3.6 GAUSS–SEIDEL ITERATION METHOD

It is another well-known iterative method for solving a system of linear equations of the form of system (3.18). In Jacobi method, the $(r + 1)$th approximation to the system (3.18) is given by Eqs. (3.21), from which we can observe that no element of $x_i^{(r+1)}$ replaces $x_i^{(r)}$ entirely for the next cycle of computation.

However, in Gauss–Seidel method, the corresponding elements of $x_i^{(r+1)}$ replaces those of $x_i^{(r)}$ as soon as they become available. Hence, it is called the method of *successive displacements*. For illustration, consider the system (3... In Gauss–Seidel iteration, the $(r + 1)$th approximation or iteration is computed from

$$
\left.
\begin{aligned}
x_1^{(r+1)} &= \frac{b_1}{a_{11}} - \frac{a_{12}}{a_{11}}x_2^{(r)} - \cdots - \frac{a_{1n}}{a_{11}}x_n^{(r)} \\
x_2^{(r+1)} &= \frac{b_2}{a_{22}} - \frac{a_{21}}{a_{22}}x_1^{(r+1)} - \cdots - \frac{a_{2n}}{a_{22}}x_n^{(r)} \\
\vdots \quad & \quad \vdots \qquad\quad \vdots \qquad\qquad\qquad \vdots \\
x_n^{(r+1)} &= \frac{b_n}{a_{nn}} - \frac{a_{n1}}{a_{nn}}x_1^{(r+1)} - \cdots - \frac{a_{n,(n-1)}}{a_{nn}}x_{n-1}^{(r+1)}
\end{aligned}
\right\} \quad (3...
$$

Thus, the general procedure can be written in the following compact form

$$
x_i^{(r+1)} = \frac{b_i}{a_{ii}} - \sum_{j=1}^{i-1}\frac{a_{ij}}{a_{ii}}x_j^{(r+1)} - \sum_{j=i+1}^{n}\frac{a_{ij}}{a_{ii}}x_j^{(r)} \quad (3.2...
$$

for all $i = 1, 2, \ldots, n$ and $r = 1, 2, \ldots$

To describe system (3.24); in the first equation, we substitute the r... approximation into the right-hand side and denote the result by $x_1^{(r+1)}$. In the second equation, we substitute $(x_1^{(r+1)}, x_3^{(r)}, \ldots, x_n^{(r)})$ and denote the result by $x_2^{(r+1)}$. In the third equation, we substitute $(x_1^{(r+1)}, x_2^{(r+1)}, x_4^{(r)}, \ldots, x_n^{(r)})$ and denote the result by $x_3^{(r+1)}$, and so on. This process is continued till we arrive at the desired result. For illustration, we consider the following example.

Example 3.7 Find the solution of the following system of equations

$$x_1 - \frac{1}{4}x_2 - \frac{1}{4}x_3 = \frac{1}{2}$$

$$-\frac{1}{4}x_1 + x_2 - \frac{1}{4}x_4 = \frac{1}{2}$$

$$-\frac{1}{4}x_1 + x_3 - \frac{1}{4}x_4 = \frac{1}{4}$$

$$-\frac{1}{4}x_2 - \frac{1}{4}x_3 + x_4 = \frac{1}{4}$$

using Gauss–Seidel method and perform the first-five iterations.

Solution The given system of equations can be rewritten as

$$\left.\begin{aligned} x_1 &= 0.5 + 0.25x_2 + 0.25x_3 \\ x_2 &= 0.5 + 0.25x_1 + 0.25x_4 \\ x_3 &= 0.25 + 0.25x_1 + 0.25x_4 \\ x_4 &= 0.25 + 0.25x_2 + 0.25x_3 \end{aligned}\right\} \quad (1)$$

Taking $x_2 = x_3 = x_4 = 0$ on the right-hand side of the first equation of system (1), we get $x_1^{(1)} = 0.5$. Taking $x_3 = x_4 = 0$ and the current value of x_1, we get

$$x_2^{(1)} = 0.5 + (0.25)\,(0.5) + 0 = 0.625$$

from the second equation of system (1). Further, we take $x_4 = 0$ and the current value of x_1, we obtain

$$x_3^{(1)} = 0.25 + (0.25)\,(0.5) + 0 = 0.375$$

from the third equation of system (1). Now, using the current values of x_2 and x_3, the fourth equation of system (1) gives

$$x_4^{(1)} = 0.25 + (0.25)\,(0.625) + (0.25)\,(0.375) = 0.5$$

The Gauss–Seidel iterations for the given set of equations can be written as

$$\begin{aligned} x_1^{(r+1)} &= 0.5 + 0.25x_2^{(r)} + 0.25x_3^{(r)} \\ x_2^{(r+1)} &= 0.5 + 0.25x_1^{(r+1)} + 0.25x_4^{(r)} \\ x_3^{(r+1)} &= 0.25 + 0.25x_1^{(r+1)} + 0.25x_4^{(r)} \\ x_4^{(r+1)} &= 0.25 + 0.25x_2^{(r+1)} + 0.25x_3^{(r+1)} \end{aligned}$$

Now, by Gauss–Seidel procedure, the second and subsequent approximations can be obtained and the sequence of the first-five approximations are tabulated as below:

Iteration number r	Variables			
	x_1	x_2	x_3	x_4
1	0.5	0.625	0.375	0.5
2	0.75	0.8125	0.5625	0.59375
3	0.84375	0.85938	0.60938	0.61719
4	0.86719	0.87110	0.62110	0.62305
5	0.87305	0.87402	0.62402	0.62451

3.7 THE RELAXATION METHOD

This [illegible]