# THE RELAXATION METHOD

This method is an iterative method and is due to southwell.

Consider the system of equations

$$\left.\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= b_2 \\ &\vdots \\ a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nn}x_n &= b_n \end{aligned}\right\} \longrightarrow (1)$$

Let $X^{(p)} = (x_1^{(p)}, x_2^{(p)}, \ldots, x_n^{(p)})^T$

be the solution vector obtained after pth iteration.

If $R_i^{(p)}$ denotes the residual of the ith equation of system (1), that is of

$$a_{i1}x_1 + a_{i2}x_2 + \cdots + a_{in}x_n = b_i$$

defined by

$$R_i^{(p)} = b_i - a_{i1}x_1^{(p)} - a_{i2}x_2^{(p)} - \cdots - a_{in}x_n^{(p)}$$

then, we can improve the solution vector succesively by reducing the largest residual to zero at that iteration. This is the basic idea of relaxation method.

To achieve the fast convergence of the procedure, we take all terms to one side and reorder the equations so that the largest coefficients in the eq$^n$. appear on the diagonal.

Now if at any iteration, $R_i$ is the largest residual in magnitude, then we give an increment

$$dx_i = -\frac{R_i}{a_{ii}}$$

to $x_i$; $a_{ii}$ being the coefficient of $x_i$. In other words, we change $x_i$ to $(x_i + dx_i)$ to relax $R_i$; That is to reduce $R_i$ to zero.

<u>Example 3.8</u> Solve the system of equations

$$\left.\begin{aligned} 6x_1 - 3x_2 + x_3 &= 11 \\ 2x_1 + x_2 - 8x_3 &= -15 \\ x_1 - 7x_2 + x_3 &= 10 \end{aligned}\right\} \longrightarrow (2)$$

by the relaxation method, starting with the vector $(0, 0, 0)$.

<u>SOLUTION</u>

~~After~~ Interchanging the second and third rows, ~~we get~~ and rewritting Eqs. (2), we get

$$\left.\begin{aligned} 0 &= 11 - 6x_1 + 3x_2 - x_3 \\ 0 &= 10 - x_1 + 7x_2 - x_3 \\ 0 &= -15 - 2x_1 - x_2 + 8x_3 \end{aligned}\right\} \longrightarrow (3)$$

Starting with initial solution vector (0,0,0)
i.e $x_1 = 0$, $x_2 = 0$, $x_3 = 0$, we find from
Eqs. 3,

$$R_1 = 11$$
$$R_2 = 10$$
$$R_3 = -15$$

where $R_3 = -15$ is largest residual in magnitude.

We introduce a change $dx_3$ in $x_3$ as

$$dx_3 = -\frac{R_3}{a_{33}} = \frac{15}{8} = 1.875$$

Similarly, we find new residuals of large magnitude and relax it to zero, and so on. This process is continued untill all the residuals are zero or very small.

The detailed calculations are shown in table.

| Iteration number | Residuals $R_1$ | $R_2$ | $R_3$ | Maximum $R_i$ | Difference $dx_i$ | Variables $x_1$ | $x_2$ | $x_3$ |
|---|---|---|---|---|---|---|---|---|
| 0 | 11 | 10 | -15 | -15 | 15/8 = 1.875 | 0 | 0 | 0 |
| 1 | 9.125 | 8.125 | 0 | 9.125 | -9.125/(-6) = 1.5288 | 0 | 0 | 1.875 |
| 2 | 0.0478 | 6.5962 | -3.0576 | 6.5962 | -6.5962/7 = -0.9423 | 1.5288 | 0 | 1.875 |
| 3 | -2.8747 | 0.0001 | -2.1153 | -2.8747 | 2.8747/(-6) = -0.4791 | 1.5288 | -0.9423 | 1.875 |
| 4 | -0.0031 | 0.4792 | -1.1571 | -1.1571 | 1.1571/8 = 0.1446 | 1.0497 ~~1.5288~~ | -0.9423 | 1.875 |
| 5 | -0.1447 | 0.3346 | 0.0003 | 0.3346 | -0.3346/7 = -0.0478 | 1.0497 | -0.9423 | 2.0196 |
| 6 | 0.2881 | 0.0000 | 0.0475 | 0.2881 | -0.2881/(-6) = 0.0480 | 1.0497 | -0.9901 | 2.0196 |
| 7 | -0.0001 | 0.048 | 0.1435 | 0.1435 | -0.1435/8 = -0.0179 | 1.0017 | -0.9901 | 2.0196 |
| 8 | 0.0178 | 0.0659 | 0.0003 | - | - | 1.0017 | -0.9901 | 2.0017 |

At this stage, residuals $R_1, R_2$ and $R_3$ are small enough, and therefore we may take corresponding values of $x_i$ at this iteration as the solution.

Hence numerical soln. to the given system is

$$x_1 = 1.0017, \quad x_2 = -0.9901, \quad x_3 = 2.0017$$

However exact solution is

$$x_1 = 1.0, \quad x_2 = -1.0, \quad x_3 = 2.0$$

<u>Exercise</u> Solve the following system of equations

$$5x - 2y + z = 13$$
$$3x + 7y - 11z = 2$$
$$x + 20y - 2z = 8$$

by relaxation method.

## MATRIX INVERSION

Consider a system of equations in the form

$$[A](X) = (B) \quad \text{—— (1)}$$

One way of writing its solution is

$$(X) = [A]^{-1}(B) \quad \text{—— (2)}$$

Thus solution to the system (1) can also be obtained if the inverse of the coefficient matrix is known.

Alternatively, if the product of two square matrices is an identity matrix, i.e, if

$$[A][B] = [I]$$

$$[B] = [A]^{-1} \text{ and } [A] = [B]^{-1}$$

Every square non-singular matrix will have an inverse.

<u>EXAMPLE</u> Use the Gaussian elimination method to find the inverse of the matrix

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 4 & 3 & -1 \\ 3 & 5 & 3 \end{bmatrix}$$

Augmented matrix is

$$\left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 4 & 3 & -1 & 0 & 1 & 0 \\ 3 & 5 & 3 & 0 & 0 & 1 \end{array}\right]$$

Stage I (Reduction to upper triangular form)

Interchanging $R_1$ and $R_2$

$$\left[\begin{array}{ccc|ccc} 4 & 3 & -1 & 0 & 1 & 0 \\ 1 & 1 & 1 & 1 & 0 & 0 \\ 3 & 5 & 3 & 0 & 0 & 1 \end{array}\right]$$

Divide $R_1$ by 4

$$\left[\begin{array}{ccc|ccc} 1 & \frac{3}{4} & -\frac{1}{4} & 0 & \frac{1}{4} & 0 \\ 1 & 1 & 1 & 1 & 0 & 0 \\ 3 & 5 & 3 & 0 & 0 & 1 \end{array}\right]$$

$$\left[\begin{array}{ccc|ccc} 1 & \frac{3}{4} & -\frac{1}{4} & 0 & \frac{1}{4} & 0 \\ 0 & \frac{1}{4} & \frac{5}{4} & 1 & -\frac{1}{4} & 0 \\ 0 & \frac{11}{4} & \frac{15}{4} & 0 & -\frac{3}{4} & 1 \end{array}\right] \quad \begin{array}{l} R_2 - R_1 \rightarrow R_2 \\ R_3 - 3R_1 \rightarrow R_3 \end{array}$$

Interchanging $R_2$ and $R_3$, we get

$$\left[\begin{array}{ccc|ccc} 1 & \frac{3}{4} & -\frac{1}{4} & 0 & \frac{1}{4} & 0 \\ 0 & \frac{11}{4} & \frac{15}{4} & 0 & -\frac{3}{4} & 1 \\ 0 & \frac{1}{4} & \frac{5}{4} & 1 & -\frac{1}{4} & 0 \end{array}\right]$$

Divide $R_2$ by $11/4$

$$\left[\begin{array}{ccc|ccc} 1 & \frac{3}{4} & -\frac{1}{4} & 0 & \frac{1}{4} & 0 \\ 0 & 1 & \frac{15}{11} & 0 & -\frac{3}{11} & \frac{4}{11} \\ 0 & \frac{1}{4} & \frac{5}{4} & 1 & -\frac{1}{4} & 0 \end{array}\right]$$

$$\left[\begin{array}{ccc|ccc} 1 & \frac{3}{4} & -\frac{1}{4} & 0 & \frac{1}{4} & 0 \\ 0 & 1 & \frac{15}{11} & 0 & -\frac{3}{11} & \frac{4}{11} \\ 0 & 0 & \frac{10}{11} & 1 & -\frac{2}{11} & -\frac{1}{11} \end{array}\right] \quad R_3 - \frac{1}{4}R_2 \to R_3$$

$$\left[\begin{array}{ccc|ccc} 1 & \frac{3}{4} & -\frac{1}{4} & 0 & \frac{1}{4} & 0 \\ 0 & 1 & \frac{15}{11} & 0 & -\frac{3}{11} & \frac{4}{11} \\ 0 & 0 & 1 & \frac{11}{10} & -\frac{1}{5} & -\frac{1}{10} \end{array}\right] \quad \frac{10}{11} R_3$$

## Stage II Reduction to an identity matrix

$$\left[\begin{array}{ccc:ccc} 1 & \frac{3}{4} & 0 & \frac{11}{40} & \frac{1}{5} & -\frac{1}{40} \\ 0 & 1 & 0 & -3/2 & 0 & 1/2 \\ 0 & 0 & 1 & 11/10 & -\frac{1}{5} & -\frac{1}{10} \end{array}\right] \begin{array}{l} R_1 - (-\frac{1}{4})R_3 \longrightarrow R_1 \\ R_2 - \frac{15}{11} R_3 \longrightarrow R_2 \end{array}$$

$$\left[\begin{array}{ccc:ccc} 1 & 0 & 0 & 7/5 & \frac{1}{5} & -\frac{2}{5} \\ 0 & 1 & 0 & -3/2 & 0 & 1/2 \\ 0 & 0 & 1 & 11/10 & -\frac{1}{5} & -\frac{1}{10} \end{array}\right] \quad R_1 - 3/4 R_2 \longrightarrow R_1$$

Thus

$$A^{-1} = \begin{bmatrix} 7/5 & 1/5 & -\frac{2}{5} \\ -3/2 & 0 & 1/2 \\ 11/10 & -1/5 & -1/10 \end{bmatrix}$$

Gauss - Jordan Method (Do it your self)

Do questions 3.11 - 3.15 from exercise