After rounding-off, we take the largest eigenvalue as $\lambda = 11.84$ and the corresponding eigenvector as

$$(X) = \begin{pmatrix} 0.44 \\ 0.76 \\ 1.00 \end{pmatrix}$$

accurate to two decimals.

4.3 JACOBI'S METHOD

Definition 4.1 An $(n \times n)$ matrix $[A]$ is said to be *orthogonal* if

$$[A]^{\mathrm{T}}\,[A] = [I], \qquad \text{i.e. } [A]^{\mathrm{T}} = [A]^{-1}$$

In order to compute all the eigenvalues and the corresponding eigenvectors of a real symmetric matrix, Jacobi's method is highly recommended. It is based on an important property from matrix theory, which states that, if $[A]$ is an $(n \times n)$ real symmetric matrix, its eigenvalues are real, and there exists an orthogonal matrix $[S]$ such that $[S^{-1}]\,[A]\,[S]$ is a diagonal matrix $[D]$. This diagonalization can be carried out by applying a series of orthogonal transformations $S_1, S_2, \ldots, S_n$, as explained below.

Let A be an $(n \times n)$ real symmetric matrix. Suppose $|\,a_{ij}\,|$ be numerically the largest element amongst the off-diagonal elements of A. We construct an orthogonal matrix S_1 defined as

$$s_{ij} = -\sin\theta, \quad s_{ji} = \sin\theta, \quad s_{ii} = \cos\theta, \quad s_{jj} = \cos\theta \tag{4.13}$$

while each of the remaining off-diagonal elements are zero, the remaining diagonal elements are assumed to be unity. Thus, we construct S_1 as under

$$\begin{array}{cc} & \begin{array}{ccccccccc} & & & i\text{th column} & & j\text{th column} & & & \\ & & & \downarrow & & \downarrow & & & \end{array} \\ S_1 = & \left[\begin{array}{ccccccccc} 1 & 0 & \cdots & 0 & \cdots & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 & \cdots & 0 & \cdots & 0 \\ \vdots & \vdots & & \vdots & & \vdots & & \vdots \\ 0 & 0 & \cdots & \cos\theta & \cdots & -\sin\theta & \cdots & 0 \\ \vdots & \vdots & & \vdots & & \vdots & & \vdots \\ 0 & 0 & \cdots & \sin\theta & \cdots & \cos\theta & \cdots & 0 \\ \vdots & \vdots & & \vdots & & \vdots & & \vdots \\ 0 & 0 & \cdots & 0 & \cdots & 0 & \cdots & 1 \end{array}\right] \begin{array}{l} \\ \\ \\ \leftarrow i\text{th row} \\ \\ \leftarrow j\text{th row} \\ \\ \\ \end{array} \end{array} \tag{4.14}$$

where $\cos\theta$, $-\sin\theta$, $\sin\theta$ and $\cos\theta$ are inserted in (i, i), (i, j), (j, i), (j, j)th positions respectively, and elsewhere it is identical with a unit matrix. Now, we compute

$$D_1 = S_1^{-1} A S_1 = S_1^{\mathrm{T}} A S_1$$

since S_1 is an orthogonal matrix, such that $S_1^{-1} = S_1^{\mathrm{T}}$. After the transformation, the elements at the positions (i, j), (j, i) get annihilated, that is, d_{ij} and d_{ji} reduce to zero, which is seen as follows:

$$\begin{bmatrix} d_{ii} & d_{ij} \\ d_{ji} & d_{jj} \end{bmatrix} = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} a_{ii} & a_{ij} \\ a_{ij} & a_{jj} \end{bmatrix} \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

$$\begin{bmatrix} a_{ii}\cos^2\theta + 2a_{ij}\sin\theta\cos\theta + a_{jj}\sin^2\theta & (a_{jj} - a_{ii})\sin\theta\cos\theta + a_{ij}\cos 2\theta \\ (a_{jj} - a_{ii})\sin\theta\cos\theta + a_{ij}\cos 2\theta & a_{ii}\sin^2\theta + a_{jj}\cos^2\theta - 2a_{ij}\sin\theta\cos\theta \end{bmatrix}$$

Therefore, $d_{ij} = 0$, only if,

$$a_{ij}\cos 2\theta + \frac{a_{jj} - a_{ii}}{2}\sin 2\theta = 0$$

That is, if

$$\tan 2\theta = \frac{2a_{ij}}{a_{ii} - a_{jj}} \tag{4.15}$$

Thus, we choose θ such that, Eq. (4.15) is satisfied, thereby, the pair of off-diagonal elements d_{ij} and d_{ji} reduces to zero.

However, though it creates a new pair of zeros, it also introduces non-zero contributions at formerly zero positions. Also, Eq. (4.15) gives four values of θ, but to get the least possible rotation, we choose $-\pi/4 \le \theta \le \pi/4$.

As a next step, the numerically largest off-diagonal element in the newly obtained rotated matrix D_1 is identified and the above procedure is repeated using another orthogonal matrix S_2 to get D_2. That is, we obtain

$$D_2 = S_2^{-1} D_1 S_2 = S_2^{T}(S_1^{T} A S_1) S_2$$

Similarly, we perform a series of such two-dimensional rotations or orthogonal transformations. After making r transformations, we obtain

$$\begin{aligned} D_r &= S_r^{-1} S_{r-1}^{-1} \cdots S_2^{-1} S_1^{-1} A S_1 S_2 \cdots S_{r-1} S_r \\ &= (S_1 S_2 \cdots S_{r-1} S_r)^{-1} A (S_1 S_2 \cdots S_{r-1} S_r) \\ &= S^{-1} A S \end{aligned} \tag{4.16}$$

where $S = S_1 S_2 \cdots S_{r-1} S_r$. Now, as $r \to \infty$, D_r approaches to a diagonal matrix, with the eigenvalues on the main diagonal. The corresponding eigenvectors are the columns of S.

It is estimated that the minimum number of rotations required to transform the given $(n \times n)$ real symmetric matrix $[A]$ into a diagonal form is $n(n-1)/2$.

Example 4.2 Find all the eigenvalues and the corresponding eigenvectors of the matrix

$$A = \begin{bmatrix} 1 & \sqrt{2} & 2 \\ \sqrt{2} & 3 & \sqrt{2} \\ 2 & \sqrt{2} & 1 \end{bmatrix}$$

by Jacobi's method.

Solution The given matrix is real and symmetric. The largest off-diagonal element is found to be $a_{13} = a_{31} = 2$. Now, we compute

$$\tan 2\theta = \frac{2a_{ij}}{a_{ii} - a_{jj}} = \frac{2a_{13}}{a_{11} - a_{33}} = \frac{4}{0} = \infty$$

which gives, $\theta = \pi/4$. Thus, we construct an orthogonal matrix S_1 as

$$S_1 = \begin{bmatrix} \cos\dfrac{\pi}{4} & 0 & -\sin\dfrac{\pi}{4} \\ 0 & 1 & 0 \\ \sin\dfrac{\pi}{4} & 0 & \cos\dfrac{\pi}{4} \end{bmatrix} = \begin{bmatrix} \dfrac{1}{\sqrt{2}} & 0 & -\dfrac{1}{\sqrt{2}} \\ 0 & 1 & 0 \\ \dfrac{1}{\sqrt{2}} & 0 & \dfrac{1}{\sqrt{2}} \end{bmatrix}$$

The first rotation gives,

$$D_1 = S_1^{-1} A S_1 = \begin{bmatrix} \dfrac{1}{\sqrt{2}} & 0 & \dfrac{1}{\sqrt{2}} \\ 0 & 1 & 0 \\ -\dfrac{1}{\sqrt{2}} & 0 & \dfrac{1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} 1 & \sqrt{2} & 2 \\ \sqrt{2} & 3 & \sqrt{2} \\ 2 & \sqrt{2} & 1 \end{bmatrix} \begin{bmatrix} \dfrac{1}{\sqrt{2}} & 0 & -\dfrac{1}{\sqrt{2}} \\ 0 & 1 & 0 \\ \dfrac{1}{\sqrt{2}} & 0 & \dfrac{1}{\sqrt{2}} \end{bmatrix}$$

$$= \begin{bmatrix} 3 & 2 & 0 \\ 2 & 3 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

we may observe that the elements d_{13} and d_{31} got annihilated. To make sure that our calculations are correct up to this step, we may also observe that the sum of the diagonal elements of D_1 is same as the sum of the diagonal elements of the original matrix A.

As a second step, we choose the largest off-diagonal element of D_1 and is found to be $d_{12} = d_{21} = 2$, and compute

$$\tan 2\theta = \frac{2d_{12}}{d_{11} - d_{22}} = \frac{4}{0} = \infty$$

which again gives $\theta = \pi/4$. Thus, we construct the second rotation matrix as

$$S_2 = \begin{bmatrix} \dfrac{1}{\sqrt{2}} & -\dfrac{1}{\sqrt{2}} & 0 \\ \dfrac{1}{\sqrt{2}} & \dfrac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

At the end of second rotation, we get

$$D_2 = S_2^{-1} D_1 S_2 = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 3 & 2 & 0 \\ 2 & 3 & 0 \\ 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 5 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix} \qquad (1)$$

which turned out to be a diagonal matrix, and therefore, we stop the computation. From (1) we notice that the eigenvalues of the given matrix are 5, 1 and −1. The eigenvectors are the column vectors of $S = S_1 S_2$. Therefore,

$$S = \begin{bmatrix} \frac{1}{\sqrt{2}} & 0 & -\frac{1}{\sqrt{2}} \\ 0 & 1 & 0 \\ \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & -\frac{1}{2} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ \frac{1}{2} & -\frac{1}{2} & \frac{1}{\sqrt{2}} \end{bmatrix}$$

Example 4.3 Find all the eigenvalues of the matrix

$$A = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix}$$

by Jacobi's method.

Solution In this example, we find that all the off-diagonal elements are of the same order of magnitude. Therefore, we can choose any one of them. Suppose, we choose a_{12} as the largest element and compute

$$\tan 2\theta = \frac{-1}{0} = \infty$$

which gives, $\theta = \pi/4$. Then $\cos\theta = \sin\theta = 1/\sqrt{2}$ and we construct an orthogonal matrix S_1 such that

$$S_1 = \begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

The first rotation gives

$$D_1 = S_1^{-1} A S_1 = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & -\frac{1}{\sqrt{2}} \\ 0 & 3 & -\frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 2 \end{bmatrix}$$

Now, we choose $d_{13} = -1/\sqrt{2}$ as the largest element of D_1 and compute

$$\tan 2\theta = \frac{2d_{13}}{d_{11} - d_{33}} = \frac{-\sqrt{2}}{1 - 2}$$

which gives, $\theta = 27° \; 22' \; 41''$.

Now we construct another orthogonal matrix S_2, such that

$$S_2 = \begin{bmatrix} 0.888 & 0 & -0.459 \\ 0 & 1 & 0 \\ 0.459 & 0 & 0.888 \end{bmatrix}$$

At the end of second rotation, we obtain

$$D_2 = S_2^{-1} D_1 S_2 = \begin{bmatrix} 0.634 & -0.325 & 0 \\ 0.325 & 3 & -0.628 \\ 0 & -0.628 & 2.365 \end{bmatrix}$$

Now, the numerically largest off-diagonal element of D_2 is found to be $d_{23} = -0.628$ and compute

$$\tan 2\theta = \frac{-2 \times 0.628}{3 - 2.365}$$

we get, $\theta = -31° \; 35' \; 24''$. Thus, the orthogonal matrix S_3 is seen to be

$$S_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0.852 & 0.524 \\ 0 & -0.524 & 0.852 \end{bmatrix}$$

At the end of third rotation, we get

$$D_3 = S_3^{-1} D_2 S_3 = \begin{bmatrix} 0.634 & -0.277 & 0 \\ 0.277 & 3.386 & 0 \\ 0 & 0 & 1.979 \end{bmatrix}$$

To reduce D_3 to a diagonal form, some more rotations are required. However, we may take 0.634, 3.386 and 1.979 as eigenvalues of the given matrix.

Example 4.4 Find all the eigenvalues and eigenvectors of the matrix:

$$A = \begin{bmatrix} 5 & 0 & 1 \\ 0 & -2 & 0 \\ 1 & 0 & 5 \end{bmatrix}$$

by Jacobi's method.

Solution The given matrix is

$$A = \begin{bmatrix} 5 & 0 & 1 \\ 0 & -2 & 0 \\ 1 & 0 & 5 \end{bmatrix}$$

In this example, the largest off-diagonal element is found to be $a_{13} = a_{31} = 1$. Now, we compute

$$\tan 2\theta = \frac{2a_{13}}{a_{11} - a_{33}} = \frac{2}{5-5} = \frac{2}{0} = \infty$$

which gives $\theta = \pi/4$. Following Jacobi's method, we construct an orthogonal matrix S_1 as

$$S_1 = \begin{bmatrix} \cos(\pi/4) & 0 & -\sin(\pi/4) \\ 0 & 1 & 0 \\ \sin(\pi/4) & 0 & \cos(\pi/4) \end{bmatrix} = \begin{bmatrix} 1/\sqrt{2} & 0 & -1/\sqrt{2} \\ 0 & 1 & 0 \\ 1/\sqrt{2} & 0 & 1/\sqrt{2} \end{bmatrix}$$

The first rotation gives

$$D_1 = S_1^{-1} A S_1 = \begin{bmatrix} 1/\sqrt{2} & 0 & 1/\sqrt{2} \\ 0 & 1 & 0 \\ -1/\sqrt{2} & 0 & 1/\sqrt{2} \end{bmatrix} \begin{bmatrix} 5 & 0 & 1 \\ 0 & -2 & 0 \\ 1 & 0 & 5 \end{bmatrix} \begin{bmatrix} 1/\sqrt{2} & 0 & -1/\sqrt{2} \\ 0 & 1 & 0 \\ 1/\sqrt{2} & 0 & 1/\sqrt{2} \end{bmatrix}$$

$$= \begin{bmatrix} 6 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 4 \end{bmatrix} \tag{1}$$

hich is a diagonal matrix and hence we stop further computation. From (1), e observe that 6, −2 and 4 are the eigenvalues of the given matrix and the rresponding eigenvectors are respectively the column vectors of

$$S_1 = \begin{bmatrix} 1/\sqrt{2} & 0 & -1/\sqrt{2} \\ 0 & 1 & 0 \\ 1/\sqrt{2} & 0 & 1/\sqrt{2} \end{bmatrix}$$

4.3 Find the dominant eigenvalue and the corresponding eigenvector of the matrix

$$\begin{bmatrix} 8 & 1 & 2 \\ 0 & 10 & -1 \\ 6 & 2 & 15 \end{bmatrix}$$

by power method with unit vector as the initial vector.

4.4 Find the largest eigenvalue and the corresponding eigenvector of the matrix

$$\begin{bmatrix} 3 & 1 & 4 \\ 0 & 2 & 6 \\ 0 & 0 & 5 \end{bmatrix}$$

by power method at the end of sixth iteration, taking unit vector as the initial vector.

4.5 Using Jacobi's method, find all the eigenvalues and eigenvectors of the Hilbert matrix

$$A = \begin{bmatrix} 1 & \frac{1}{2} & \frac{1}{3} \\ \frac{1}{2} & \frac{1}{3} & \frac{1}{4} \\ \frac{1}{3} & \frac{1}{4} & \frac{1}{5} \end{bmatrix}.$$

Give result after two rotations.

4.6 Use Jacobi's method to find all the eigenvalues and the corresponding eigenvectors of the matrix

$$A = \begin{bmatrix} 4 & 3 & 2 & 1 \\ 3 & 4 & 3 & 2 \\ 2 & 3 & 4 & 3 \\ 1 & 2 & 3 & 4 \end{bmatrix}$$

4.7 Find all the eigenvalues and the corresponding eigenvectors of the matrix

(i) $A = \begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & 2 \\ 2 & 2 & 1 \end{bmatrix}$ (ii) $A = \begin{bmatrix} 3 & 2 & 1 \\ 2 & 3 & 2 \\ 1 & 2 & 3 \end{bmatrix}$

by Jacobi's method. Give results at the end of third rotation.

4.8 Find the dominant eigenvalue of

(i) $A = \begin{bmatrix} 1 & 2 \\ 3 & 2 \end{bmatrix}$ (ii) $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$

and the corresponding eigenvector.