &= \Delta^2 y_0 \quad \text{or} \quad \Delta y_1 = \Delta y_0 + \Delta^2 y_0 \\ \Delta y_2 - \Delta y_1 &= \Delta^2 y_1 \quad \text{or} \quad \Delta y_2 = \Delta y_1 + \Delta^2 y_1 \end{aligned}\right\} \rightarrow (2)$$

and so on

Similarly

$$\left.\begin{aligned} \Delta^2 y_1 - \Delta^2 y_0 &= \Delta^3 y_0 \quad \text{or} \quad \Delta^2 y_1 = \Delta^2 y_0 + \Delta^3 y_0 \\ \Delta^2 y_2 - \Delta^2 y_1 &= \Delta^3 y_1 \quad \text{or} \quad \Delta^2 y_2 = \Delta^2 y_1 + \Delta^3 y_1 \end{aligned}\right\} \rightarrow (3)$$

Now from Eqs. (2) & (3), we can write

$$\Delta y_2 = \Delta y_0 + \Delta^2 y_0 + \Delta^2 y_0 + \Delta^3 y_0$$

$$\Delta y_2 = \Delta y_0 + 2\Delta^2 y_0 + \Delta^3 y_0 \longrightarrow (4)$$

From Eqs. (1) & (4), we can write

$$y_3 = y_2 + \Delta y_2$$

$$= (y_1 + \Delta y_1) + (\Delta y_0 + 2\Delta^2 y_0 + \Delta^3 y_0)$$

$$= (y_0 + \Delta y_0) + (\Delta y_0 + \Delta^2 y_0) + (\Delta y_0 + 2\Delta^2 y_0 + \Delta^3 y_0)$$

$$= y_0 + 3\Delta y_0 + 3\Delta^2 y_0 + \Delta^3 y_0$$

$$y_3 = (1 + \Delta)^3 y_0$$

Similarly

$$y_1 = (1+\Delta) y_0, \qquad y_2 = (1+\Delta)^2 y_0$$

$$y_3 = (1+\Delta)^3 y_0$$

Continuing this procedure, we can write

$$y_n = (1+\Delta)^n y_0$$

Therefore

$$y_n = y_0 + {}^nC_1 \Delta y_0 + {}^nC_2 \Delta^2 y_0 + \cdots + \Delta^n y_0$$

or

$$y_n = \sum_{i=0}^{n} {}^nC_i \, \Delta^i y_0$$

Backward differences $\longrightarrow$ Do yourself.

CENTRAL DIFFERENCES

Central difference operator δ is denoted by δ and the subscript of δy for any difference is represented as the average of the subscript of the two members of the difference.

Thus, we write

$$\delta y_{1/2} = y_1 \,, \quad \delta y_{3/2} = y_2 - y_1 \quad \text{etc}$$

In general

$$\delta y_i = y_{i+(1/2)} - y_{i-(1/2)}$$

Higher order differences are defined as follows

$$\delta^2 y_i = \delta y_{i+(1/2)} - \delta y_{i-(1/2)}$$

$$\delta^n y_i = \delta^{n-1} y_{i+(1/2)} - \delta^{n-1} y_{i-(1/2)}$$

x	y	δy	$\delta^2 y$	$\delta^3 y$	$\delta^4 y$	$\delta^5 y$	$\delta^6 y$
x_0	y_0						
		$\delta y_{1/2}$					
x_1	y_1		$\delta^2 y_1$				
x_2	y_2	$\delta y_{3/2}$		$\delta^3 y_{3/2}$			
			$\delta^2 y_2$		$\delta^4 y_2$		
x_3	y_3	$\delta y_{5/2}$				$\delta^5 y_{5/2}$	
				$\delta^3 y_{5/2}$			
x_4	y_4	$\delta y_{7/2}$	$\delta^2 y_3$		$\delta^4 y_3$		$\delta^6 y_3$
x_5	y_5	$\delta y_{9/2}$	$\delta^2 y_4$	$\delta^3 y_{7/2}$		$\delta^5 y_{7/2}$	
					$\delta^4 y_4$		
x_6	y_6	$\delta y_{11/2}$	$\delta^2 y_5$	$\delta^3 y_{9/2}$			

ALTERNATE NOTATION FOR FINITE DIFFERENCE OPERATORS

Let $y = f(x)$ be a functional relation between x and y, which is also denoted by y_x.

Consider consecutive values of x as $x,\ x+h,\ x+2h,\ x+3h$ etc (differing by h)

Corresponding values of y are y, y_x, y_{x+h}, y_{x+2h}, y_{x+3h} etc

Differences of these values can be written as

$$\Delta y_x = y_{x+h} - y_x = f(x+h) - f(x)$$

$$\Delta^2 y_x = \Delta y_{x+h} - \Delta y_x$$

Similarly

$$\nabla y_x = y_x - y_{x-h} = f(x) - f(x-h)$$

Also

$$\delta y_x = y_{x+(h/2)} - y_{x-(h/2)} = f(x+h/2) - f(x-h/2)$$