# Chapter 3

# Solution of Linear System of Equations and Matrix Inversion

## 3.1 INTRODUCTION

Many real-life problems in engineering give rise to a system of linear equations. For example, such systems occur in certain applications of statistical analysis and in finding the numerical solution of partial differential equations and so on. It is therefore, natural to seek efficient methods for solving these equations numerically.

The general form of a system of $m$ linear equations in $n$ unknowns $x_1, x_2, x_3, \ldots, x_n$ can be represented in matrix form as under:

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & a_{m3} & \cdots & a_{mn} \end{bmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{pmatrix} \tag{3.1}$$

Using matrix notation, the above system can be written in compact form as

$$[A]\,(X) = (B) \tag{3.2}$$

The solution of the system of equations (3.2) gives $n$ unknown values $x_1, x_2, \ldots, x_n$, which satisfy the system simultaneously. If $m > n$, we may not be able to find a solution, in principle, which satisfy all the equations. If $m < n$, the system usually will have an infinite number of solutions. However, in this chapter, we shall restrict to the case $m = n$. In this case, if $|A| \neq 0$, then the system will have a unique solution, while, if $|A| = 0$, then there exists no solution.

Various numerical methods are available for finding the solution of the system of equations (3.2), and they are classified as *direct* and *iterative methods*. In direct methods, we get the solution of the system after performing all the steps involved in the procedure. The direct methods consist of *elimination methods* and *decomposition methods*. In this chapter, under elimination methods, we consider, *Gaussian elimination* and *Gauss–Jordan elimination methods*. *Crout's reduction* also known as *Cholesky's reduction* is considered under decomposition methods. Under iterative methods, the initial approximate solution is assumed to be known and is improved towards the exact solution in an iterative way. We consider *Jacobi, Gauss–Seidel* and *relaxation methods* under iterative methods. All these methods are easily adoptable to computers and can be used to solve even hundred or more simultaneous linear equations.

## 3.2 GAUSSIAN ELIMINATION METHOD

In the Gaussian elimination method, the solution to the system of Eqs. (3.2) is obtained in two stages. In the first stage, the given system of equations is reduced to an equivalent upper triangular form using elementary transformations. In the second stage, the upper triangular system is solved using back substitution procedure by which we obtain the solution in the order $x_n$, $x_{n-1}$, $x_{n-2}$, ..., $x_2$, $x_1$.

This method is explained by considering a system of $n$ equations in $n$ unknowns in the form as follows

$$\left.\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= b_2 \\ \vdots \qquad \vdots \qquad\qquad \vdots \quad &\quad \vdots \\ a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nn}x_n &= b_n \end{aligned}\right\} \quad (3.3)$$

*Stage I:* We divide the first equation by $a_{11}$ and then subtract this equation multiplied by $a_{21}$, $a_{31}$, ..., $a_{n1}$ from the 2nd, 3rd, ..., $n$ th equation. Then the system (3.3) reduces to the following form:

$$\left.\begin{aligned} x_1 + a'_{12}x_2 + \cdots + a'_{1n}x_n &= b'_1 \\ a'_{22}x_2 + \cdots + a'_{2n}x_n &= b'_2 \\ \vdots \qquad\qquad \vdots \quad &\quad \vdots \\ a'_{n2}x_2 + \cdots + a'_{nn}x_n &= b'_n \end{aligned}\right\} \quad (3.4)$$

Here, we can observe that the last $(n - 1)$ equations are independent of $x_1$, that is, $x_1$ is eliminated from the last $(n - 1)$ equations.

This procedure is repeated with the second equation of (3.4), that is, we divide the second equation by $a'_{22}$ and then $x_2$ is eliminated from 3rd, 4th, ..., $n$th equations of (3.4). The same procedure is repeated again and again till the given system assumes the following upper triangular form:

$$\left.\begin{aligned} c_{11}x_1 + c_{12}x_2 + \cdots + c_{1n}x_n &= d_1 \\ c_{22}x_2 + \cdots + c_{2n}x_n &= d_2 \\ \vdots \quad &\quad \vdots \\ c_{nn}x_n &= d_n \end{aligned}\right\} \quad (3.5)$$

*Stage II:* Now, the values of the unknowns are determined by back substitution procedure, in which we obtain $x_n$ from the last equation of (3.5) and then substituting this value of $x_n$ in the preceding equation, we get the value of $x_{n-1}$. Continuing this way, we can find the values of all other unknowns in the order $x_n, x_{n-1}, \ldots, x_2, x_1$.

In this method, we observe that the determinant of the coefficient matrix is obtained as a by-product, that is,

$$|A| = c_{11}c_{22}\cdots c_{nn} \tag{3.6}$$

To familiarize with the method, we consider the following example:

**Example 3.1** Solve the following system of equations using Gaussian elimination method

$$2x + 3y - z = 5$$

$$4x + 4y - 3z = 3$$

$$-2x + 3y - z = 1$$

***Solution*** The given system of equations is solved in two stages.

*Stage I (Reduction to upper-triangular form):* We divide the first equation by 2 and then subtract the resulting equation (multiplied by 4 and −2) from the second and third equations respectively. Thus, we eliminate $x$ from the 2nd and 3rd equations. The resulting new system is given by

$$\left.\begin{aligned} x + \frac{3}{2}y - \frac{z}{2} &= \frac{5}{2} \\ -2y - z &= -7 \\ 6y - 2z &= 6 \end{aligned}\right\} \tag{1}$$

Now, we divide the second equation of (1) by −2 and eliminate $y$ from the last equation and the modified system is given by

$$\left.\begin{aligned} x + \frac{3}{2}y - \frac{z}{2} &= \frac{5}{2} \\ y + \frac{z}{2} &= \frac{7}{2} \\ -5z &= -15 \end{aligned}\right\} \tag{2}$$

*Stage II (Back substitution):* From the last equation of (2), we immediately get

$$z = 3 \tag{3}$$

using this value of $z$, the second equation of (2) gives

$$y = \frac{7}{2} - \frac{3}{2} = 2 \tag{4}$$

Using these values of $y$ and $z$ in the first equation of (2), we get

$$x = \frac{5}{2} + \frac{3}{2} - 3 = 1 \tag{5}$$

Thus, the solution of the given system is given by Eqs. (3) – (5).

**Partial and full pivoting**

The Gaussian elimination method fails if any one of the pivot elements becomes zero. In such a situation, we rewrite the equations in a different order to avoid zero pivots. Changing the order of equations is called *pivoting*.

We now introduce the concept of partial pivoting. In this technique, if the pivot $a_{ii}$ happens to be zero, then the $i$th column elements are searched for the numerically largest element. Let the $j$th row ($j > i$) contains this element, then we interchange the $i$th equation with the $j$th equation and proceed for elimination. This process is continued whenever pivots become zero during elimination. For example, let us examine the solution of the following simple system

$$10^{-5}x_1 + x_2 = 1$$
$$x_1 + x_2 = 2$$

Using Gaussian elimination method with and without partial pivoting, assuming that we require the solution accurate to only four decimal places. The solution by Gaussian elimination gives $x_1 = 0$, $x_2 = 1$. If we use partial pivoting, the system takes the form

$$x_1 + x_2 = 2$$
$$10^{-5}x_1 + x_2 = 1$$

Using Gaussian elimination method, the solution is found to be $x_1 = 1$, $x_2 = 1$, which is a meaningful and perfect result.

In full pivoting which is also known as *complete pivoting*, we interchange rows as well as columns, such that the largest element in the matrix of the system becomes the pivot element. In this process, the position of the unknown variables also get changed. Full pivoting, in fact, is more complicated than the partial pivoting. Partial pivoting is preferred for hand computation.

**Example 3.2** Solve the system of equations

$$x + y + z = 7$$
$$3x + 3y + 4z = 24$$
$$2x + y + 3z = 16$$

by Gaussian elimination method with partial pivoting.

***Solution*** In matrix notation, the given system can be written as

$$\begin{bmatrix} 1 & 1 & 1 \\ 3 & 3 & 4 \\ 2 & 1 & 3 \end{bmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 7 \\ 24 \\ 16 \end{pmatrix} \tag{1}$$

To start with, we observe that the pivot element $a_{11} = 1$ ($\neq 0$). However, a gl[illegible] at the first column reveals that the numerically largest element is 3 which [illegible] the second row. Hence, we interchange the first row with the second row[illegible] then proceed for elimination. Thus, Eq. (1) takes the form

$$\begin{bmatrix} 3 & 3 & 4 \\ 1 & 1 & 1 \\ 2 & 1 & 3 \end{bmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 24 \\ 7 \\ 16 \end{pmatrix}$$

after partial pivoting.

*Stage I (Reduction to upper triangular form):* By dividing the first row of system (2) by 3 and then subtracting the resulting row, multiplied by 1 a[illegible] from the second and third rows of the system (2), we get

$$\begin{bmatrix} 1 & 1 & \frac{4}{3} \\ & & -\frac{1}{3} \\ & -1 & \frac{1}{3} \end{bmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 8 \\ -1 \\ 0 \end{pmatrix}$$

The second row in Eq. (3) cannot be used as the pivot row, as $a_{22}$ [illegible] Interchanging the second and third rows, we obtain

$$\begin{bmatrix} 1 & 1 & \frac{4}{3} \\ & -1 & \frac{1}{3} \\ & & -\frac{1}{3} \end{bmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 8 \\ 0 \\ -1 \end{pmatrix}$$

which is in the upper triangular form.

*Stage II (Back substitution):* From the last row of Eq. (4), we at once g[illegible]

$$z = 3$$

The second row of Eq. (4) with this value of $z$ gives

$$-y + 1 = 0 \quad \text{or} \quad y = 1$$

Using these values of $y$ and $z$, the first row of Eq. (4) gives

$$x + 1 + 4 = 8 \quad \text{or} \quad x = 3$$

Thus, Eqs. (5)–(7) constitute the solution to the given system of equation[illegible]

**Example 3.3** Solve by Gaussian elimination method with partial pivoting, [illegible] following system of equations:

$$0x_1 + 4x_2 + 2x_3 + 8x_4 = 24$$
$$4x_1 + 10x_2 + 5x_3 + 4x_4 = 32$$
$$4x_1 + 5x_2 + 6.5x_3 + 2x_4 = 26$$
$$9x_1 + 4x_2 + 4x_3 + 0x_4 = 21$$

***Solution*** In matrix notation, the given system can be written as

$$\begin{bmatrix} 0 & 4 & 2 & 8 \\ 4 & 10 & 5 & 4 \\ 4 & 5 & 6.5 & 2 \\ 9 & 4 & 4 & 0 \end{bmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 24 \\ 32 \\ 26 \\ 21 \end{pmatrix} \tag{1}$$

o start with, we observe that the pivot row, that is, the first row has a zero pivot lement ($a_{11} = 0$). This row should be interchanged with any row following it, which on becoming a pivot row should not have a zero pivot element. While nterchanging rows it is better to interchange with a row having largest pivotal lement. Thus, we interchange the first and fourth rows, which is called partial ivoting and get,

$$\begin{bmatrix} 9 & 4 & 4 & 0 \\ 4 & 10 & 5 & 4 \\ 4 & 5 & 6.5 & 2 \\ 0 & 4 & 2 & 8 \end{bmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 21 \\ 32 \\ 26 \\ 24 \end{pmatrix} \tag{2}$$

We observe that, in partial pivoting, the unknown vector remains unaltered, while he right-hand side vector gets changed.

Now, we shall carry out Gaussian elimination process in two stages.

*Stage I (Reduction to upper-triangular form):* In this stage, by dividing the first row of the system (2) by 9 and then subtracting this resulting row, multiplied by 4 and 4 from the second and third rows of Eq. (2), we get

$$\begin{bmatrix} 1 & \frac{4}{9} & \frac{4}{9} & 0 \\ 0 & 8.2222 & 3.2222 & 4 \\ 0 & 3.2222 & 4.7222 & 2 \\ 0 & 4 & 2 & 8 \end{bmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 2.3333 \\ 22.6666 \\ 16.6666 \\ 24 \end{pmatrix} \tag{3}$$

Now, we divide the second pivot row by 8.2222 and subtract the resultant row multiplied by 3.2222 and 4 from the third and fourth rows of Eq. (3) to get

$$\begin{bmatrix} 1 & \frac{4}{9} & \frac{4}{9} & 0 \\ 0 & 1 & 0.3919 & 0.4865 \\ 0 & 0 & 3.4594 & 0.4324 \\ 0 & 0 & 0.4324 & 6.0540 \end{bmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 2.3333 \\ 2.7568 \\ 7.7836 \\ 12.9728 \end{pmatrix} \tag{4}$$

Finally, we divide the third pivot row by 3.4594 and subtract the resultant multiplied by 0.4324 from fourth row of Eq. (4), thus getting the u triangular form

$$\begin{bmatrix} 1 & \frac{4}{9} & \frac{4}{9} & 0 \\ 0 & 1 & 0.3919 & 0.4865 \\ 0 & 0 & 1 & 0.1250 \\ 0 & 0 & 0 & 5.9999 \end{bmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 2.3333 \\ 2.7568 \\ 2.2500 \\ 11.9999 \end{pmatrix}$$

*Stage II (Back substitution):* From the last row of Eq. (5), we immediately $x_4 = 2.0000$. Using this value of $x_4$ into the third row of Eq. (5), we obta

$$x_3 + 0.25 = 2.25 \quad \text{or} \quad x_3 = 2.0000$$

Similarly, we get

$$x_2 = 1.0000, \quad x_1 = 1.0000$$

Thus, the solution of the given system is given by

$$x_1 = 1.0, \quad x_2 = 1.0, \quad x_3 = 2.0, \quad x_4 = 2.0$$

## 3.3 GAUSS–JORDAN ELIMINATION METHOD

This method is a variation of Gaussian elimination method. In this method, elements above and below the diagonal are simultaneously made zero thereby the given system is reduced to an equivalent diagonal form us elementary transformations. Then the solution of the resulting diagonal syst can be readily obtained.

Sometimes, we normalize the pivot row with respect to the pivot eleme before elimination. Partial pivoting is also used whenever the pivot elem becomes zero.

This method is illustrated through the following examples.

**Example 3.4** Solve the system of equations

$$\left.\begin{aligned} x + 2y + z &= 8 \\ 2x + 3y + 4z &= 20 \\ 4x + 3y + 2z &= 16 \end{aligned}\right\} \quad ($$

using Gauss–Jordan elimination method.

***Solution*** In matrix notation, the given system (1) can be written as

$$\begin{bmatrix} 1 & 2 & 1 \\ 2 & 3 & 4 \\ 4 & 3 & 2 \end{bmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 8 \\ 20 \\ 16 \end{pmatrix} \quad (2$$

We subtract the first row multiplied by 2 and 4 from the second and third rows respectively of Eq. (2), and eliminate $x$

$$\begin{bmatrix} 1 & 2 & 1 \\ 0 & -1 & 2 \\ 0 & -5 & -2 \end{bmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 8 \\ 4 \\ -16 \end{pmatrix} \tag{3}$$

Now, we eliminate $y$ from the first and third rows using the second row. Thus, we get

$$\begin{bmatrix} 1 & 0 & 5 \\ 0 & -1 & 2 \\ 0 & 0 & -12 \end{bmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 16 \\ 4 \\ -36 \end{pmatrix} \tag{4}$$

Before, eliminating $z$ from the first and second row, normalizing the third row with respect to the pivot element, we get

$$\begin{bmatrix} 1 & 0 & 5 \\ 0 & -1 & 2 \\ 0 & 0 & 1 \end{bmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 16 \\ 4 \\ 3 \end{pmatrix} \tag{5}$$

Using the third row of Eq. (5), eliminating $z$ from the first and second rows of Eq. (5), we obtain

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ -2 \\ 3 \end{pmatrix} \tag{6}$$

From Eq. (16), we get the solution directly as $x = 1, y = 2, z = 3$.

## 3.4 CROUT'S REDUCTION METHOD

This method is based on the fact that the coefficient matrix $[A]$ of the system of equations (3.3) can be decomposed into the product of two matrices $[L]$ and $[U]$, where $[L]$ is a lower-triangular matrix and $[U]$ is an upper-triangular matrix with 1's on its main diagonal. The rules for getting $[L]$ and $[U]$ can be obtained from the fact

$$[L]\,[U] = [A] \tag{3.7}$$

For the purpose of illustration, let us consider a $(3 \times 3)$ general matrix in the form

$$\begin{bmatrix} l_{11} & 0 & 0 \\ l_{21} & l_{22} & 0 \\ l_{31} & l_{32} & l_{33} \end{bmatrix} \begin{bmatrix} 1 & u_{12} & u_{13} \\ 0 & 1 & u_{23} \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \tag{3.8}$$